

Elliptic Curves

(PARI-GP version 2.15.3)

An elliptic curve is initially given by 5-tuple $v = [a_1, a_2, a_3, a_4, a_6]$ attached to Weierstrass model or simply $[a_4, a_6]$. It must be converted to an *ell* struct.

Initialize *ell* struct over domain D **E** = `ellinit(v, {D = 1})`
over **Q** $D = 1$
over **F_p** $D = p$
over **F_q**, $q = p^f$ $D = \text{ffgen}([p, f])$
over **Q_p**, precision n $D = O(p^n)$
over **C**, current bitprecision $D = 1.0$
over number field K $D = \text{nf}$

Points are $[x, y]$, the origin is $[0]$. Struct members accessed as **E.member**:

- All domains: **E.a1,a2,a3,a4,a6, b2,b4,b6,b8, c4,c6, disc, j**
- E defined over **R** or **C**
 x -coords. of points of order 2 **E.roots**
periods / quasi-periods **E.omega, E.eta**
volume of complex lattice **E.area**

- E defined over **Q_p**
residual characteristic **E.p**
If $|j|_p > 1$: Tate's $[u^2, u, q, [a, b], \mathcal{L}]$ **E.tate**
- E defined over **F_q**
characteristic **E.p**
 $\#E(\mathbf{F}_q)/\text{cyclic structure/generators}$ **E.no, E.cyc, E.gen**

- E defined over **Q**
generators of $E(\mathbf{Q})$ (require `elldata`) **E.gen**
 $[a_1, a_2, a_3, a_4, a_6]$ from j -invariant `ellfromj(j)`
cubic/quartic/biquadratic to Weierstrass `ellfromeqn(eq)`
add points $P + Q / P - Q$ `elladd(E, P, Q), ellsub`

- negate point `ellneg(E, P)`
- compute $n \cdot P$ `ellmul(E, P, n)`
- sum of Galois conjugates of P `elltrace(E, P)`
- check if P is on E `ellisoncurve(E, P)`
- order of torsion point P `ellorder(E, P)`
- y -coordinates of point(s) for x `ellordinate(E, x)`
- $[\varphi(z), \varphi'(z)] \in E(\mathbf{C})$ attached to $z \in \mathbf{C}$ `ellztopoint(E, z)`
- $z \in \mathbf{C}$ such that $P = [\varphi(z), \varphi'(z)]$ `ellpointtoz(E, P)`
- $z \in \bar{\mathbf{Q}}^*/q^{\mathbf{Z}}$ to $P \in E(\bar{\mathbf{Q}}_p)$ `ellztopoint(E, z)`
- $P \in E(\bar{\mathbf{Q}}_p)$ to $z \in \bar{\mathbf{Q}}^*/q^{\mathbf{Z}}$ `ellpointtoz(E, P)`

- Change of Weierstrass models, using** $v = [u, r, s, t]$
- change curve E using v `ellchangecurve(E, v)`
- change point P using v `ellchangepoint(P, v)`
- change point P using inverse of v `ellchangepointinv(P, v)`

- Twists and isogenies**
- quadratic twist `elltwist(E, d)`
- n -division polynomial $f_n(x)$ `elldivpol(E, n, {x})`
- $[n]P = (\phi_n \psi_n : \omega_n : \psi_n^3)$; return (ϕ_n, ψ_n^2) `ellxn(E, n, {x})`
- isogeny from E to E/G `ellisogeny(E, G)`
- apply isogeny to g (point or isogeny) `ellisogenyapply(f, g)`
- torsion subgroup with generators `elltors(E)`

- Formal group**
- formal exponential, n terms `ellformalexp(E, {n}, {x})`
- formal logarithm, n terms `ellformallog(E, {n}, {x})`
- $\log_E(-x(P)/y(P)) \in \mathbf{Q}_p$; $P \in E(\mathbf{Q}_p)$ `ellpadiclog(E, p, n, P)`
- P in the formal group `ellformalpoint(E, {n}, {x})`
- $[\omega/dt, x\omega/dt]$ `ellformaldifferential(E, {n}, {x})`
- $w = -1/y$ in parameter $-x/y$ `ellformalw(E, {n}, {x})`

