

**User's Guide**  
**to**  
**the PARI library**  
**(version 2.15.3)**

The PARI Group

Institut de Mathématiques de Bordeaux, UMR 5251 du CNRS.  
Université de Bordeaux, 351 Cours de la Libération  
F-33405 TALENCE Cedex, FRANCE  
e-mail: `pari@math.u-bordeaux.fr`

**Home Page:**  
<http://pari.math.u-bordeaux.fr/>

Copyright © 2000–2022 The PARI Group

Permission is granted to make and distribute verbatim copies of this manual provided the copyright notice and this permission notice are preserved on all copies.

Permission is granted to copy and distribute modified versions, or translations, of this manual under the conditions for verbatim copying, provided also that the entire resulting derived work is distributed under the terms of a permission notice identical to this one.

PARI/GP is Copyright © 2000–2022 The PARI Group

PARI/GP is free software; you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation. It is distributed in the hope that it will be useful, but WITHOUT ANY WARRANTY WHATSOEVER.

## Table of Contents

<b>Chapter 4: Programming PARI in Library Mode . . . . .</b>	<b>13</b>
4.1 Introduction: initializations, universal objects . . . . .	13
4.2 Important technical notes . . . . .	14
4.2.1 Backward compatibility . . . . .	14
4.2.2 Types . . . . .	14
4.2.3 Type recursivity . . . . .	15
4.2.4 Variations on basic functions . . . . .	15
4.2.5 Portability: 32-bit / 64-bit architectures . . . . .	16
4.2.6 Using <code>malloc</code> / <code>free</code> . . . . .	17
4.3 Garbage collection . . . . .	17
4.3.1 Why and how . . . . .	17
4.3.2 Variants . . . . .	20
4.3.3 Examples . . . . .	20
4.3.4 Comments . . . . .	24
4.4 Creation of PARI objects, assignments, conversions . . . . .	24
4.4.1 Creation of PARI objects . . . . .	24
4.4.2 Sizes . . . . .	26
4.4.3 Assignments . . . . .	26
4.4.4 Copy . . . . .	27
4.4.5 Clones . . . . .	27
4.4.6 Conversions . . . . .	28
4.5 Implementation of the PARI types . . . . .	28
4.5.1 Type <code>t_INT</code> (integer) . . . . .	29
4.5.2 Type <code>t_REAL</code> (real number) . . . . .	30
4.5.3 Type <code>t_INTMOD</code> . . . . .	31
4.5.4 Type <code>t_FRAC</code> (rational number) . . . . .	31
4.5.5 Type <code>t_FFELT</code> (finite field element) . . . . .	31
4.5.6 Type <code>t_COMPLEX</code> (complex number) . . . . .	31
4.5.7 Type <code>t_PADIC</code> ( $p$ -adic numbers) . . . . .	32
4.5.8 Type <code>t_QUAD</code> (quadratic number) . . . . .	32
4.5.9 Type <code>t_POLMOD</code> (polmod) . . . . .	32
4.5.10 Type <code>t_POL</code> (polynomial) . . . . .	32
4.5.11 Type <code>t_SER</code> (power series) . . . . .	33
4.5.12 Type <code>t_RFRAC</code> (rational function) . . . . .	34
4.5.13 Type <code>t_QFB</code> (binary quadratic form) . . . . .	34
4.5.14 Type <code>t_VEC</code> and <code>t_COL</code> (vector) . . . . .	34
4.5.15 Type <code>t_MAT</code> (matrix) . . . . .	34
4.5.16 Type <code>t_VECSMALL</code> (vector of small integers) . . . . .	34
4.5.17 Type <code>t_STR</code> (character string) . . . . .	34
4.5.18 Type <code>t_ERROR</code> (error context) . . . . .	34
4.5.19 Type <code>t_CLOSURE</code> (closure) . . . . .	34
4.5.20 Type <code>t_INFINITY</code> (infinity) . . . . .	34
4.5.21 Type <code>t_LIST</code> (list) . . . . .	34
4.6 PARI variables . . . . .	35
4.6.1 Multivariate objects . . . . .	35
4.6.2 Creating variables . . . . .	35

4.6.3 Comparing variables . . . . .	37
4.7 Input and output . . . . .	38
4.7.1 Input . . . . .	38
4.7.2 Output to screen or file, output to string . . . . .	39
4.7.3 Errors . . . . .	40
4.7.4 Warnings . . . . .	41
4.7.5 Debugging output . . . . .	41
4.7.6 Timers and timing output . . . . .	42
4.8 Iterators, Numerical integration, Sums, Products . . . . .	43
4.8.1 Iterators . . . . .	43
4.8.2 Iterating over primes . . . . .	44
4.8.3 Parallel iterators . . . . .	45
4.8.4 Numerical analysis . . . . .	47
4.9 Catching exceptions . . . . .	47
4.9.1 Basic use . . . . .	47
4.9.2 Advanced use . . . . .	48
4.10 A complete program . . . . .	49
<b>Chapter 5: Technical Reference Guide: the basics . . . . .</b>	<b>53</b>
5.1 Initializing the library . . . . .	53
5.1.1 General purpose . . . . .	53
5.1.2 Technical functions . . . . .	54
5.1.3 Notions specific to the GP interpreter . . . . .	56
5.1.4 Public callbacks . . . . .	57
5.1.5 Configuration variables . . . . .	58
5.1.6 Utility functions . . . . .	58
5.1.7 Saving and restoring the GP context . . . . .	59
5.1.8 GP history . . . . .	59
5.2 Handling GENs . . . . .	60
5.2.1 Allocation . . . . .	60
5.2.2 Length conversions . . . . .	61
5.2.3 Read type-dependent information . . . . .	62
5.2.4 Eval type-dependent information . . . . .	63
5.2.5 Set type-dependent information . . . . .	64
5.2.6 Type groups . . . . .	65
5.2.7 Accessors and components . . . . .	65
5.3 Global numerical constants . . . . .	66
5.3.1 Constants related to word size . . . . .	66
5.3.2 Masks used to implement the GEN type . . . . .	66
5.3.3 $\log 2$ , $\pi$ . . . . .	67
5.4 Iterating over small primes, low-level interface . . . . .	67
5.5 Handling the PARI stack . . . . .	68
5.5.1 Allocating memory on the stack . . . . .	68
5.5.2 Stack-independent binary objects . . . . .	69
5.5.3 Garbage collection . . . . .	70
5.5.4 Garbage collection: advanced use . . . . .	72
5.5.5 Debugging the PARI stack . . . . .	73
5.5.6 Copies . . . . .	73
5.5.7 Simplify . . . . .	73
5.6 The PARI heap . . . . .	73

5.6.1 Introduction . . . . .	73
5.6.2 Public interface . . . . .	74
5.6.3 Implementation note . . . . .	74
5.7 Handling user and temp variables . . . . .	75
5.7.1 Low-level . . . . .	75
5.7.2 User variables . . . . .	75
5.7.3 Temporary variables . . . . .	75
5.8 Adding functions to PARI . . . . .	76
5.8.1 Nota Bene . . . . .	76
5.8.2 Coding guidelines . . . . .	76
5.8.3 GP prototypes, parser codes . . . . .	77
5.8.4 Integration with <b>gp</b> as a shared module . . . . .	79
5.8.5 Library interface for <b>install</b> . . . . .	80
5.8.6 Integration by patching <b>gp</b> . . . . .	80
5.9 Globals related to PARI configuration . . . . .	81
5.9.1 PARI version numbers . . . . .	81
5.9.2 Miscellaneous . . . . .	81
<b>Chapter 6: Arithmetic kernel: Level 0 and 1 . . . . .</b>	<b>83</b>
6.1 Level 0 kernel (operations on ulongs) . . . . .	83
6.1.1 Micro-kernel . . . . .	83
6.1.2 Modular kernel . . . . .	84
6.1.3 Modular kernel with “precomputed inverse” . . . . .	85
6.1.4 Switching between FL_XXX and standard operators . . . . .	87
6.2 Level 1 kernel (operations on longs, integers and reals) . . . . .	88
6.2.1 Creation . . . . .	88
6.2.2 Assignment . . . . .	89
6.2.3 Copy . . . . .	89
6.2.4 Conversions . . . . .	90
6.2.5 Integer parts . . . . .	91
6.2.6 2-adic valuations and shifts . . . . .	91
6.2.7 From <b>t_INT</b> to bits or digits in base $2^k$ and back . . . . .	92
6.2.8 Integer valuation . . . . .	93
6.2.9 Generic unary operators . . . . .	94
6.2.10 Comparison operators . . . . .	94
6.2.11 Generic binary operators . . . . .	96
6.2.12 Exact division and divisibility . . . . .	98
6.2.13 Division with integral operands and <b>t_REAL</b> result . . . . .	99
6.2.14 Division with remainder . . . . .	99
6.2.15 Modulo to longs . . . . .	100
6.2.16 Powering, Square root . . . . .	101
6.2.17 GCD, extended GCD and LCM . . . . .	102
6.2.18 Continued fractions and convergents . . . . .	103
6.2.19 Pseudo-random integers . . . . .	103
6.2.20 Modular operations . . . . .	103
6.2.21 Extending functions to vector inputs . . . . .	106
6.2.22 Miscellaneous arithmetic functions . . . . .	107
<b>Chapter 7: Level 2 kernel . . . . .</b>	<b>109</b>
7.1 Naming scheme . . . . .	109
7.2 Coefficient ring . . . . .	111

7.3 Modular arithmetic . . . . .	112
7.3.1 FpC / FpV, FpM . . . . .	112
7.3.2 Flc / Flv, Flm . . . . .	116
7.3.3 F2c / F2v, F2m . . . . .	119
7.3.4 F3c / F3v, F3m . . . . .	121
7.3.5 FlxqV, FlxqC, FlxqM . . . . .	122
7.3.6 FpX . . . . .	122
7.3.7 FpXQ, Fq . . . . .	127
7.3.8 FpXQ . . . . .	128
7.3.9 Fq . . . . .	129
7.3.10 FpXn . . . . .	131
7.3.11 FpXC, FpXM . . . . .	131
7.3.12 FpXX, FpXY . . . . .	131
7.3.13 FpXQX, FqX . . . . .	132
7.3.14 FpXQXn, FqXn . . . . .	134
7.3.15 FpXQXQ, FqXQ . . . . .	135
7.3.16 Flx . . . . .	138
7.3.17 FlxV . . . . .	143
7.3.18 FlxM . . . . .	143
7.3.19 FlxT . . . . .	143
7.3.20 Flxn . . . . .	144
7.3.21 Flxq . . . . .	144
7.3.22 FlxX . . . . .	146
7.3.23 FlxXV, FlxXC, FlxXM . . . . .	147
7.3.24 FlxqX . . . . .	148
7.3.25 FlxqXQ . . . . .	151
7.3.26 FlxqXn . . . . .	152
7.3.27 F2x . . . . .	152
7.3.28 F2xq . . . . .	154
7.3.29 F2xn . . . . .	155
7.3.30 F2xqV, F2xqM . . . . .	155
7.3.31 F2xX . . . . .	155
7.3.32 F2xXV/F2xXC . . . . .	156
7.3.33 F2xqX . . . . .	156
7.3.34 F2xqXQ . . . . .	157
7.3.35 Functions returning objects with <code>t_INTMOD</code> coefficients . . . . .	158
7.3.36 Slow Chinese remainder theorem over $\mathbf{Z}$ . . . . .	159
7.3.37 Fast remainders . . . . .	161
7.3.38 Fast Chinese remainder theorem over $\mathbf{Z}$ . . . . .	162
7.3.39 Rational reconstruction . . . . .	163
7.3.40 Zp . . . . .	164
7.3.41 ZpM . . . . .	164
7.3.42 ZpX . . . . .	164
7.3.43 ZpXQ . . . . .	166
7.3.44 Zq . . . . .	166
7.3.45 ZpXQM . . . . .	166
7.3.46 ZpXQX . . . . .	166
7.3.47 ZqX . . . . .	167
7.3.48 Other $p$ -adic functions . . . . .	167

7.3.49	Conversions involving single precision objects	169
7.4	Higher arithmetic over <b>Z</b> : primes, factorization	172
7.4.1	Pure powers	172
7.4.2	Factorization	173
7.4.3	Coprime factorization	175
7.4.4	Checks attached to arithmetic functions	176
7.4.5	Incremental integer factorization	177
7.4.6	Integer core, squarefree factorization	177
7.4.7	Primes, primality and compositeness tests	178
7.4.8	Iterators over primes	179
7.5	Integral, rational and generic linear algebra	180
7.5.1	<b>ZC</b> / <b>ZV</b> , <b>ZM</b>	180
7.5.2	<b>QM</b>	184
7.5.3	<b>Qevproj</b>	184
7.5.4	<b>zv</b> , <b>zm</b>	185
7.5.5	<b>ZMV</b> / <b>zmV</b> (vectors of <b>ZM</b> / <b>zm</b> )	186
7.5.6	<b>QC</b> / <b>QV</b> , <b>QM</b>	186
7.5.7	<b>RgC</b> / <b>RgV</b> , <b>RgM</b>	186
7.5.8	<b>ZG</b>	191
7.5.9	Sparse and blackbox linear algebra	192
7.5.10	Obsolete functions	193
7.6	Integral, rational and generic polynomial arithmetic	193
7.6.1	<b>ZX</b>	193
7.6.2	Resultants	197
7.6.3	<b>ZXV</b>	197
7.6.4	<b>ZXT</b>	197
7.6.5	<b>ZXQ</b>	197
7.6.6	<b>ZXn</b>	198
7.6.7	<b>ZXQM</b>	198
7.6.8	<b>ZXQX</b>	198
7.6.9	<b>ZXX</b>	198
7.6.10	<b>QX</b>	199
7.6.11	<b>QXQ</b>	199
7.6.12	<b>QXQX</b>	200
7.6.13	<b>QXQM</b>	201
7.6.14	<b>ZX</b>	201
7.6.15	<b>RgX</b>	201
7.6.16	<b>RgXn</b>	207
7.6.17	<b>RgXnV</b>	208
7.6.18	<b>RgXQ</b>	208
7.6.19	<b>RgXQV</b> , <b>RgXQC</b>	209
7.6.20	<b>RgXQM</b>	209
7.6.21	<b>RgXQX</b>	209
<b>Chapter 8:</b>	<b>Black box algebraic structures</b>	<b>209</b>
8.1	Black box groups	210
8.1.1	Black box groups with pairing	212
8.1.2	Functions returning black box groups	212
8.2	Black box fields	213
8.2.1	Functions returning black box fields	214