Curves over finite fields, Pairings

random point on E `random(E)`
 $\#E(\mathbf{F}_q)$ `ellcard(E)`
 $\#E(\mathbf{F}_q)$ with almost prime order `ellsea(E, {tors})`
structure $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$ of $E(\mathbf{F}_q)$ `ellgroup(E)`
is E supersingular? `ellissupersingular(E)`
Weil pairing of m -torsion pts P, Q `ellweilpairing(E, P, Q, m)`
Tate pairing of P, Q ; P m -torsion `elltatepairing(E, P, Q, m)`
Discrete log, find n s.t. $P = [n]Q$ `elllog(E, P, Q, {ord})`

Curves over Q

- Reduction, minimal model**
- minimal model of E/\mathbf{Q} `ellminimalmodel(E, {\&v})`
- quadratic twist of minimal conductor `ellminimaltwist(E)`
- $[k]P$ with good reduction `ellnonsingularmultiple(E, P)`
- E supersingular at p ? `ellissupersingular(E, p)`
- affine points of naïve height $\leq h$ `ellratpoints(E, h)`

- Complex heights**
- canonical height of P `ellheight(E, P)`
- canonical bilinear form taken at P, Q `ellheight(E, P, Q)`
- height regulator matrix for pts in L `ellheightmatrix(E, L)`

- p -adic heights**
- cyclotomic p -adic height of $P \in E(\mathbf{Q})$ `ellpadicheight(E, p, n, P)`
- \dots bilinear form at $P, Q \in E(\mathbf{Q})$ `ellpadicheight(E, p, n, P, Q)`
- \dots matrix at vector for pts in L `ellpadicheightmatrix(E, p, n, L)`
- \dots regulator for canonical height `ellpadicregulator(E, p, n, Q)`
- Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$ `ellpadicfrobenius(E, p, n)`
- slope of unit eigenvector of Frobenius `ellpads2(E, p, n)`

- Isogenous curves**
- matrix of isogeny degrees for **Q**-isog. curves `ellisomat(E)`
- tree of prime degree isogenies `ellisotree(E)`
- a modular equation of prime degree N `ellmodulareqn(N)`

- L -function**
- p -th coeff a_p of L -function, p prime `ellap(E, p)`
- k -th coeff a_k of L -function `ellak(E, k)`
- $L(E, s)$ (using less memory than `lfun`) `elllseries(E, s)`
- $L^{(r)}(E, 1)$ (using less memory than `lfun`) `elll1(E, r)`
- a Heegner point on E of rank 1 `ellheegner(E)`
- order of vanishing at 1 `ellanalyticrank(E, {eps})`
- root number for $L(E, \cdot)$ at p `ellrootno(E, {p})`
- modular parametrization of E `elltaniyama(E)`
- degree of modular parametrization `ellmoddegree(E)`
- compare with $H^1(X_0(N), \mathbf{Z})$ (for $E' \rightarrow E$) `ellweilcurve(E)`

- p -adic L function $L_p^{(r)}(E, d, \chi^s)$ `ellpadicL(E, p, n, {s}, {r}, {d})`
- BSD conjecture for $L_p^{(r)}(E_D, \chi^0)$ `ellpadicbsd(E, p, n, {D = 1})`
- Iwasawa invariants for $L_p(E_D, \tau^i)$ `ellpadiclambda(E, p, D, i)`

- Rational points**
- attempt to compute $E(\mathbf{Q})$ `ellrank(E, {effort}, {points})`
- initialize for later `ellrank` calls, `ellrankinit(E)`
- saturate $\langle P_1, \dots, P_n \rangle$ wrt. primes $\leq B$ `ellsaturation(E, P, B)`
- 2-covers of the curve E `ell2cover(E)`