8.3 Black box algebra . . . . .	214
8.3.1 Functions returning black box algebras . . . . .	215
8.4 Black box ring . . . . .	215
8.5 Black box free $\mathbf{Z}_p$ -modules . . . . .	216
<b>Chapter 9: Operations on general PARI objects . . . . .</b>	<b>217</b>
9.1 Assignment . . . . .	217
9.2 Conversions . . . . .	217
9.2.1 Scalars . . . . .	217
9.2.2 Modular objects / lifts . . . . .	219
9.2.3 Between polynomials and coefficient arrays . . . . .	219
9.3 Constructors . . . . .	222
9.3.1 Clean constructors . . . . .	222
9.3.2 Unclean constructors . . . . .	224
9.3.3 From roots to polynomials . . . . .	227
9.4 Integer parts . . . . .	228
9.5 Valuation and shift . . . . .	228
9.6 Comparison operators . . . . .	229
9.6.1 Generic . . . . .	229
9.6.2 Comparison with a small integer . . . . .	229
9.7 Miscellaneous Boolean functions . . . . .	230
9.7.1 Obsolete . . . . .	231
9.8 Sorting . . . . .	231
9.8.1 Basic sort . . . . .	231
9.8.2 Indirect sorting . . . . .	231
9.8.3 Generic sort and search . . . . .	232
9.8.4 Further useful comparison functions . . . . .	233
9.9 Division . . . . .	233
9.10 Divisibility, Euclidean division . . . . .	234
9.11 GCD, content and primitive part . . . . .	235
9.11.1 Generic . . . . .	235
9.11.2 Over the rationals . . . . .	235
9.12 Generic arithmetic operators . . . . .	237
9.12.1 Unary operators . . . . .	237
9.12.2 Binary operators . . . . .	237
9.13 Generic operators: product, powering, factorback . . . . .	238
9.14 Matrix and polynomial norms . . . . .	240
9.15 Substitution and evaluation . . . . .	241
<b>Chapter 10: Miscellaneous mathematical functions . . . . .</b>	<b>243</b>
10.1 Fractions . . . . .	243
10.2 Binomials . . . . .	243
10.3 Real numbers . . . . .	243
10.4 Complex numbers . . . . .	244
10.5 Quadratic numbers and binary quadratic forms . . . . .	244
10.6 Polynomials . . . . .	245
10.7 Power series . . . . .	246
10.8 Functions to handle $\mathbf{t\_FFELT}$ . . . . .	246
10.8.1 FFX . . . . .	249
10.8.2 FFM . . . . .	250
10.8.3 FFXQ . . . . .	251



10.9 Transcendental functions . . . . .	251
10.9.1 Transcendental functions with <b>t_REAL</b> arguments . . . . .	251
10.9.2 Other complex transcendental functions . . . . .	252
10.9.3 Modular functions . . . . .	254
10.9.4 Transcendental functions with <b>t_PADIC</b> arguments . . . . .	254
10.9.5 Cached constants . . . . .	254
10.9.6 Obsolete functions . . . . .	255
10.10 Permutations . . . . .	255
10.11 Small groups . . . . .	256
<b>Chapter 11: Standard data structures . . . . .</b>	<b>261</b>
11.1 Character strings . . . . .	261
11.1.1 Functions returning a <b>char *</b> . . . . .	261
11.1.2 Functions returning a <b>t_STR</b> . . . . .	262
11.1.3 Dynamic strings . . . . .	262
11.2 Output . . . . .	263
11.2.1 Output contexts . . . . .	263
11.2.2 Default output context . . . . .	263
11.2.3 PARI colors . . . . .	264
11.2.4 Obsolete output functions . . . . .	264
11.3 Files . . . . .	265
11.3.1 pariFILE . . . . .	265
11.3.2 Temporary files . . . . .	266
11.4 Errors . . . . .	266
11.4.1 Internal errors, “system” errors . . . . .	266
11.4.2 Syntax errors, type errors . . . . .	267
11.4.3 Overflows . . . . .	268
11.4.4 Errors triggered intentionally . . . . .	269
11.4.5 Mathematical errors . . . . .	270
11.4.6 Miscellaneous functions . . . . .	271
11.5 Hashtables . . . . .	271
11.6 Dynamic arrays . . . . .	273
11.6.1 Initialization . . . . .	273
11.6.2 Adding elements . . . . .	274
11.6.3 Accessing elements . . . . .	274
11.6.4 Stack of stacks . . . . .	274
11.6.5 Public interface . . . . .	274
11.7 Vectors and Matrices . . . . .	275
11.7.1 Access and extract . . . . .	275
11.7.2 Componentwise operations . . . . .	276
11.7.3 Low-level vectors and columns functions . . . . .	277
11.8 Vectors of small integers . . . . .	278
11.8.1 <b>t_VECSMALL</b> . . . . .	278
11.8.2 Vectors of <b>t_VECSMALL</b> . . . . .	279
<b>Chapter 12: Functions related to the GP interpreter . . . . .</b>	<b>281</b>
12.1 Handling closures . . . . .	281
12.1.1 Functions to evaluate <b>t_CLOSURE</b> . . . . .	281
12.1.2 Functions to handle control flow changes . . . . .	282
12.1.3 Functions to deal with lexical local variables . . . . .	282
12.1.4 Functions returning new closures . . . . .	283

12.1.5 Functions used by the gp debugger (break loop)	283
12.1.6 Standard wrappers for iterators	283
12.2 Defaults	284
12.3 Records and Lazy vectors	287
<b>Chapter 13: Algebraic Number Theory</b>	<b>289</b>
13.1 General Number Fields	289
13.1.1 Number field types	289
13.1.2 Extracting info from a <b>nf</b> structure	291
13.1.3 Extracting info from a <b>bnf</b> structure	292
13.1.4 Extracting info from a <b>bnr</b> structure	293
13.1.5 Extracting info from an <b>rnf</b> structure	293
13.1.6 Extracting info from a <b>bid</b> structure	294
13.1.7 Extracting info from a <b>znstar</b> structure	295
13.1.8 Inserting info in a number field structure	295
13.1.9 Increasing accuracy	296
13.1.10 Number field arithmetic	297
13.1.11 Number field arithmetic for linear algebra	299
13.1.12 Cyclotomic field arithmetic for linear algebra	300
13.1.13 Cyclotomic trace	300
13.1.14 Elements in factored form	301
13.1.15 Ideal arithmetic	302
13.1.16 Maximal ideals	305
13.1.17 Decomposition groups	307
13.1.18 Reducing modulo maximal ideals	307
13.1.19 Valuations	308
13.1.20 Signatures	309
13.1.21 Complex embeddings	310
13.1.22 Maximal order and discriminant, conversion to <b>nf</b> structure	311
13.1.23 Computing in the class group	312
13.1.24 Floating point embeddings, the $T_2$ quadratic form	313
13.1.25 Ideal reduction, low level	314
13.1.26 Ideal reduction, high level	315
13.1.27 Class field theory	316
13.1.28 Abelian maps	318
13.1.29 Grunwald–Wang theorem	318
13.1.30 Relative equations, Galois conjugates	318
13.1.31 Units	320
13.1.32 Obsolete routines	320
13.2 Galois extensions of <b>Q</b>	322
13.2.1 Extracting info from a <b>gal</b> structure	322
13.2.2 Miscellaneous functions	322
13.3 Quadratic number fields and quadratic forms	323
13.3.1 Checks	323
13.3.2 Class number	323
13.3.3 <b>t_QFB</b>	324
13.3.4 Efficient real quadratic forms	326
13.4 Linear algebra over <b>Z</b>	327
13.4.1 Hermite and Smith Normal Forms	327
13.4.2 The LLL algorithm	331

13.4.3 Linear dependencies . . . . .	333
13.4.4 Reduction modulo matrices . . . . .	333
13.5 Finite abelian groups and characters . . . . .	334
13.5.1 Abstract groups . . . . .	334
13.5.2 Dirichlet characters . . . . .	335
13.6 Hecke characters . . . . .	336
13.7 Central simple algebras . . . . .	336
13.7.1 Initialization . . . . .	336
13.7.2 Type checks . . . . .	337
13.7.3 Shallow accessors . . . . .	337
13.7.4 Other low-level functions . . . . .	338
<b>Chapter 14: Elliptic curves and arithmetic geometry . . . . .</b>	<b>339</b>
14.1 Elliptic curves . . . . .	339
14.1.1 Types of elliptic curves . . . . .	339
14.1.2 Type checking . . . . .	339
14.1.3 Extracting info from an <code>ell</code> structure . . . . .	340
14.1.4 Points . . . . .	344
14.1.5 Change of variables . . . . .	344
14.1.6 Generic helper functions . . . . .	344
14.1.7 Functions to handle elliptic curves over finite fields . . . . .	345
14.2 Arithmetic on elliptic curve over a finite field in simple form . . . . .	345
14.2.1 Helper functions . . . . .	345
14.2.2 Elliptic curves over $\mathbf{F}_p$ , $p > 3$ . . . . .	346
14.2.3 <code>FpE</code> . . . . .	346
14.2.4 <code>Fle</code> . . . . .	347
14.2.5 <code>FpJ</code> . . . . .	348
14.2.6 <code>Flj</code> . . . . .	348
14.2.7 Elliptic curves over $\mathbf{F}_{2^n}$ . . . . .	349
14.2.8 <code>F2xqE</code> . . . . .	349
14.2.9 Elliptic curves over $\mathbf{F}_q$ , small characteristic $p > 2$ . . . . .	350
14.2.10 <code>FlxqE</code> . . . . .	350
14.2.11 Elliptic curves over $\mathbf{F}_q$ , large characteristic . . . . .	351
14.2.12 <code>FpXQE</code> . . . . .	351
14.3 Functions related to modular polynomials . . . . .	352
14.3.1 Functions related to modular invariants . . . . .	352
14.4 Other curves . . . . .	353
<b>Chapter 15: <math>L</math>-functions . . . . .</b>	<b>355</b>
15.1 Accessors . . . . .	355
15.2 Conversions and constructors . . . . .	356
15.3 Variants of GP functions . . . . .	357
15.4 Inverse Mellin transforms of Gamma products . . . . .	357
<b>Chapter 16: Modular symbols . . . . .</b>	<b>359</b>
<b>Chapter 17: Modular forms . . . . .</b>	<b>361</b>
17.1 Implementation of public data structures . . . . .	361
17.1.1 Accessors for modular form spaces . . . . .	361
17.1.2 Accessors for individual modular forms . . . . .	362
17.1.3 Nebentypus . . . . .	363
17.1.4 Miscellaneous functions . . . . .	363
<b>Chapter 18: Plots . . . . .</b>	<b>365</b>

18.1 Highlevel functions . . . . .	365
18.2 Function . . . . .	366
18.2.1 Obsolete functions . . . . .	366
18.3 Dump rectwindows to a PostScript or SVG file . . . . .	367
18.4 Technical functions exported for convenience . . . . .	367
<b>Appendix A: A Sample program and Makefile . . . . .</b>	<b>369</b>
<b>Appendix B: PARI and threads . . . . .</b>	<b>371</b>
Index . . . . .	374

## Chapter 4:

# Programming PARI in Library Mode

The *User's Guide to Pari/GP* gives in three chapters a general presentation of the system, of the `gp` calculator, and detailed explanation of high level PARI routines available through the calculator. The present manual assumes general familiarity with the contents of these chapters and the basics of ANSI C programming, and focuses on the usage of the PARI library. In this chapter, we introduce the general concepts of PARI programming and describe useful general purpose functions; the following chapters describes all public low or high-level functions, underlying or extending the GP functions seen in Chapter 3 of the User's guide.

### 4.1 Introduction: initializations, universal objects.

To use PARI in library mode, you must write a C program and link it to the PARI library. See the installation guide or the Appendix to the *User's Guide to Pari/GP* on how to create and install the library and include files. A sample Makefile is presented in Appendix A, and a more elaborate one in `examples/Makefile`. The best way to understand how programming is done is to work through a complete example. We will write such a program in Section 4.10. Before doing this, a few explanations are in order.

First, one must explain to the outside world what kind of objects and routines we are going to use. This is done\* with the directive

```
#include <pari/pari.h>
```

In particular, this defines the fundamental type for all PARI objects: the type **GEN**, which is simply a pointer to `long`.

Before any PARI routine is called, one must initialize the system, and in particular the PARI stack which is both a scratchboard and a repository for computed objects. This is done with a call to the function

```
void pari_init(size_t size, ulong maxprime)
```

The first argument is the number of bytes given to PARI to work with, and the second is the upper limit on a precomputed prime number table; `size` should not reasonably be taken below 500000 but you may set `maxprime = 0`, although the system still needs to precompute all primes up to about  $2^{16}$ . For lower-level variants allowing finer control, e.g. preventing PARI from installing its own error or signal handlers, see Section 5.1.2.

We have now at our disposal:

- a PARI *stack* containing nothing. This is a big connected chunk of `size` bytes of memory, where all computations take place. In large computations, intermediate results quickly clutter up memory so some kind of garbage collecting is needed. Most systems do garbage collecting when the memory is getting scarce, and this slows down the performance. PARI takes a different approach,

---

\* This assumes that PARI headers are installed in a directory which belongs to your compiler's search path for header files. You might need to add flags like `-I/usr/local/include` or modify `C_INCLUDE_PATH`.

admittedly more demanding on the programmer: you must do your own cleaning up when the intermediate results are not needed anymore. We will see later how (and when) this is done.

- the following *universal objects* (by definition, objects which do not belong to the stack): the integers 0, 1,  $-1$ , 2 and  $-2$  (respectively called `gen_0`, `gen_1`, `gen_m1`, `gen_2` and `gen_m2`), the fraction  $\frac{1}{2}$  (`ghalf`). All of these are of type `GEN`.

- a *heap* which is just a linked list of permanent universal objects. For now, it contains exactly the ones listed above. You will probably very rarely use the heap yourself; and if so, only as a collection of copies of objects taken from the stack (called clones in the sequel). Thus you need not bother with its internal structure, which may change as PARI evolves. Some complex PARI functions create clones for special garbage collecting purposes, usually destroying them when returning.

- a table of primes (in fact of *differences* between consecutive primes), called `diffptr`, of type `byteptr` (pointer to `unsigned char`). Its use is described in Section 5.4 later. Using it directly is deprecated, high-level iterators provide a cleaner and more flexible interface, see Section 4.8.2 (such iterators use the private prime table, but extend it dynamically).

- access to all the built-in functions of the PARI library. These are declared to the outside world when you include `pari.h`, but need the above things to function properly. So if you forget the call to `pari_init`, you will get a fatal error when running your program.

## 4.2 Important technical notes.

**4.2.1 Backward compatibility.** The PARI function names evolved over time, and deprecated functions are eventually deleted. The file `pariold.h` contains macros implementing a weak form of backward compatibility. In particular, whenever the name of a documented function changes, a `#define` is added to this file so that the old name expands to the new one (provided the prototype didn't change also).