- Elldata package, Cremona's database:**
- db code "11a1" \leftrightarrow [`conductor, class, index`] `ellconvertname(s)`
- generators of Mordell-Weil group `ellgenerators(E)`
- look up E in database `ellidentify(E)`
- all curves matching criterion `ellsearch(N)`
- loop over curves with cond. from a to b `forell(E, a, b, seq)`

Curves over number field K

coeff a_p of L -function `ellap(E, p)`
Kodaira type of \mathfrak{p} -fiber of E `elllocalred(E, p)`
integral model of E/K `ellintegralmodel(E, {\&v})`
minimal model of E/K `ellminimalmodel(E, {\&v})`
minimal discriminant of E/K `ellminimaldisc(E)`
cond, min mod, Tamagawa num $[N, v, c]$ `ellglobalred(E)`
global Tamagawa number `elltamagawa(E)`
 $P \in E(K)$ n -divisible? $[n]Q = P$ `ellisdivisible(E, P, n, {\&Q})`

L -function

A domain $D = [c, w, h]$ in initialization mean we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w, |\Im(s)| < h; D = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $D = [1/2, 0, h]$ (critical line up to height h).
vector of first n a_k 's in L -function `ellan(E, n)`
init $L^{(k)}(E, s)$ for $k \leq n$ **L** = `lfuninit(E, D, {n = 0})`
compute $L(E, s)$ (n -th derivative) `lfun(L, s, {n = 0})`
 $L(E, 1, r)/(r! \cdot R \cdot \#Sha)$ assuming BSD `ellbsd(E)`

Other curves of small genus

A hyperelliptic curve C is given by a pair $[P, Q]$ ($y^2 + Qy = P$ with $Q^2 + 4P$ squarefree or a single squarefree polynomial P ($y^2 = P$)).
check if $[x, y]$ is on C `hyperellisoncurve(C, [x, y])`
discriminant of C `hyperelldisc(C)`
Cremona-Stoll reduction `hyperellred(C)`
apply $m = [e, [a, b; c, d], H]$ to model `hyperellchangecurve(C, m)`
minimal discriminant of integral C `hyperellminimaldisc(C)`
minimal model of integral C `hyperellminimalmodel(C)`
reduction of $y^2 + Qy = P$ (genus 2) `genus2red(C, {p})`
affine rational points of height $\leq h$ `hyperellratpoints(C, h)`
find a rational point on a conic, ${}^t xGx = 0$ `qfsolve(G)`
 $[H, U]$ such that $H = cU^tGU$ has minimat `defminimize(G)`
quadratic Hilbert symbol (at p) `hilbert(x, y, {p})`
all solutions in \mathbf{Q}^3 of ternary form `qfparam(G, x)`
 $P, Q \in \mathbf{F}_q[X]$; char. poly. of Frobenius `hyperellcharpoly(Q)`
matrix of Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1$ `hyperellpadicfrobenius`

Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$ or *ell* struct (**E.omega**), $\tau = \omega_1/\omega_2$.
arithmetic-geometric mean `agm(x, y)`
elliptic j -function $1/q + 744 + \dots$ `ellj(x)`
Weierstrass $\sigma/\wp/\zeta$ function `ellsigma(w, z), ellwp, ellzeta`
periods/quasi-periods `ellperiods(E, {flag}), elleta(w)`
 $(2i\pi/\omega_2)^k E_k(\tau)$ `elleisnum(w, k, {flag})`
modified Dedekind η func. $\prod(1 - q^n)$ `eta(x, {flag})`
Dedekind sum $s(h, k)$ `sumdedekind(h, k)`
Jacobi sine theta function `theta(q, z)`
 k -th derivative at $z=0$ of $\theta(q, z)$ `thetanullk(q, k)`
Weber's f functions `weber(x, {flag})`
modular pol. of level N `polmodular(N, {inv = j})`
Hilbert class polynomial for $\mathbf{Q}(\sqrt{D})$ `polclass(D, {inv = j})`

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