This file is included by `pari.h`, but a large section is commented out by default. Define `PARI_OLD_NAMES` before including `pari.h` to pollute your namespace with lots of obsolete names like `un*`: that might enable you to compile old programs without having to modify them. The preferred way to do that is to add `-DPARI_OLD_NAMES` to your compiler `CFLAGS`, so that you don't need to modify the program files themselves.

Of course, it's better to fix the program if you can!

### 4.2.2 Types.

Although PARI objects all have the C type `GEN`, we will freely use the word **type** to refer to PARI dynamic subtypes: `t_INT`, `t_REAL`, etc. The declaration

```
GEN x;
```

declares a C variable of type `GEN`, but its “value” will be said to have type `t_INT`, `t_REAL`, etc. The meaning should always be clear from the context.

---

\* For (long)`gen_1`. Since 2004 and version 2.2.9, typecasts are completely unnecessary in PARI programs.

### 4.2.3 Type recursivity.

Conceptually, most PARI types are recursive. But the **GEN** type is a pointer to **long**, not to **GEN**. So special macros must be used to access **GEN**'s components. The simplest one is **gel**(*V*, *i*), where **el** stands for **e**lement, to access component number *i* of the **GEN** *V*. This is a valid **lvalue** (may be put on the left side of an assignment), and the following two constructions are exceedingly frequent

```
gel(V, i) = x;  
x = gel(V, i);
```

where **x** and **V** are **GEN**s. This macro accesses and modifies directly the components of *V* and do not create a copy of the coefficient, contrary to all the library *functions*.

More generally, to retrieve the values of elements of lists of ... of lists of vectors we have the **gmael** macros (for **m**ultidimensional **a**rray **e**lement). The syntax is **gmael***n*(*V*, *a*<sub>1</sub>, ..., *a*<sub>*n*</sub>), where *V* is a **GEN**, the *a*<sub>*i*</sub> are indexes, and *n* is an integer between 1 and 5. This stands for *x*[*a*<sub>1</sub>][*a*<sub>2</sub>]...[*a*<sub>*n*</sub>], and returns a **GEN**. The macros **gel** (resp. **gmael**) are synonyms for **gmael1** (resp. **gmael2**).

Finally, the macro **gcoeff**(*M*, *i*, *j*) has exactly the meaning of *M*[*i*,*j*] in GP when *M* is a matrix. Note that due to the implementation of **t\_MATs** as horizontal lists of vertical vectors, **gcoeff**(*x*,*y*) is actually equivalent to **gmael**(*y*,*x*). One should use **gcoeff** in matrix context, and **gmael** otherwise.

**4.2.4 Variations on basic functions.** In the library syntax descriptions in Chapter 3, we have only given the basic names of the functions. For example **gadd**(*x*,*y*) assumes that *x* and *y* are **GEN**s, and *creates* the result *x*+*y* on the PARI stack. For most of the basic operators and functions, many other variants are available. We give some examples for **gadd**, but the same is true for all the basic operators, as well as for some simple common functions (a complete list is given in Chapter 6):

**GEN** **gaddgs**(**GEN** *x*, **long** *y*)

**GEN** **gaddsg**(**long** *x*, **GEN** *y*)

In the following one, *z* is a preexisting **GEN** and the result of the corresponding operation is put into *z*. The size of the PARI stack does not change:

**void** **gaddz**(**GEN** *x*, **GEN** *y*, **GEN** *z*)

(This last form is inefficient in general and deprecated outside of PARI kernel programming.) Low level kernel functions implement these operators for specialized arguments and are also available: Level 0 deals with operations at the word level (**longs** and **ulongs**), Level 1 with **t\_INT** and **t\_REAL** and Level 2 with the rest (modular arithmetic, polynomial arithmetic and linear algebra). Here are some examples of Level 1 functions:

**GEN** **addii**(**GEN** *x*, **GEN** *y*): here *x* and *y* are **GEN**s of type **t\_INT** (this is not checked).

**GEN** **addr**(**GEN** *x*, **GEN** *y*): here *x* and *y* are **GEN**s of type **t\_REAL** (this is not checked).

There also exist functions **addir**, **addri**, **mpadd** (whose two arguments can be of type **t\_INT** or **t\_REAL**), **addis** (to add a **t\_INT** and a **long**) and so on.

The Level 1 names are self-explanatory once you know that **i** stands for a **t\_INT**, **r** for a **t\_REAL**, **mp** for **i** or **r**, **s** for a signed C long integer, **u** for an unsigned C long integer; finally the suffix **z** means that the result is not created on the PARI stack but assigned to a preexisting **GEN** object passed as an extra argument. Chapter 6 gives a description of these low-level functions.

Level 2 names are more complicated, see Section 7.1 for all the gory details, and we content ourselves with a simple example used to implement `t_INTMOD` arithmetic:

`GEN Fp_add(GEN x, GEN y, GEN m)`: returns the sum of  $x$  and  $y$  modulo  $m$ . Here  $x, y, m$  are `t_INTs` (this is not checked). The operation is more efficient if the inputs  $x, y$  are reduced modulo  $m$ , but this is not a necessary condition.

**Important Note.** These specialized functions are of course more efficient than the generic ones, but note the hidden danger here: the types of the objects involved (which is not checked) must be severely controlled, e.g. using `addii` on a `t_FRAC` argument will cause disasters. Type mismatches may corrupt the PARI stack, though in most cases they will just immediately overflow the stack. Because of this, the PARI philosophy of giving a result which is as exact as possible, enforced for generic functions like `gadd` or `gmul`, is dropped in kernel routines of Level 1, where it is replaced by the much simpler rule: the result is a `t_INT` if and only if all arguments are integer types (`t_INT` but also `C long` and `ulong`) and a `t_REAL` otherwise. For instance, multiplying a `t_REAL` by a `t_INT` always yields a `t_REAL` if you use `mulir`, where `gmul` returns the `t_INT gen_0` if the integer is 0.

#### 4.2.5 Portability: 32-bit / 64-bit architectures.

PARI supports both 32-bit and 64-bit based machines, but not simultaneously! The library is compiled assuming a given architecture, and some of the header files you include (through `pari.h`) will have been modified to match the library.

Portable macros are defined to bypass most machine dependencies. If you want your programs to run identically on 32-bit and 64-bit machines, you have to use these, and not the corresponding numeric values, whenever the precise size of your `long` integers might matter. Here are the most important ones:

	64-bit	32-bit	
<code>BITS_IN_LONG</code>	64	32	
<code>LONG_IS_64BIT</code>	defined	undefined	
<code>DEFAULTPREC</code>	3	4	( $\approx 19$ decimal digits, see formula below)
<code>MEDDEFAULTPREC</code>	4	6	( $\approx 38$ decimal digits)
<code>BIGDEFAULTPREC</code>	5	8	( $\approx 57$ decimal digits)

For instance, suppose you call a transcendental function, such as

`GEN gexp(GEN x, long prec).`

The last argument `prec` is an integer  $\geq 3$ , corresponding to the default floating point precision required. It is *only* used if `x` is an exact object, otherwise the relative precision is determined by the precision of `x`. Since the parameter `prec` sets the size of the inexact result counted in (`long`) *words* (including codewords), the same value of `prec` will yield different results on 32-bit and 64-bit machines. Real numbers have two codewords (see Section 4.5), so the formula for computing the bit accuracy is

$$\text{bit\_accuracy}(\text{prec}) = (\text{prec} - 2) * \text{BITS\_IN\_LONG}$$

(this is actually the definition of an inline function). The corresponding accuracy expressed in decimal digits would be

$$\text{bit\_accuracy}(\text{prec}) * \log(2) / \log(10).$$

For example if the value of `prec` is 5, the corresponding accuracy for 32-bit machines is  $(5 - 2) * \log(2^{32}) / \log(10) \approx 28$  decimal digits, while for 64-bit machines it is  $(5 - 2) * \log(2^{64}) / \log(10) \approx 57$  decimal digits.



Thus, you must take care to change the `prec` parameter you are supplying according to the bit size, either using the default precisions given by the various `DEFAULTPRECS`, or by using conditional constructs of the form:

```
#ifndef LONG_IS_64BIT
    prec = 4;
#else
    prec = 6;
#endif
```

which is in this case equivalent to the statement `prec = MEDDEFAULTPREC;`.

Note that for parity reasons, half the accuracies available on 32-bit architectures (the odd ones) have no precise equivalents on 64-bit machines.

**4.2.6 Using `malloc` / `free`.** You should make use of the PARI stack as much as possible, and avoid allocating objects using the customary functions. If you do, you should use, or at least have a very close look at, the following wrappers:

`void* pari_malloc(size_t size)` calls `malloc` to allocate `size` bytes and returns a pointer to the allocated memory. If the request fails, an error is raised. The `SIGINT` signal is blocked until `malloc` returns, to avoid leaving the system stack in an inconsistent state.

`void* pari_realloc(void* ptr, size_t size)` as `pari_malloc` but calls `realloc` instead of `malloc`.

`void pari_realloc_ip(void** ptr, size_t size)` equivalent to `*ptr= realloc(*ptr, size)`, while blocking `SIGINT`.

`void* pari_calloc(size_t size)` as `pari_malloc`, setting the memory to zero.

`void pari_free(void* ptr)` calls `free` to liberate the memory space pointed to by `ptr`, which must have been allocated by `malloc` (`pari_malloc`) or `realloc` (`pari_realloc`). The `SIGINT` signal is blocked until `free` returns.

If you use the standard `libc` functions instead of our wrappers, then your functions will be subtly incompatible with the `gp` calculator: when the user tries to interrupt a computation, the calculator may crash (if a system call is interrupted at the wrong time).

## 4.3 Garbage collection.

### 4.3.1 Why and how.

As we have seen, `pari_init` allocates a big range of addresses, the *stack*, that are going to be used throughout. Recall that all PARI objects are pointers. Except for a few universal objects, they all point at some part of the stack.

The stack starts at the address `bot` and ends just before `top`. This means that the quantity

$$(\text{top} - \text{bot}) / \text{sizeof}(\text{long})$$

is (roughly) equal to the `size` argument of `pari_init`. The PARI stack also has a “current stack pointer” called `avma`, which stands for **a**vailable **m**emory **a**ddress. These three variables are global (declared by `pari.h`). They are of type `pari_sp`, which means *pari stack pointer*.

The stack is oriented upside-down: the more recent an object, the closer to `bot`. Accordingly, initially `avma = top`, and `avma` gets *decremented* as new objects are created. As its name indicates, `avma` always points just *after* the first free address on the stack, and `(GEN)avma` is always (a pointer to) the latest created object. When `avma` reaches `bot`, the stack overflows, aborting all computations, and an error message is issued. To avoid this *you* need to clean up the stack from time to time, when intermediate objects are not needed anymore. This is called “*garbage collecting*.”

We are now going to describe briefly how this is done. We will see many concrete examples in the next subsection.

- First, PARI routines do their own garbage collecting, which means that whenever a documented function from the library returns, only its result(s) have been added to the stack, possibly up to a very small overhead (undocumented ones may not do this). In particular, a PARI function that does not return a `GEN` does not clutter the stack. Thus, if your computation is small enough (e.g. you call few PARI routines, or most of them return `long` integers), then you do not need to do any garbage collecting. This is probably the case in many of your subroutines. Of course the objects that were on the stack *before* the function call are left alone. Except for the ones listed below, PARI functions only collect their own garbage.

- It may happen that all objects that were created after a certain point can be deleted — for instance, if the final result you need is not a `GEN`, or if some search proved futile. Then, it is enough to record the value of `avma` just *before* the first garbage is created, and restore it upon exit:

```

pari_sp av = avma; /* record initial avma */

garbage ...
set_avma(av); /* restore it */

```

All objects created in the `garbage` zone will eventually be overwritten: they should no longer be accessed after `avma` has been restored. Think of the `set_avma` call as a simple `avma = av` restoring the `avma` value.

- If you want to destroy (i.e. give back the memory occupied by) the *latest* PARI object on the stack (e.g. the latest one obtained from a function call), you can use the function

```
void cgiv(GEN z)
```

where `z` is the object you want to give back. This is equivalent to the above where the initial `av` is computed from `z`.

- Unfortunately life is not so simple, and sometimes you will want to give back accumulated garbage *during* a computation without losing recent data. We shall start with the lowest level function to get a feel for the underlying mechanisms, we shall describe simpler variants later:

`GEN gerepile(pari_sp ltop, pari_sp lbot, GEN q)`. This function cleans up the stack between `ltop` and `lbot`, where `lbot < ltop`, and returns the updated object `q`. This means:

- 1) we translate (copy) all the objects in the interval `[avma, lbot]`, so that its right extremity abuts the address `ltop`. Graphically

```

      bot          avma  lbot          ltop    top
End of stack |-----[+++++[---/--/--/--/--|+++++] Start
              free memory          garbage

```

becomes:

```

      bot          avma  ltop    top
End of stack |-----[+++++[+++++] Start
              free memory

```

where ++ denote significant objects, -- the unused part of the stack, and --- the garbage we remove.

2) The function then inspects all the PARI objects between `avma` and `lbot` (i.e. the ones that we want to keep and that have been translated) and looks at every component of such an object which is not a codeword. Each such component is a pointer to an object whose address is either

- between `avma` and `lbot`, in which case it is suitably updated,
- larger than or equal to `ltop`, in which case it does not change, or
- between `lbot` and `ltop` in which case `gerepile` raises an error (“significant pointers lost in gerepile”).

3) `avma` is updated (we add `ltop - lbot` to the old value).

4) We return the (possibly updated) object `q`: if `q` initially pointed between `avma` and `lbot`, we return the updated address, as in 2). If not, the original address is still valid, and is returned!

As stated above, no component of the remaining objects (in particular `q`) should belong to the erased segment `[lbot, ltop[`, and this is checked within `gerepile`. But beware as well that the addresses of the objects in the translated zone change after a call to `gerepile`, so you must not access any pointer which previously pointed into the zone below `ltop`. If you need to recover more than one object, use the `gerepileall` function below.

**Remark.** As a consequence of the preceding explanation, if a PARI object is to be relocated by `gerepile` then, apart from universal objects, the chunks of memory used by its components should be in consecutive memory locations. All `GENs` created by documented PARI functions are guaranteed to satisfy this. This is because the `gerepile` function knows only about *two connected zones*: the garbage that is erased (between `lbot` and `ltop`) and the significant pointers that are copied and updated. If there is garbage interspersed with your objects, disaster occurs when we try to update them and consider the corresponding “pointers”. In most cases of course the said garbage is in fact a bunch of other `GENs`, in which case we simply waste time copying and updating them for nothing. But be wary when you allow objects to become disconnected.

In practice this is achieved by the following programming idiom:

```

ltop = avma; garbage(); lbot = avma; q = anything();
return gerepile(ltop, lbot, q); /* returns the updated q */

```

or directly

```

ltop = avma; garbage(); lbot = avma;
return gerepile(ltop, lbot, anything());

```

Beware that

```
ltop = avma; garbage();
return gerepile(ltop, avma, anything())
```

might work, but should be frowned upon. We cannot predict whether `avma` is evaluated after or before the call to `anything()`: it depends on the compiler. If we are out of luck, it is *after* the call, so the result belongs to the garbage zone and the `gerepile` statement becomes equivalent to `set_avma(ltop)`. Thus we return a pointer to random garbage.

### 4.3.2 Variants.

`GEN gerepileupto(pari_sp ltop, GEN q)`. Cleans the stack between `ltop` and the *connected* object `q` and returns `q` updated. For this to work, `q` must have been created *before* all its components, otherwise they would belong to the garbage zone! Unless mentioned otherwise, documented PARI functions guarantee this.

`GEN gerepilecopy(pari_sp ltop, GEN x)`. Functionally equivalent to, but more efficient than

```
gerepileupto(ltop, gcopy(x))
```

In this case, the `GEN` parameter `x` need not satisfy any property before the garbage collection: it may be disconnected, components created before the root, and so on. Of course, this is about twice slower than either `gerepileupto` or `gerepile`, because `x` has to be copied to a clean stack zone first. This function is a special case of `gerepileall` below, where  $n = 1$ .

`void gerepileall(pari_sp ltop, int n, ...)`. To cope with complicated cases where many objects have to be preserved. The routine expects  $n$  further arguments, which are the *addresses* of the `GENs` you want to preserve:

```
pari_sp ltop = avma;
...; y = ...; ... x = ...; ...;
gerepileall(ltop, 2, &x, &y);
```

It cleans up the most recent part of the stack (between `ltop` and `avma`), updating all the `GENs` added to the argument list. A copy is done just before the cleaning to preserve them, so they do not need to be connected before the call. With `gerepilecopy`, this is the most robust of the `gerepile` functions (the less prone to user error), hence the slowest.

`void gerepileallsp(pari_sp ltop, pari_sp lbot, int n, ...)`. More efficient, but trickier than `gerepileall`. Cleans the stack between `lbot` and `ltop` and updates the `GENs` pointed at by the elements of `gp` without any further copying. This is subject to the same restrictions as `gerepile`, the only difference being that more than one address gets updated.

### 4.3.3 Examples.

#### 4.3.3.1 gerepile.

Let `x` and `y` be two preexisting PARI objects and suppose that we want to compute  $x^2 + y^2$ . This is done using the following program:

```
GEN x2 = gsqr(x);
GEN y2 = gsqr(y), z = gadd(x2,y2);
```

The `GEN z` indeed points at the desired quantity. However, consider the stack: it contains as unnecessary garbage `x2` and `y2`. More precisely it contains (in this order) `z`, `y2`, `x2`. (Recall that, since the stack grows downward from the top, the most recent object comes first.)

It is not possible to get rid of  $x_2$ ,  $y_2$  before  $z$  is computed, since they are used in the final operation. We cannot record `avma` before  $x_2$  is computed and restore it later, since this would destroy  $z$  as well. It is not possible either to use the function `cgiv` since  $x_2$  and  $y_2$  are not at the bottom of the stack and we do not want to give back  $z$ .

But using `gerepile`, we can give back the memory locations corresponding to  $x_2$ ,  $y_2$ , and move the object  $z$  upwards so that no space is lost. Specifically:

```
pari_sp ltop = avma; /* remember the current top of the stack */
GEN x2 = gsqr(x);
GEN y2 = gsqr(y);
pari_sp lbot = avma; /* the bottom of the garbage pile */
GEN z = gadd(x2, y2); /* z is now the last object on the stack */
z = gerepile(ltop, lbot, z);
```

Of course, the last two instructions could also have been written more simply:

```
z = gerepile(ltop, lbot, gadd(x2,y2));
```

In fact `gerepileupto` is even simpler to use, because the result of `gadd` is the last object on the stack and `gadd` is guaranteed to return an object suitable for `gerepileupto`:

```
ltop = avma;
z = gerepileupto(ltop, gadd(gsqr(x), gsqr(y)));
```

Make sure you understand exactly what has happened before you go on!

**Remark on assignments and `gerepile`.** When the tree structure and the size of the PARI objects which will appear in a computation are under control, one may allocate sufficiently large objects at the beginning, use assignment statements, then simply restore `avma`. Coming back to the above example, note that *if* we know that  $x$  and  $y$  are of type real fitting into `DEFAULTPREC` words, we can program without using `gerepile` at all:

```
z = cgetr(DEFAULTPREC); ltop = avma;
gaffect(gadd(gsqr(x), gsqr(y)), z);
set_avma(ltop);
```

This is often *slower* than a craftily used `gerepile` though, and certainly more cumbersome to use. As a rule, assignment statements should generally be avoided.

**Variations on a theme.** it is often necessary to do several `gerepiles` during a computation. However, the fewer the better. The only condition for `gerepile` to work is that the garbage be connected. If the computation can be arranged so that there is a minimal number of connected pieces of garbage, then it should be done that way.

For example suppose we want to write a function of two `GEN` variables  $x$  and  $y$  which creates the vector  $[x^2 + y, y^2 + x]$ . Without garbage collecting, one would write:

```
p1 = gsqr(x); p2 = gadd(p1, y);
p3 = gsqr(y); p4 = gadd(p3, x);
z = mkvec2(p2, p4); /* not suitable for gerepileupto! */
```

This leaves a dirty stack containing (in this order)  $z$ ,  $p_4$ ,  $p_3$ ,  $p_2$ ,  $p_1$ . The garbage here consists of  $p_1$  and  $p_3$ , which are separated by  $p_2$ . But if we compute  $p_3$  *before*  $p_2$  then the garbage becomes connected, and we get the following program with garbage collecting:

```

ltop = avma; p1 = gsqr(x); p3 = gsqr(y);
lbot = avma; z = cgetg(3, t_VEC);
gel(z, 1) = gadd(p1,y);
gel(z, 2) = gadd(p3,x); z = gerepile(ltop,lbot,z);

```

Finishing by `z = gerepileupto(ltop, z)` would be ok as well. Beware that

```

ltop = avma; p1 = gadd(gsqr(x), y); p3 = gadd(gsqr(y), x);
z = cgetg(3, t_VEC);
gel(z, 1) = p1;
gel(z, 2) = p3; z = gerepileupto(ltop,z); /* WRONG */

```

is a disaster since `p1` and `p3` are created before `z`, so the call to `gerepileupto` overwrites them, leaving `gel(z, 1)` and `gel(z, 2)` pointing at random data! The following does work:

```

ltop = avma; p1 = gsqr(x); p3 = gsqr(y);
lbot = avma; z = mkvec2(gadd(p1,y), gadd(p3,x));
z = gerepile(ltop,lbot,z);

```

but is very subtly wrong in the sense that `z = gerepileupto(ltop, z)` would *not* work. The reason being that `mkvec2` creates the root `z` of the vector *after* its arguments have been evaluated, creating the components of `z` too early; `gerepile` does not care, but the created `z` is a time bomb which will explode on any later `gerepileupto`. On the other hand

```

ltop = avma; z = cgetg(3, t_VEC);
gel(z, 1) = gadd(gsqr(x), y);
gel(z, 2) = gadd(gsqr(y), x); z = gerepileupto(ltop,z); /* INEFFICIENT */

```

leaves the results of `gsqr(x)` and `gsqr(y)` on the stack (and lets `gerepileupto` update them for naught). Finally, the most elegant and efficient version (with respect to time and memory use) is as follows

```

z = cgetg(3, t_VEC);
ltop = avma; gel(z, 1) = gerepileupto(ltop, gadd(gsqr(x), y));
ltop = avma; gel(z, 2) = gerepileupto(ltop, gadd(gsqr(y), x));

```

which avoids updating the container `z` and cleans up its components individually, as soon as they are computed.

**One last example.** Let us compute the product of two complex numbers  $x$  and  $y$ , using the  $3M$  method which requires 3 multiplications instead of the obvious 4. Let  $z = x*y$ , and set  $x = x_r + i*x_i$  and similarly for  $y$  and  $z$ . We compute  $p_1 = x_r * y_r$ ,  $p_2 = x_i * y_i$ ,  $p_3 = (x_r + x_i) * (y_r + y_i)$ , and then we have  $z_r = p_1 - p_2$ ,  $z_i = p_3 - (p_1 + p_2)$ . The program is as follows:

```

ltop = avma;
p1 = gmul(gel(x,1), gel(y,1));
p2 = gmul(gel(x,2), gel(y,2));
p3 = gmul(gadd(gel(x,1), gel(x,2)), gadd(gel(y,1), gel(y,2)));
p4 = gadd(p1,p2);
lbot = avma; z = cgetg(3, t_COMPLEX);
gel(z, 1) = gsub(p1,p2);
gel(z, 2) = gsub(p3,p4); z = gerepile(ltop,lbot,z);

```

**Exercise.** Write a function which multiplies a matrix by a column vector. Hint: start with a `cgetg` of the result, and use `gerepile` whenever a coefficient of the result vector is computed. You can look at the answer in `src/basemath/RgV.c:RgM_RgC_mul()`.

#### 4.3.3.2 `gerepileall`.

Let us now see why we may need the `gerepileall` variants. Although it is not an infrequent occurrence, we do not give a specific example but a general one: suppose that we want to do a computation (usually inside a larger function) producing more than one PARI object as a result, say two for instance. Then even if we set up the work properly, before cleaning up we have a stack which has the desired results `z1`, `z2` (say), and then connected garbage from `lbot` to `ltop`. If we write

```
z1 = gerepile(ltop, lbot, z1);
```

then the stack is cleaned, the pointers fixed up, but we have lost the address of `z2`. This is where we need the `gerepileall` function:

```
gerepileall(ltop, 2, &z1, &z2)
```

copies `z1` and `z2` to new locations, cleans the stack from `ltop` to the old `avma`, and updates the pointers `z1` and `z2`. Here we do not assume anything about the stack: the garbage can be disconnected and `z1`, `z2` need not be at the bottom of the stack. If all of these assumptions are in fact satisfied, then we can call `gerepilemanysp` instead, which is usually faster since we do not need the initial copy (on the other hand, it is less cache friendly).

A most important usage is “random” garbage collection during loops whose size requirements we cannot (or do not bother to) control in advance:

```
pari_sp av = avma;
GEN x, y;
while (...)
{
    garbage(); x = anything();
    garbage(); y = anything(); garbage();
    if (gc_needed(av,1)) /* memory is running low (half spent since entry) */
        gerepileall(av, 2, &x, &y);
}
```

Here we assume that only `x` and `y` are needed from one iteration to the next. As it would be costly to call `gerepile` once for each iteration, we only do it when it seems to have become necessary.

More precisely, the macro `stack_lim(av,n)` denotes an address where  $2^{n-1}/(2^{n-1} + 1)$  of the remaining stack space since reference point `av` is exhausted (1/2 for  $n = 1$ , 2/3 for  $n = 2$ ). The test `gc_needed(av,n)` becomes true whenever `avma` drops below that address.

#### 4.3.4 Comments.

First, `gerepile` has turned out to be a flexible and fast garbage collector for number-theoretic computations, which compares favorably with more sophisticated methods used in other systems. Our benchmarks indicate that the price paid for using `gerepile` and `gerepile`-related copies, when properly used, is usually less than 1% of the total running time, which is quite acceptable!

Second, it is of course harder on the programmer, and quite error-prone if you do not stick to a consistent PARI programming style. If all seems lost, just use `gerepilecopy` (or `gerepileall`) to fix up the stack for you. You can always optimize later when you have sorted out exactly which routines are crucial and what objects need to be preserved and their usual sizes.

If you followed us this far, congratulations, and rejoice: the rest is much easier.

### 4.4 Creation of PARI objects, assignments, conversions.

**4.4.1 Creation of PARI objects.** The basic function which creates a PARI object is

`GEN cgetg(long l, long t)`  $l$  specifies the number of longwords to be allocated to the object, and  $t$  is the type of the object, in symbolic form (see Section 4.5 for the list of these). The precise effect of this function is as follows: it first creates on the PARI *stack* a chunk of memory of size `length` longwords, and saves the address of the chunk which it will in the end return. If the stack has been used up, a message to the effect that “the PARI stack overflows” is printed, and an error raised. Otherwise, it sets the type and length of the PARI object. In effect, it fills its first codeword (`z[0]`). Many PARI objects also have a second codeword (types `t_INT`, `t_REAL`, `t_PADIC`, `t_POL`, and `t_SER`). In case you want to produce one of those from scratch, which should be exceedingly rare, *it is your responsibility to fill this second codeword*, either explicitly (using the macros described in Section 4.5), or implicitly using an assignment statement (using `gaffect`).

Note that the length argument  $l$  is predetermined for a number of types: 3 for types `t_INTMOD`, `t_FRAC`, `t_COMPLEX`, `t_POLMOD`, `t_RFRAC`, 4 for type `t_QUAD`, and 5 for type `t_PADIC` and `t_QFB`. However for the sake of efficiency, `cgetg` does not check this: disasters will occur if you give an incorrect length for those types.

**Notes.** 1) The main use of this function is create efficiently a constant object, or to prepare for later assignments (see Section 4.4.3). Most of the time you will use `GEN` objects as they are created and returned by PARI functions. In this case you do not need to use `cgetg` to create space to hold them.

2) For the creation of leaves, i.e. `t_INT` or `t_REAL`,

```
GEN cgeti(long length)
```

```
GEN cgetr(long length)
```

should be used instead of `cgetg(length, t_INT)` and `cgetg(length, t_REAL)` respectively. Finally

```
GEN cgetc(long prec)
```

creates a `t_COMPLEX` whose real and imaginary part are `t_REALs` allocated by `cgetr(prec)`.



**Examples.** 1) Both `z = cgeti(DEFAULTPREC)` and `cgetg(DEFAULTPREC, t_INT)` create a `t_INT` whose “precision” is `bit_accuracy(DEFAULTPREC) = 64`. This means `z` can hold rational integers of absolute value less than  $2^{64}$ . Note that in both cases, the second codeword is *not* filled. Of course we could use numerical values, e.g. `cgeti(4)`, but this would have different meanings on different machines as `bit_accuracy(4)` equals 64 on 32-bit machines, but 128 on 64-bit machines.

2) The following creates a *complex number* whose real and imaginary parts can hold real numbers of precision `bit_accuracy(MEDDEFAULTPREC) = 96` bits:

```
z = cgetg(3, t_COMPLEX);
gel(z, 1) = cgetr(MEDDEFAULTPREC);
gel(z, 2) = cgetr(MEDDEFAULTPREC);
```

or simply `z = cgetc(MEDDEFAULTPREC)`.

3) To create a matrix object for  $4 \times 3$  matrices:

```
z = cgetg(4, t_MAT);
for(i=1; i<4; i++) gel(z, i) = cgetg(5, t_COL);
```

or simply `z = zeromatcopy(4, 3)`, which further initializes all entries to `gen_0`.

These last two examples illustrate the fact that since PARI types are recursive, all the branches of the tree must be created. The function `cgetg` creates only the “root”, and other calls to `cgetg` must be made to produce the whole tree. For matrices, a common mistake is to think that `z = cgetg(4, t_MAT)` (for example) creates the root of the matrix: one needs also to create the column vectors of the matrix (obviously, since we specified only one dimension in the first `cgetg`!). This is because a matrix is really just a row vector of column vectors (hence a priori not a basic type), but it has been given a special type number so that operations with matrices become possible.

Finally, to facilitate input of constant objects when speed is not paramount, there are four `varargs` functions:

`GEN mkintn(long n, ...)` returns the nonnegative `t_INT` whose development in base  $2^{32}$  is given by the following  $n$  32bit-words (`unsigned int`).

```
mkintn(3, a2, a1, a0);
```

returns  $a_2 2^{64} + a_1 2^{32} + a_0$ .

`GEN mkpoln(long n, ...)` Returns the `t_POL` whose  $n$  coefficients (`GEN`) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial  $X^2 + 2X$  (in variable 0, use `setvarn` if you want other variable numbers). Beware that  $n$  is the number of coefficients, hence *one more* than the degree.

`GEN mkvecn(long n, ...)` returns the `t_VEC` whose  $n$  coefficients (`GEN`) follow.

`GEN mkcoln(long n, ...)` returns the `t_COL` whose  $n$  coefficients (`GEN`) follow.

**Warning.** Contrary to the policy of general PARI functions, the latter three functions do *not* copy their arguments, nor do they produce an object a priori suitable for `gerepileupto`. For instance

```
/* gerepile-safe: components are universal objects */
z = mkvecn(3, gen_1, gen_0, gen_2);
/* not OK for gerepileupto: stoi(3) creates component before root */
z = mkvecn(3, stoi(3), gen_0, gen_2);
/* NO! First vector component x is destroyed */
x = gclone(gen_1);
z = mkvecn(3, x, gen_0, gen_2);
guncclone(x);
```

The following function is also available as a special case of `mkintn`:

`GEN uu32toi(ulong a, ulong b)`

Returns the GEN equal to  $2^{32}a + b$ , *assuming* that  $a, b < 2^{32}$ . This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

#### 4.4.2 Sizes.

`long gsizeword(GEN x)` returns the total number of BITS\_IN\_LONG-bit words occupied by the tree representing `x`.

`long gsizebyte(GEN x)` returns the total number of bytes occupied by the tree representing `x`, i.e. `gsizeword(x)` multiplied by `sizeof(long)`. This is normally useless since PARI functions use a number of *words* as input for lengths and precisions.

**4.4.3 Assignments.** Firstly, if `x` and `y` are both declared as `GEN` (i.e. pointers to something), the ordinary C assignment `y = x` makes perfect sense: we are just moving a pointer around. However, physically modifying either `x` or `y` (for instance, `x[1] = 0`) also changes the other one, which is usually not desirable.

**Very important note.** Using the functions described in this paragraph is inefficient and often awkward: one of the `gerepile` functions (see Section 4.3) should be preferred. See the paragraph end for one exception to this rule.

The general PARI assignment function is the function `gaffect` with the following syntax:

```
void gaffect(GEN x, GEN y)
```

Its effect is to assign the PARI object `x` into the *preexisting* object `y`. Both `x` and `y` must be *scalar* types. For convenience, vector or matrices of scalar types are also allowed.

This copies the whole structure of `x` into `y` so many conditions must be met for the assignment to be possible. For instance it is allowed to assign a `t_INT` into a `t_REAL`, but the converse is forbidden. For that, you must use the truncation or rounding function of your choice, e.g. `mpfloor`.

It can also happen that `y` is not large enough or does not have the proper tree structure to receive the object `x`. For instance, let `y` the zero integer with length equal to 2; then `y` is too small to accommodate any nonzero `t_INT`. In general common sense tells you what is possible, keeping in mind the PARI philosophy which says that if it makes sense it is valid. For instance, the assignment of an imprecise object into a precise one does *not* make sense. However, a change in precision of imprecise objects is allowed, even if it *increases* its accuracy: we complement the “mantissa” with

infinitely many 0 digits in this case. (Mantissa between quotes, because this is not restricted to `t_REALs`, it also applies for  $p$ -adics for instance.)

All functions ending in “z” such as **gaddz** (see Section 4.2.4) implicitly use this function. In fact what they exactly do is record **avma** (see Section 4.3), perform the required operation, **gaffect** the result to the last operand, then restore the initial **avma**.

You can assign ordinary C long integers into a PARI object (not necessarily of type `t_INT`) using

```
void gaffsg(long s, GEN y)
```

**Note.** Due to the requirements mentioned above, it is usually a bad idea to use **gaffect** statements. There is one exception: for simple objects (e.g. leaves) whose size is controlled, they can be easier to use than **gerepile**, and about as efficient.

**Coercion.** It is often useful to coerce an inexact object to a given precision. For instance at the beginning of a routine where precision can be kept to a minimum; otherwise the precision of the input is used in all subsequent computations, which is inefficient if the latter is known to thousands of digits. One may use the **gaffect** function for this, but it is easier and more efficient to call

`GEN gtotfp(GEN x, long prec)` converts the complex number  $x$  (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of length `prec`.

**4.4.4 Copy.** It is also very useful to copy a PARI object, not just by moving around a pointer as in the `y = x` example, but by creating a copy of the whole tree structure, without pre-allocating a possibly complicated `y` to use with **gaffect**. The function which does this is called **gcopy**. Its syntax is:

```
GEN gcopy(GEN x)
```

and the effect is to create a new copy of  $x$  on the PARI stack.

Sometimes, on the contrary, a quick copy of the skeleton of  $x$  is enough, leaving pointers to the original data in  $x$  for the sake of speed instead of making a full recursive copy. Use `GEN shallowcopy(GEN x)` for this. Note that the result is not suitable for **gerepileupto** !

Make sure at this point that you understand the difference between `y = x`, `y = gcopy(x)`, `y = shallowcopy(x)` and **gaffect**( $x, y$ ).

**4.4.5 Clones.** Sometimes, it is more efficient to create a *persistent* copy of a PARI object. This is not created on the stack but on the heap, hence unaffected by **gerepile** and friends. The function which does this is called **gclone**. Its syntax is:

```
GEN gclone(GEN x)
```

A clone can be removed from the heap (thus destroyed) using

```
void gunclone(GEN x)
```

No PARI object should keep references to a clone which has been destroyed!

**4.4.6 Conversions.** The following functions convert C objects to PARI objects (creating them on the stack as usual):

`GEN stoi(long s):` C long integer (“small”) to `t_INT`.

`GEN dbltor(double s):` C double to `t_REAL`. The accuracy of the result is 19 decimal digits, i.e. a type `t_REAL` of length `DEFAULTPREC`, although on 32-bit machines only 16 of them are significant.

We also have the converse functions:

`long itos(GEN x):` `x` must be of type `t_INT`,

`double rtodbl(GEN x):` `x` must be of type `t_REAL`,

as well as the more general ones:

`long gtolong(GEN x),`

`double gtodouble(GEN x).`

## 4.5 Implementation of the PARI types.

We now go through each type and explain its implementation. Let `z` be a `GEN`, pointing at a PARI object. In the following paragraphs, we will constantly mix two points of view: on the one hand, `z` is treated as the C pointer it is, on the other, as PARI’s handle on some mathematical entity, so we will shamelessly write `z ≠ 0` to indicate that the *value* thus represented is nonzero (in which case the *pointer* `z` is certainly not `NULL`). We offer no apologies for this style. In fact, you had better feel comfortable juggling both views simultaneously in your mind if you want to write correct PARI programs.

Common to all the types is the first codeword `z[0]`, which we do not have to worry about since this is taken care of by `cgetg`. Its precise structure depends on the machine you are using, but it always contains the following data: the *internal type number* attached to the symbolic type name, the *length* of the root in longwords, and a technical bit which indicates whether the object is a clone or not (see Section 4.4.5). This last one is used by `gp` for internal garbage collecting, you will not have to worry about it.

Some types have a second codeword, different for each type, which we will soon describe as we will shortly consider each of them in turn.

The first codeword is handled through the following *macros*:

`long typ(GEN z)` returns the type number of `z`.

`void settyp(GEN z, long n)` sets the type number of `z` to `n` (you should not have to use this function if you use `cgetg`).

`long lg(GEN z)` returns the length (in longwords) of the root of `z`.

`long setlg(GEN z, long l)` sets the length of `z` to `l`; you should not have to use this function if you use `cgetg`.

`void lg_increase(GEN z)` increase the length of `z` by 1; you should not have to use this function if you use `cgetg`.

`long isclone(GEN z)` is `z` a clone?

`void setisclone(GEN z)` sets the *clone* bit.

`void unsetisclone(GEN z)` clears the *clone* bit.

**Important remark.** For the sake of efficiency, none of the codeword-handling macros check the types of their arguments even when there are stringent restrictions on their use. It is trivial to create invalid objects, or corrupt one of the “universal constants” (e.g. setting the sign of `gen_0` to 1), and they usually provide negligible savings. Use higher level functions whenever possible.

**Remark.** The clone bit is there so that `gunclone` can check it is deleting an object which was allocated by `gclone`. Miscellaneous vector entries are often cloned by `gp` so that a GP statement like `v[1] = x` does not involve copying the whole of `v`: the component `v[1]` is deleted if its clone bit is set, and is replaced by a clone of `x`. Don’t set/unset yourself the clone bit unless you know what you are doing: in particular *never* set the clone bit of a vector component when the said vector is scheduled to be uncloned. Hackish code may abuse the clone bit to tag objects for reasons unrelated to the above instead of using proper data structures. Don’t do that.

**4.5.1 Type `t_INT` (integer).** this type has a second codeword `z[1]` which contains the following information:

the sign of `z`: coded as 1, 0 or  $-1$  if  $z > 0$ ,  $z = 0$ ,  $z < 0$  respectively.

the *effective length* of `z`, i.e. the total number of significant longwords. This means the following: apart from the integer 0, every integer is “normalized”, meaning that the most significant mantissa longword is nonzero. However, the integer may have been created with a longer length. Hence the “length” which is in `z[0]` can be larger than the “effective length” which is in `z[1]`.

This information is handled using the following macros:

`long signe(GEN z)` returns the sign of `z`.

`void setsigne(GEN z, long s)` sets the sign of `z` to `s`.

`long lgefint(GEN z)` returns the effective length of `z`.

`void setlgefint(GEN z, long l)` sets the effective length of `z` to `l`.

The integer 0 can be recognized either by its sign being 0, or by its effective length being equal to 2. Now assume that  $z \neq 0$ , and let

$$|z| = \sum_{i=0}^n z_i B^i, \quad \text{where } z_n \neq 0 \text{ and } B = 2^{\text{BITS\_IN\_LONG}}.$$

With these notations,  $n$  is `lgefint(z) - 3`, and the mantissa of `z` may be manipulated via the following interface:

`GEN int_MSW(GEN z)` returns a pointer to the most significant word of `z`,  $z_n$ .

`GEN int_LSW(GEN z)` returns a pointer to the least significant word of `z`,  $z_0$ .

`GEN int_W(GEN z, long i)` returns the  $i$ -th significant word of `z`,  $z_i$ . Accessing the  $i$ -th significant word for  $i > n$  yields unpredictable results.

`GEN int_W_lg(GEN z, long i, long lz)` returns the  $i$ -th significant word of `z`,  $z_i$ , assuming `lgefint(z)` is `lz` ( $= n + 3$ ). Accessing the  $i$ -th significant word for  $i > n$  yields unpredictable results.

`GEN int_precW(GEN z)` returns the previous (less significant) word of `z`,  $z_{i-1}$  assuming `z` points to  $z_i$ .

`GEN int_nextW(GEN z)` returns the next (more significant) word of  $z$ ,  $z_{i+1}$  assuming  $z$  points to  $z_i$ .

Unnormalized integers, such that  $z_n$  is possibly 0, are explicitly forbidden. To enforce this, one may write an arbitrary mantissa then call

```
void int_normalize(GEN z, long known0)
```

normalizes in place a nonnegative integer (such that  $z_n$  is possibly 0), assuming at least the first `known0` words are zero.

For instance a binary `and` could be implemented in the following way:

```
GEN AND(GEN x, GEN y) {
    long i, lx, ly, lout;
    long *xp, *yp, *outp; /* mantissa pointers */
    GEN out;
    if (!signe(x) || !signe(y)) return gen_0;
    lx = lgefint(x); xp = int_LSW(x);
    ly = lgefint(y); yp = int_LSW(y); lout = min(lx,ly); /* > 2 */
    out = cgeti(lout); out[1] = evalsigne(1) | evallgefint(lout);
    outp = int_LSW(out);
    for (i=2; i < lout; i++)
    {
        *outp = (*xp) & (*yp);
        outp = int_nextW(outp);
        xp = int_nextW(xp);
        yp = int_nextW(yp);
    }
    if (!int_MSW(out)) out = int_normalize(out, 1);
    return out;
}
```

This low-level interface is mandatory in order to write portable code since PARI can be compiled using various multiprecision kernels, for instance the native one or GNU MP, with incompatible internal structures (for one thing, the mantissa is oriented in different directions).

**4.5.2 Type `t_REAL` (real number).** this type has a second codeword `z[1]` which also encodes its sign, obtained or set using the same functions as for a `t_INT`, and a binary exponent. This exponent is handled using the following macros:

`long expo(GEN z)` returns the exponent of  $z$ . This is defined even when  $z$  is equal to zero.

`void setexpo(GEN z, long e)` sets the exponent of  $z$  to  $e$ .

Note the functions:

`long gexpo(GEN z)` which tries to return an exponent for  $z$ , even if  $z$  is not a real number.

`long gsigne(GEN z)` which returns a sign for  $z$ , even when  $z$  is a real number of type `t_INT`, `t_FRAC` or `t_REAL`, an infinity (`t_INFINITY`) or a `t_QUAD` of positive discriminant.

The real zero is characterized by having its sign equal to 0. If  $z$  is not equal to 0, then it is represented as  $2^e M$ , where  $e$  is the exponent, and  $M \in [1, 2[$  is the mantissa of  $z$ , whose digits are

stored in  $z[2], \dots, z[\lg(z) - 1]$ . For historical reasons, the `prec` parameter attached to floating point functions is measured in `BITS_IN_LONG`-bit words and is equal to the length of  $x$ : yes, this includes the two code words and depends on `sizeof(long)`. For clarity we advise to use `bit_accuracy`, which computes the true length of the mantissa in bits, and convert between bits and `prec` using the `prec2nbits` and `nbits2prec` macros. But keep in mind that the accuracy of `t_REAL` actually increases by increments of `BITS_IN_LONG`bits.

More precisely, let  $m$  be the integer  $(z[2], \dots, z[\lg(z)-1])$  in base  $2^{\text{BITS\_IN\_LONG}}$ ; here,  $z[2]$  is the most significant longword and is normalized, i.e. its most significant bit is 1. Then we have  $M := m/2^{\text{bit\_accuracy}(\lg(z))-1-\text{expo}(z)}$ .

`GEN mantissa_real(GEN z, long *e)` returns the mantissa  $m$  of  $z$ , and sets  $*e$  to the exponent  $\text{bit\_accuracy}(\lg(z)) - 1 - \text{expo}(z)$ , so that  $z = m/2^e$ .

Thus, the real number 3.5 to accuracy  $\text{bit\_accuracy}(\lg(z))$  is represented as  $z[0]$  (encoding `type = t_REAL`,  $\lg(z)$ ),  $z[1]$  (encoding `sign = 1`, `expo = 1`),  $z[2] = 0xe0000000$ ,  $z[3] = \dots = z[\lg(z) - 1] = 0x0$ .

**4.5.3 Type `t_INTMOD`.**  $z[1]$  points to the modulus, and  $z[2]$  at the number representing the class  $z$ . Both are separate `GEN` objects, and both must be `t_INT`s, satisfying the inequality  $0 \leq z[2] < z[1]$ .

**4.5.4 Type `t_FRAC` (rational number).**  $z[1]$  points to the numerator  $n$ , and  $z[2]$  to the denominator  $d$ . Both must be of type `t_INT` such that  $n \neq 0$ ,  $d > 0$  and  $(n, d) = 1$ .

**4.5.5 Type `t_FFELT` (finite field element).** (Experimental)

Components of this type should normally not be accessed directly. Instead, finite field elements should be created using `ffgen`.

The second codeword  $z[1]$  determines the storage format of the element, among

- `t_FF_FpXQ`:  $A=z[2]$  and  $T=z[3]$  are `FpX`,  $p=z[4]$  is a `t_INT`, where  $p$  is a prime number,  $T$  is irreducible modulo  $p$ , and  $\deg A < \deg T$ . This represents the element  $A \pmod{T}$  in  $\mathbf{F}_p[X]/T$ .
- `t_FF_Flxq`:  $A=z[2]$  and  $T=z[3]$  are `Flx`,  $l=z[4]$  is a `t_INT`, where  $l$  is a prime number,  $T$  is irreducible modulo  $l$ , and  $\deg A < \deg T$ . This represents the element  $A \pmod{T}$  in  $\mathbf{F}_l[X]/T$ .
- `t_FF_F2xq`:  $A=z[2]$  and  $T=z[3]$  are `F2x`,  $l=z[4]$  is the `t_INT` 2,  $T$  is irreducible modulo 2, and  $\deg A < \deg T$ . This represents the element  $A \pmod{T}$  in  $\mathbf{F}_2[X]/T$ .

**4.5.6 Type `t_COMPLEX` (complex number).**  $z[1]$  points to the real part, and  $z[2]$  to the imaginary part. The components  $z[1]$  and  $z[2]$  must be of type `t_INT`, `t_REAL` or `t_FRAC`. For historical reasons `t_INTMOD` and `t_PADIC` are also allowed (the latter for  $p = 2$  or congruent to 3 mod 4 only), but one should rather use the more general `t_POLMOD` construction.

**4.5.7 Type `t_PADIC` (*p*-adic numbers).** this type has a second codeword `z[1]` which contains the following information: the *p*-adic precision (the exponent of *p* modulo which the *p*-adic unit corresponding to `z` is defined if `z` is not 0), i.e. one less than the number of significant *p*-adic digits, and the exponent of `z`. This information can be handled using the following functions:

`long precp(GEN z)` returns the *p*-adic precision of `z`. This is 0 if `z = 0`.

`void setprecp(GEN z, long l)` sets the *p*-adic precision of `z` to `l`.

`long valp(GEN z)` returns the *p*-adic valuation of `z` (i.e. the exponent). This is defined even if `z` is equal to 0.

`void setvalp(GEN z, long e)` sets the *p*-adic valuation of `z` to `e`.

In addition to this codeword, `z[2]` points to the prime *p*, `z[3]` points to  $p^{\text{precp}(z)}$ , and `z[4]` points to `at_INT` representing the *p*-adic unit attached to `z` modulo `z[3]` (and to zero if `z` is zero). To summarize, if  $z \neq 0$ , we have the equality:

$$z = p^{\text{valp}(z)} * (z[4] + O(z[3])), \quad \text{where } z[3] = O(p^{\text{precp}(z)}).$$

**4.5.8 Type `t_QUAD` (quadratic number).** `z[1]` points to the canonical polynomial *P* defining the quadratic field (as output by `quadpoly`), `z[2]` to the “real part” and `z[3]` to the “imaginary part”. The latter are of type `t_INT`, `t_FRAC`, `t_INTMOD`, or `t_PADIC` and are to be taken as the coefficients of `z` with respect to the canonical basis  $(1, X)$  of  $\mathbf{Q}[X]/(P(X))$ . Exact complex numbers may be implemented as quadratics, but `t_COMPLEX` is in general more versatile (`t_REAL` components are allowed) and more efficient.

Operations involving a `t_QUAD` and `t_COMPLEX` are implemented by converting the `t_QUAD` to a `t_REAL` (or `t_COMPLEX` with `t_REAL` components) to the accuracy of the `t_COMPLEX`. As a consequence, operations between `t_QUAD` and *exact* `t_COMPLEX`s are not allowed.

**4.5.9 Type `t_POLMOD` (polmod).** as for `t_INTMOD`s, `z[1]` points to the modulus, and `z[2]` to a polynomial representing the class of `z`. Both must be of type `t_POL` in the same variable, satisfying the inequality  $\deg z[2] < \deg z[1]$ . However, `z[2]` is allowed to be a simplification of such a polynomial, e.g. a scalar. This is tricky considering the hierarchical structure of the variables; in particular, a polynomial in variable of *lesser* priority (see Section 4.6) than the modulus variable is valid, since it is considered as the constant term of a polynomial of degree 0 in the correct variable. On the other hand a variable of *greater* priority is not acceptable.

**4.5.10 Type `t_POL` (polynomial).** this type has a second codeword. It contains a “*sign*”: 0 if the polynomial is equal to 0, and 1 if not (see however the important remark below) and a *variable number* (e.g. 0 for *x*, 1 for *y*, etc. . .).

These data can be handled with the following macros: **signe** and **setsigne** as for `t_INT` and `t_REAL`,

`long varn(GEN z)` returns the variable number of the object `z`,

`void setvarn(GEN z, long v)` sets the variable number of `z` to `v`.

The variable numbers encode the relative priorities of variables, we will give more details in Section 4.6. Note also the function `long gvar(GEN z)` which tries to return a variable number for `z`, even if `z` is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE`, which has lower priority than any other variable number.



The components  $z[2], z[3], \dots, z[\lg(z)-1]$  point to the coefficients of the polynomial *in ascending order*, with  $z[2]$  being the constant term and so on.

For a `t_POL` of nonzero sign, `degpol`, `leading_coeff`, `constant_coeff`, return its degree, and a pointer to the leading, resp. constant, coefficient with respect to the main variable. Note that no copy is made on the PARI stack so the returned value is not safe for a basic `gerepile` call. Applied to any other type than `t_POL`, the result is unspecified. Those three functions are still defined when the sign is 0, see Section 5.2.7 and Section 10.6.

`long degree(GEN x)` returns the degree of  $x$  with respect to its main variable even when  $x$  is not a polynomial (a rational function for instance). By convention, the degree of a zero polynomial is  $-1$ .

**Important remark.** The leading coefficient of a `t_POL` may be equal to zero:

- it is not allowed to be an exact rational 0, such as `gen_0`;
- an exact nonrational 0, like `Mod(0,2)`, is possible for constant polynomials, i.e. of length 3 and no other coefficient: this carries information about the base ring for the polynomial;
- an inexact 0, like `0.E-38` or `0(3^5)`, is always possible. Inexact zeroes do not correspond to an actual 0, but to a very small coefficient according to some metric; we keep them to give information on how much cancellation occurred in previous computations.

A polynomial disobeying any of these rules is an invalid *unnormalized* object. We advise *not* to use low-level constructions to build a `t_POL` coefficient by coefficient, such as

```
GEN T = cgetg(4, t_POL);
T[1] = evalvarn(0);
gel(T, 2) = x;
gel(T, 3) = y;
```

But if you do and it is not clear whether the result will be normalized, call

`GEN normalizepol(GEN x)` applied to an unnormalized `t_POL`  $x$  (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes  $x$  correctly in place and returns  $x$ . This function sets `signe` (to 0 or 1) properly.

**Caveat.** A consequence of the remark above is that zero polynomials are characterized by the fact that their sign is 0. It is in general incorrect to check whether `lg(x)` is 2 or `degpol(x) < 0`, although both tests are valid when the coefficient types are under control: for instance, when they are guaranteed to be `t_INTs` or `t_FRACs`. The same remark applies to `t_SERs`.

**4.5.11 Type `t_SER` (power series).** This type also has a second codeword, which encodes a “*sign*”, i.e. 0 if the power series is 0, and 1 if not, a *variable number* as for polynomials, and an *exponent*. This information can be handled with the following functions: **signe**, **setsigne**, **varn**, **setvarn** as for polynomials, and **valp**, **setvalp** for the exponent as for  $p$ -adic numbers. Beware: do *not* use **expo** and **setexpo** on power series.

The coefficients  $z[2], z[3], \dots, z[\lg(z)-1]$  point to the coefficients of  $z$  in ascending order. As for polynomials (see remark there), the sign of a `t_SER` is 0 if and only if all its coefficients are equal to 0. (The leading coefficient cannot be an integer 0.) A series whose coefficients are integers equal to zero is represented as  $O(x^n)$  (`zeroser(vx, n)`). A series whose coefficients are exact zeroes, but not all of them integers (e.g. an `t_INTMOD` such as `Mod(0,2)`) is represented as  $z * x^{n-1} + O(x^n)$ , where  $z$  is the 0 of the base ring, as per `Rg_get_0`.

Note that the exponent of a power series can be negative, i.e. we are then dealing with a Laurent series (with a finite number of negative terms).

**4.5.12 Type `t_RFRAC` (rational function).** `z[1]` points to the numerator  $n$ , and `z[2]` on the denominator  $d$ . The denominator must be of type `t_POL`, with variable of higher priority than the numerator. The numerator  $n$  is not an exact 0 and  $(n, d) = 1$  (see `gred_rfac2`).

**4.5.13 Type `t_QFB` (binary quadratic form).** `z[1]`, `z[2]`, `z[3]` point to the three coefficients of the form, and `z[4]` point to the form discriminant. All four are of type `t_INT`.

**4.5.14 Type `t_VEC` and `t_COL` (vector).** `z[1]`, `z[2]`, ..., `z[lg(z)-1]` point to the components of the vector.

**4.5.15 Type `t_MAT` (matrix).** `z[1]`, `z[2]`, ..., `z[lg(z)-1]` point to the column vectors of  $z$ , i.e. they must be of type `t_COL` and of the same length.

**4.5.16 Type `t_VECSMALL` (vector of small integers).** `z[1]`, `z[2]`, ..., `z[lg(z)-1]` are ordinary signed long integers. This type is used instead of a `t_VEC` of `t_INTs` for efficiency reasons, for instance to implement efficiently permutations, polynomial arithmetic and linear algebra over small finite fields, etc.

**4.5.17 Type `t_STR` (character string).**

`char * GSTR(z) (= (z+1))` points to the first character of the (NULL-terminated) string.

**4.5.18 Type `t_ERROR` (error context).** This type holds error messages, as well as details about the error, as returned by the exception handling system. The second codeword `z[1]` contains the error type (an `int`, as passed to `pari_err`). The subsequent words `z[2]`, ..., `z[lg(z)-1]` are `GENs` containing additional data, depending on the error type.

**4.5.19 Type `t_CLOSURE` (closure).** This type holds GP functions and closures, in compiled form. The internal detail of this type is subject to change each time the GP language evolves. Hence we do not describe it here and refer to the Developer's Guide. However functions to create or to evaluate `t_CLOSUREs` are documented in Section 12.1.

`long closure_arity(GEN C)` returns the arity of the `t_CLOSURE`.

`long closure_is_variadic(GEN C)` returns 1 if the closure  $C$  is variadic, 0 else.

**4.5.20 Type `t_INFINITY` (infinity).**

This type has a single `t_INT` component, which is either 1 or  $-1$ , corresponding to  $+\infty$  and  $-\infty$  respectively.

`GEN mkmoo()` returns  $-\infty$

`GEN mkoo()` returns  $\infty$

`long inf_get_sign(GEN x)` returns 1 if  $x$  is  $+\infty$ , and  $-1$  if  $x$  is  $-\infty$ .

**4.5.21 Type `t_LIST` (list).** this type was introduced for specific `gp` use and is rather inefficient compared to a straightforward linked list implementation (it requires more memory, as well as many unnecessary copies). Hence we do not describe it here and refer to the Developer's Guide.

**Implementation note.** For the types including an exponent (or a valuation), we actually store a biased nonnegative exponent (bit-ORing the biased exponent to the codeword), obtained by adding a constant to the true exponent: either `HIGHXPOBIT` (for `t_REAL`) or `HIGHVALPBIT` (for `t_PADIC` and `t_SER`). Of course, this is encapsulated by the exponent/valuation-handling macros and needs not concern the library user.

## 4.6 PARI variables.

### 4.6.1 Multivariate objects.

We now consider variables and formal computations. As we have seen in Section 4.5, the codewords for types `t_POL` and `t_SER` encode a “variable number”. This is an integer, ranging from 0 to `MAXVARN`. Relative priorities may be ascertained using

```
int varncmp(long v, long w)
```

which is  $> 0$ ,  $= 0$ ,  $< 0$  whenever  $v$  has lower, resp. same, resp. higher priority than  $w$ .

The way an object is considered in formal computations depends entirely on its “principal variable number” which is given by the function

```
long gvar(GEN z)
```

which returns a variable number for  $z$ , even if  $z$  is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE` which has lower priority than any valid variable number. The variable number of a recursive type which is not a polynomial or power series is the variable number with highest priority among its components. But for polynomials and power series only the “outermost” number counts (we directly access `varn(x)` in the codewords): the representation is not symmetrical at all.

Under `gp`, one needs not worry too much since the interpreter defines the variables as it sees them\* and do the right thing with the polynomials produced.

But in library mode, they are tricky objects if you intend to build polynomials yourself (and not just let PARI functions produce them, which is less efficient). For instance, it does not make sense to have a variable number occur in the components of a polynomial whose main variable has a lower priority, even though PARI cannot prevent you from doing it.

**4.6.2 Creating variables.** A basic difficulty is to “create” a variable. Some initializations are needed before you can use a given integer  $v$  as a variable number.

Initially, this is done for 0 and 1 (the variables `x` and `y` under `gp`), and  $2, \dots, 9$  (printed as `t2`,  $\dots$  `t9`), with decreasing priority.

---

\* The first time a given identifier is read by the GP parser a new variable is created, and it is assigned a strictly lower priority than any variable in use at this point. On startup, before any user input has taken place, ‘x’ is defined in this way and has initially maximal priority (and variable number 0).

**4.6.2.1 User variables.** When the program starts, `x` (number 0) and `y` (number 1) are the only available variables, numbers 2 to 9 (decreasing priority) are reserved for building polynomials with predictable priorities.

To define further ones, you may use

```
GEN varhigher(const char *s)
```

```
GEN varlower(const char *s)
```

to recover a monomial of degree 1 in a new variable, which is guaranteed to have higher (resp. lower) priority than all existing ones at the time of the function call. The variable is printed as `s`, but is not part of GP's interpreter: it is not a symbol bound to a value.

On the other hand

`long fetch_user_var(char *s)`: inspects the user variable whose name is the string pointed to by `s`, creating it if needed, and returns its variable number.

```
long v = fetch_user_var("y");
GEN gy = pol_x(v);
```

The function raises an exception if the name is already in use for an `installed` or built-in function, or an alias. This function is mostly useless since it returns a variable with unpredictable priority. Don't use it to create new variables.

**Caveat.** You can use `gp_read_str` (see Section 4.7.1) to execute a GP command and create GP variables on the fly as needed:

```
GEN gy = gp_read_str("'y"); /* returns pol_x(v), for some v */
long v = varn(gy);
```

But please note the quote `'y` in the above. Using `gp_read_str("y")` might work, but is dangerous, especially when programming functions to be used under `gp`. The latter reads the value of `y`, as *currently* known by the `gp` interpreter, possibly creating it in the process. But if `y` has been modified by previous `gp` commands (e.g. `y = 1`), then the value of `gy` is not what you expected it to be and corresponds instead to the current value of the `gp` variable (e.g. `gen_1`).

`GEN fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered `v`. Returns `NULL` if that variable number is unknown to the interpreter, e.g. it is a user variable. Note that this may not be the same as `pol_x(v)` if assignments have been performed in the interpreter.

**4.6.2.2 Temporary variables.** You can create temporary variables using

`long fetch_var()` returns a new variable with *lower* priority than any variable currently in use.

`long fetch_var_higher()` returns a new variable with *higher* priority than any variable currently in use.

After the statement `v = fetch_var()`, you can use `pol_1(v)` and `pol_x(v)`. The variables created in this way have no identifier assigned to them though, and are printed as `tnumber`. You can assign a name to a temporary variable, after creating it, by calling the function

```
void name_var(long n, char *s)
```

after which the output machinery will use the name `s` to represent the variable number `n`. The GP parser will *not* recognize it by that name, however, and calling this on a variable known to `gp`

raises an error. Temporary variables are meant to be used as free variables to build polynomials and power series, and you should never assign values or functions to them as you would do with variables under `gp`. For that, you need a user variable.

All objects created by `fetch_var` are on the heap and not on the stack, thus they are not subject to standard garbage collecting (they are not destroyed by a `gerepile` or `set_avma(ltop)` statement). When you do not need a variable number anymore, you can delete it using

```
long delete_var()
```

which deletes the *latest* temporary variable created and returns the variable number of the previous one (or simply returns 0 if none remain). Of course you should make sure that the deleted variable does not appear anywhere in the objects you use later on. Here is an example:

```
long first = fetch_var();
long n1 = fetch_var();
long n2 = fetch_var(); /* prepare three variables for internal use */
...
/* delete all variables before leaving */
do { num = delete_var(); } while (num && num <= first);
```

The (dangerous) statement

```
while (delete_var()) /* empty */;
```

removes all temporary variables in use.

#### 4.6.3 Comparing variables.

Let us go back to `varncmp`. There is an interesting corner case, when one of the compared variables (from `gvar`, say) is `NO_VARIABLE`. In this case, `varncmp` declares it has lower priority than any other variable; of course, comparing `NO_VARIABLE` with itself yields 0 (same priority);

In addition to `varncmp` we have

`long varnmax(long v, long w)` given two variable numbers (possibly `NO_VARIABLE`), returns the variable with the highest priority. This function always returns a valid variable number unless it is comparing `NO_VARIABLE` to itself.

`long varnmin(long x, long y)` given two variable numbers (possibly `NO_VARIABLE`), returns the variable with the lowest priority. Note that when comparing a true variable with `NO_VARIABLE`, this function returns `NO_VARIABLE`, which is not a valid variable number.

## 4.7 Input and output.

Two important aspects have not yet been explained which are specific to library mode: input and output of PARI objects.

### 4.7.1 Input.

For input, PARI provides several powerful high level functions which enable you to input your objects as if you were under `gp`. In fact, it *is* essentially the GP syntactical parser.

There are two similar functions available to parse a string:

```
GEN gp_read_str(const char *s)
```

```
GEN gp_read_str_multiline(const char *s, char *last)
```

Both functions read the whole string `s`. The function `gp_read_str` ignores newlines: it assumes that the input is one expression and returns the result of this expression.

The function `gp_read_str_multiline` processes the text in the same way as the GP command `read`: newlines are significant and can be used to separate expressions. The return value is that of the last nonempty expression evaluated.

In `gp_read_str_multiline`, if `last` is not NULL, then `*last` receives the last character from the *filtered* input: this can be used to check if the last character was a semi-colon (to hide the output in interactive usage). If (and only if) the input contains no statements, then `*last` is set to 0.

For both functions, `gp`'s metacommands *are* recognized.

Two variants allow to specify a default precision while evaluating the string:

```
GEN gp_read_str_prec(const char *s, long prec) As gp_read_str, but set the precision to prec words while evaluating s.
```

```
GEN gp_read_str_bitprec(const char *s, long bitprec) As gp_read_str, but set the precision to bitprec bits while evaluating s.
```

**Note.** The obsolete form

```
GEN readseq(char *t)
```

still exists for backward compatibility (assumes filtered input, without spaces or comments). Don't use it.

To read a GEN from a file, you can use the simpler interface

```
GEN gp_read_stream(FILE *file)
```

which reads a character string of arbitrary length from the stream `file` (up to the first complete expression sequence), applies `gp_read_str` to it, and returns the resulting GEN. This way, you do not have to worry about allocating buffers to hold the string. To interactively input an expression, use `gp_read_stream(stdin)`. Return NULL when there are no more expressions to read (we reached EOF).

Finally, you can read in a whole file, as in GP's `read` statement

```
GEN gp_read_file(char *name)
```

As usual, the return value is that of the last nonempty expression evaluated. There is one technical exception: if `name` is a *binary* file (from `writebin`) containing more than one object, a `t_VEC` containing them all is returned. This is because binary objects bypass the parser, hence reading them has no useful side effect.

#### 4.7.2 Output to screen or file, output to string.

General output functions return nothing but print a character string as a side effect. Low level routines are available to write on PARI output stream `pari_outfile` (`stdout` by default):

`void pari_putc(char c):` write character `c` to the output stream.

`void pari_puts(char *s):` write `s` to the output stream.

`void pari_flush():` flush output stream; most streams are buffered by default, this command makes sure that all characters output so are actually written.

`void pari_printf(const char *fmt, ...):` the most versatile such function. `fmt` is a character string similar to the one `printf` uses. In there, `%` characters have a special meaning, and describe how to print the remaining operands. In addition to the standard format types (see the GP function `printf`), you can use the *length modifier* `P` (for PARI of course!) to specify that an argument is a GEN. For instance, the following are valid conversions for a GEN argument

```
%Ps      convert to char* (will print an arbitrary GEN)
%P.10s    convert to char*, truncated to 10 chars
%P.2f     convert to floating point format with 2 decimals
%P4d      convert to integer, field width at least 4
```

```
pari_printf("x[%d] = %Ps is not invertible!\n", i, gel(x,i));
```

Here `i` is an `int`, `x` a GEN which is not a leaf (presumably a vector, or a polynomial) and this would insert the value of its *i*-th GEN component: `gel(x,i)`.

Simple but useful variants to `pari_printf` are

`void output(GEN x)` prints `x` in raw format, followed by a newline and a buffer flush. This is more or less equivalent to

```
pari_printf("%Ps\n", x);
pari_flush();
```

`void outmat(GEN x)` as above except if `x` is a `t_MAT`, in which case a multi-line display is used to display the matrix. This is prettier for small dimensions, but quickly becomes unreadable and cannot be pasted and reused for input. If all entries of `x` are small integers, you may use the recursive features of `%Pd` and obtain the same (or better) effect with

```
pari_printf("%Pd\n", x);
pari_flush();
```

A variant like `"%5Pd"` would improve alignment by imposing 5 chars for each coefficient. Similarly if all entries are to be converted to floats, a format like `"%5.1Pf"` could be useful.

These functions write on (PARI's idea of) standard output, and must be used if you want your functions to interact nicely with `gp`. In most programs, this is not a concern and it is more flexible to write to an explicit `FILE*`, or to recover a character string:

`void pari_fprintf(FILE *file, const char *fmt, ...)` writes the remaining arguments to stream `file` according to the format specification `fmt`.

`char* pari_sprintf(const char *fmt, ...)` produces a string from the remaining arguments, according to the PARI format `fmt` (see `printf`). This is the `libpari` equivalent of `strprintf`, and returns a `malloc`'ed string, which must be freed by the caller. Note that contrary to the analogous `sprintf` in the `libc` you do not provide a buffer (leading to all kinds of buffer overflow concerns); the function provided is actually closer to the GNU extension `asprintf`, although the latter has a different interface.

Simple variants of `pari_sprintf` convert a `GEN` to a `malloc`'ed ASCII string, which you must still `free` after use:

`char* GENTostr(GEN x)`, using the current default output format (`prettymat` by default).

`char* GENToTeXstr(GEN x)`, suitable for inclusion in a `TeX` file.

Note that we have `va_list` analogs of the functions of `printf` type seen so far:

`void pari_vprintf(const char *fmt, va_list ap)`

`void pari_vfprintf(FILE *file, const char *fmt, va_list ap)`

`char* pari_vsprintf(const char *fmt, va_list ap)`

### 4.7.3 Errors.

If you want your functions to issue error messages, you can use the general error handling routine `pari_err`. The basic syntax is

```
pari_err(e_MISC, "error message");
```

This prints the corresponding error message and exit the program (in library mode; go back to the `gp` prompt otherwise). You can also use it in the more versatile guise

```
pari_err(e_MISC, format, ...);
```

where `format` describes the format to use to write the remaining operands, as in the `pari_printf` function. For instance:

```
pari_err(e_MISC, "x[%d] = %Ps is not invertible!", i, gel(x,i));
```

The simple syntax seen above is just a special case with a constant format and no remaining arguments. The general syntax is

```
void pari_err(numerr, ...)
```

where `numerr` is a codeword which specifies the error class and what to do with the remaining arguments and what message to print. For instance, if `x` is a `GEN` with internal type `t_STR`, say, `pari_err(e_TYPE, "extgcd", x)` prints the message:

```
*** incorrect type in extgcd (t_STR),
```

See Section 11.4 for details. In the `libpari` code itself, the general-purpose `e_MISC` is used sparingly: it is so flexible that the corresponding error contexts (`t_ERROR`) become hard to use reliably. Other more rigid error types are generally more useful: for instance the error context attached to the `e_TYPE` exception above is precisely documented and contains `"extgcd"` and `x` (not only its type) as readily available components.



#### 4.7.4 Warnings.

To issue a warning, use

`void pari_warn(warnerr, ...)` In that case, of course, we do *not* abort the computation, just print the requested message and go on. The basic example is

```
pari_warn(warner, "Strategy 1 failed. Trying strategy 2")
```

which is the exact equivalent of `pari_err(e_MISC,...)` except that you certainly do not want to stop the program at this point, just inform the user that something important has occurred; in particular, this output would be suitably highlighted under `gp`, whereas a simple `printf` would not.

The valid *warning* keywords are `warner` (general), `warnprec` (increasing precision), `warnmem` (garbage collecting) and `warnfile` (error in file operation), used as follows:

```
pari_warn(warnprec, "bnfinit", newprec);
pari_warn(warnmem, "bnfinit");
pari_warn(warnfile, "close", "afile"); /* error when closing "afile" */
```

#### 4.7.5 Debugging output.

For debugging output, you can use the standard output functions, `output` and `pari_printf` mainly. Corresponding to the `gp` metacommand `\x`, you can also output the hexadecimal tree attached to an object:

`void dbgGEN(GEN x, long nb = -1)`, displays the recursive structure of `x`. If `nb = -1`, the full structure is printed, otherwise the leaves (nonrecursive components) are truncated to `nb` words.

The function `output` is vital under debuggers, since none of them knows how to print PARI objects by default. Seasoned PARI developers add the following `gdb` macro to their `.gdbinit`:

```
define oo
  call output((GEN)$arg0)
end
define xx
  call dbgGEN($arg0,-1)
end
```

Typing `i x` at a breakpoint in `gdb` then prints the value of the `GEN x` (provided the optimizer has not put it into a register, but it is rarely a good idea to debug optimized code).

The global variables **DEBUGLEVEL** and **DEBUGMEM** (corresponding to the default **debug** and **debugmem**) are used throughout the PARI code to govern the amount of diagnostic and debugging output, depending on their values. You can use them to debug your own functions, especially if you install the latter under `gp`. Note that **DEBUGLEVEL** is redefined in each code module, attaching it to a particular debug domain (see `setdebug`).

`void setalldbg(long L)` sets all **DEBUGLEVEL** incarnations (all debug domains) to `L`.

`void dbg_pari_heap(void)` print debugging statements about the PARI stack, heap, and number of variables used. Corresponds to `\s` under `gp`.

#### 4.7.6 Timers and timing output.

To handle timings in a reentrant way, PARI defines a dedicated data type, `pari_timer`, together with the following methods:

`void timer_start(pari_timer *T)` start (or reset) a timer.

`long timer_delay(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Resets the timer as a side effect. Assume  $T$  was started by `timer_start`.

`long timer_get(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Does *not* reset the timer. Assume  $T$  was started by `timer_start`.

`void walltimer_start(pari_timer *T)` start a timer, as if it had been started at the Unix epoch (see `getwalltime`).

`long walltimer_delay(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last checked. Assume  $T$  was started by `walltimer_start`.

`long walltimer_get(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Does *not* reset the timer. Assume  $T$  was started by `walltimer_start`.

`long timer_printf(pari_timer *T, char *format, ...)` This diagnostics function is equivalent to the following code

```
err_printf("Time ")
... prints remaining arguments according to format ...
err_printf(": %ld", timer_delay(T));
```

Resets the timer as a side effect.

They are used as follows:

```
pari_timer T;
timer_start(&T); /* initialize timer */
...
printf("Total time: %ldms\n", timer_delay(&T));
```

or

```
pari_timer T;
timer_start(&T);
for (i = 1; i < 10; i++) {
    ...
    timer_printf(&T, "for i = %ld (L[i] = %Ps)", i, gel(L,i));
}
```

The following functions provided the same functionality, in a nonreentrant way, and are now deprecated.

`long timer(void)`

`long timer2(void)`

`void msgtimer(const char *format, ...)`

The following function implements `gp`'s timer and should not be used in `libpari` programs:  
`long gettime(void)` equivalent to `timer_delay(T)` attached to a private timer  $T$ .

## 4.8 Iterators, Numerical integration, Sums, Products.

**4.8.1 Iterators.** Since it is easier to program directly simple loops in library mode, some GP iterators are mainly useful for GP programming. Here are the others:

- **fordiv** is a trivial iteration over a list produced by **divisors**.

- **forell**, **forqfvec** and **forsubgroup** are currently not implemented as an iterator but as a procedure with callbacks.

**void forell(void \*E, long fun(void\*, GEN), GEN a, GEN b, long flag)** goes through the same curves as **forell(ell,a,b,,flag)**, calling **fun(E, ell)** for each curve **ell**, stopping if **fun** returns a nonzero value.

**void forqfvec(void \*E, long (\*fun)(void \*, GEN, GEN, double), GEN q, GEN b)**  
: Evaluate **fun(E,U,v,m)** on all  $v$  such that  $q(Uv) < b$ , where  $U$  is a **t\_MAT**,  $v$  is a **t\_VECSMALL** and  $m = q(v)$  is a C double. The function **fun** must return 0, unless **forqfvec** should stop, in which case, it should return 1.

**void forqfvec1(void \*E, long (\*fun)(void \*, GEN), GEN q, GEN b)**: Evaluate **fun(E,v)** on all  $v$  such that  $q(v) < b$ , where  $v$  is a **t\_COL**. The function **fun** must return 0, unless **forqfvec** should stop, in which case, it should return 1.

**void forsubgroup(void \*E, long fun(void\*, GEN), GEN G, GEN B)** goes through the same subgroups as **forsubgroup(H = G, B,)**, calling **fun(E, H)** for each subgroup  $H$ , stopping if **fun** returns a nonzero value.

- **forprime** and **forprimestep**, iterators over primes and primes in arithmetic progressions, for which we refer you to the next subsection.

- **forcomposite**, we provide an iterator over composite integers:

**int forcomposite\_init(forcomposite\_t \*T, GEN a, GEN b)** initialize an iterator  $T$  over composite integers in  $[a, b]$ ; over composites  $\geq a$  if  $b = \text{NULL}$ . We must have  $a \geq 0$ . Return 0 if the range is known to be empty from the start (as if  $b < a$  or  $b < 0$ ), and return 1 otherwise.

**GEN forcomposite\_next(forcomposite\_t \*T)** returns the next composite in the range, assuming that  $T$  was initialized by **forcomposite\_init**.

- **forvec**, for which we provide a convenient iterator. To initialize the analog of **forvec(X = v, ..., flag)**, call

**int forvec\_init(forvec\_t \*T, GEN v, long flag)** initialize an iterator  $T$  over the vectors generated by **forvec(X = v, ..., flag)**. This returns 0 if this vector list is empty, and 1 otherwise.

**GEN forvec\_next(forvec\_t \*T)** returns the next element in the **forvec** sequence, or **NULL** if we are done. The return value must be used immediately or copied since the next call to the iterator destroys it: the relevant vector is updated in place. The iterator works hard to not use up PARI stack, and is more efficient when all lower bounds in the initialization vector  $v$  are integers. In that case, the cost is linear in the number of tuples enumerated, and you can expect to run over more than  $10^9$  tuples per minute. If speed is critical and all integers involved would fit in  $C$  longs, write a simple direct backtracking algorithm yourself.

- **forpart** is a variant of **forvec** which iterates over partitions. See the documentation of the **forpart** GP function for details. This function is available as a loop with callbacks:

```
void forpart(void *data, long (*call)(void*, GEN), long k, GEN a, GEN n)
```

It is also available as an iterator:

`void forpart_init(forpart_t *T, long k, GEN a, GEN n)` initializes an iterator over the partitions of  $k$ , with length restricted by  $n$ , and components restricted by  $a$ , either of which can be set to `NULL` to run without restriction.

`GEN forpart_next(forpart_t *T)` returns the next partition, or `NULL` when all partitions have been exhausted.

`GEN forpart_prev(forpart_t *T)` returns the previous partition, or `NULL` when all partitions have been exhausted.

In both cases, the partition must be used or copied before the next call since it is returned from a state array which will be modified in place. You may *not* mix calls to `forpart_next` and `forpart_prev`: the first one called determines the ordering used to iterate over the partitions; you can not go back since the `forpart_t` structure is used in incompatible ways.

- `forperm` to loop over permutations of  $k$ . See the documentation of the `forperm` GP function for details. This function is available as an iterator:

`void forperm_init(forperm_t *T, GEN k)` initializes an iterator over the permutations of  $k$  (`t_INT`, `t_VEC` or `t_VECSMALL`).

`GEN forperm_next(forperm_t *T)` returns the next permutation as a `t_VECSMALL` or `NULL` when all permutations have been exhausted. The permutation must be used or copied before the next call since it is returned from a state array which will be modified in place.

- `forsubset` to loop over subsets. See the documentation of the `forsubset` GP function for details. This function is available as two iterators:

```
void forallsubset_init(forsubset_t *T, long n)
```

```
void forksubset_init(forsubset_t *T, long n, long k)
```

It is also available in generic form:

`void forsubset_init(forsubset_t *T, GEN nk)` where `nk` is either a `t_INT`  $n$  or a `t_VEC` with two integral components  $[n, k]$ .

In all three cases, `GEN forsubset_next(forsubset_t *T)` returns the next subset as a `t_VECSMALL` or `NULL` when all subsets have been exhausted.

#### 4.8.2 Iterating over primes.

The library provides a high-level iterator, which stores its (private) data in a `struct forprime_t` and runs over arbitrary ranges of primes, without ever overflowing.

The iterator has two flavors, one providing the successive primes as `ulongs`, the other as `GEN`. They are initialized as follows, where we expect to run over primes  $\geq a$  and  $\leq b$ :

`int u_forprime_init(forprime_t *T, ulong a, ulong b)` for the `ulong` variant, where  $b = \text{ULONG\_MAX}$  means we will run through all primes representable in a `ulong` type.

`int forprime_init(forprime_t *T, GEN a, GEN b)` for the `GEN` variant, where  $b = \text{NULL}$  means  $+\infty$ .

`int forprimestep_init(forprime_t *T, GEN a, GEN b, GEN q)` initialize an iterator  $T$  over primes in an arithmetic progression,  $p \geq a$  and  $p \leq b$  (where  $b = \text{NULL}$  means  $+\infty$ ). The argument  $q$  is either a `t_INT` ( $p \equiv a \pmod{q}$ ) or a `t_INTMOD` `Mod(c,N)` and we restrict to that congruence class.

All variants return 1 on success, and 0 if the iterator would run over an empty interval (if  $a > b$ , for instance). They allocate the `forprime_t` data structure on the PARI stack.

The successive primes are then obtained using

`GEN forprime_next(forprime_t *T)`, returns `NULL` if no more primes are available in the interval and the next suitable prime as a `t_INT` otherwise.

`ulong u_forprime_next(forprime_t *T)`, returns 0 if no more primes are available in the interval and fitting in an `ulong` and the next suitable prime otherwise.

These two functions leave alone the PARI stack, and write their state information in the preallocated `forprime_t` struct. The typical usage is thus:

```
forprime_t T;
GEN p;
pari_sp av = avma, av2;
forprime_init(&T, gen_2, stoi(1000));
av2 = avma;
while ( (p = forprime_next(&T)) )
{
    ...
    if ( prime_is_OK(p) ) break;
    set_avma(av2); /* delete garbage accumulated in this iteration */
}
set_avma(av); /* delete all */
```

Of course, the final `set_avma(av)` could be replaced by a `gerepile` call. Beware that swapping the `av2 = avma` and `forprime_init` call would be incorrect: the first `set_avma(av2)` would delete the `forprime_t` structure!

### 4.8.3 Parallel iterators.

Theses iterators loops over the values of a `t_CLOSURE` taken at some data, where the evaluations are done in parallel.

- **parfor**. To initialize the analog of `parfor(i = a, b, ...)`, call

`void parfor_init(parfor_t *T, GEN a, GEN b, GEN code)` initialize an iterator over the evaluation of `code` on the integers between  $a$  and  $b$ .

`GEN parfor_next(parfor_t *T)` returns a `t_VEC` `[i,code(i)]` where  $i$  is one of the integers and `code(i)` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parfor_next` must not be called anymore with the same initialization.

`void parfor_stop(parfor_t *T)` needs to be called when leaving the iterator before `parfor_next` returned `NULL`.

The following returns an integer  $1 \leq i \leq N$  such that `fun(i)` is not zero, or `NULL`.

`GEN`

```

parfirst(GEN fun, GEN N)
{
    parfor_t T;
    GEN e;
    parfor_init(&T, gen_1, N, fun);
    while ((e = parfor_next(&T)))
    {
        GEN i = gel(e,1), funi = gel(e,2);
        if (!gequal0(funi))
        { /* found: stop the iterator and return the index */
            parfor_stop(&T);
            return i;
        }
    }
    return NULL; /* not found */
}

```

- **parforeach.** To initialize the analog of `parforeach(V, X, ...)`, call

`void parforeach_init(parforeach_t *T, GEN V, GEN code)` initialize an iterator over the evaluation of code on the components of  $V$ .

`GEN parforeach_next(parforeach_t *T)` returns a `t_VEC [V[i],code(V[i])]` where  $V[i]$  is one of the components of  $V$  and `code(V[i])` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parforprime_next` must not be called anymore with the same initialization.

`void parforeach_stop(parforeach_t *T)` needs to be called when leaving the iterator before `parforeach_next` returned `NULL`.

- **parforprime.** To initialize the analog of `parforprime(p = a, b, ...)`, call

`void parforprime_init(parforprime_t *T, GEN a, GEN b, GEN code)` initialize an iterator over the evaluation of code on the prime numbers between  $a$  and  $b$ .

- **parforprimestep.** To initialize the analog of `parforprimestep(p = a, b, q, ...)`, call

`void parforprimestep_init(parforprime_t *T, GEN a, GEN b, GEN q, GEN code)` initialize an iterator over the evaluation of code on the prime numbers between  $a$  and  $b$  in the congruence class defined by  $q$ .

`GEN parforprime_next(parforprime_t *T)` returns a `t_VEC [p,code(p)]` where  $p$  is one of the prime numbers and `code(p)` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parforprime_next` must not be called anymore with the same initialization.

`void parforprime_stop(parforprime_t *T)` needs to be called when leaving the iterator before `parforprime_next` returned `NULL`.

- **parforvec.** To initialize the analog of `parforvec(X = V, ..., flag)`, call

`void parforvec_init(parforvec_t *T, GEN V, GEN code, long flag)` initialize an iterator over the evaluation of code on the vectors specified by  $V$  and `flag`, see `forvec` for detail.

`GEN parforvec_next(parforvec_t *T)` returns a `t_VEC [v,code(v)]` where  $v$  is one of the vectors and `code(v)` is the evaluation, `NULL` when all data have been exhausted. Once it happens, `parforvec_next` must not be called anymore with the same initialization.

`void parforvec_stop(parforvec_t *T)` needs to be called when leaving the iterator before `parforvec_next` returned `NULL`.

#### 4.8.4 Numerical analysis.

Numerical routines code a function (to be integrated, summed, zeroed, etc.) with two parameters named

```
void *E;
GEN (*eval)(void*, GEN)
```

The second is meant to contain all auxiliary data needed by your function. The first is such that `eval(x, E)` returns your function evaluated at `x`. For instance, one may code the family of functions  $f_t : x \rightarrow (x + t)^2$  via

```
GEN fun(void *t, GEN x) { return gsqr(gadd(x, (GEN)t)); }
```

One can then integrate  $f_1$  between  $a$  and  $b$  with the call

```
intnum((void*)stoi(1), &fun, a, b, NULL, prec);
```

Since you can set `E` to a pointer to any `struct` (typecast to `void*`) the above mechanism handles arbitrary functions. For simple functions without extra parameters, you may set `E = NULL` and ignore that argument in your function definition.

### 4.9 Catching exceptions.

#### 4.9.1 Basic use.

PARI provides a mechanism to trap exceptions generated via `pari_err` using the `pari_CATCH` construction. The basic usage is as follows

```
pari_CATCH(err_code) {
    recovery branch
}
pari_TRY {
    main branch
}
pari_ENDCATCH
```

This fragment executes the main branch, then the recovery branch *if* exception `err_code` is thrown, e.g. `e_TYPE`. See Section 11.4 for the description of all error classes. The special error code `CATCH_ALL` is available to catch all errors.

One can replace the `pari_TRY` keyword by `pari_RETRY`, in which case once the recovery branch is run, we run the main branch again, still catching the same exceptions.

## Restrictions.

- Such constructs can be nested without adverse effect, the innermost handler catching the exception.

- It is *valid* to leave either branch using `pari_err`.

- It is *invalid* to use C flow control instructions (`break`, `continue`, `return`) to directly leave either branch without seeing the `pari_ENDCATCH` keyword. This would leave an invalid structure in the exception handler stack, and the next exception would crash.

- In order to leave using `break`, `continue` or `return`, one must precede the keyword by a call to

`void pari_CATCH_reset()` disable the current handler, allowing to leave without adverse effect.

### 4.9.2 Advanced use.

In the recovery branch, the exception context can be examined via the following helper routines:

`GEN pari_err_last()` returns the exception context, as a `t_ERROR`. The exception *E* returned by `pari_err_last` can be rethrown, using

```
pari_err(0, E);
```

`long err_get_num(GEN E)` returns the error symbolic name. E.g `e_TYPE`.

`GEN err_get_compo(GEN E, long i)` error *i*-th component, as documented in Section 11.4.

For instance

```
pari_CATCH(CATCH_ALL) { /* catch everything */
    GEN x, E = pari_err_last();
    long code = err_get_num(E);
    if (code != e_INV) pari_err(0, E); /* unexpected error, rethrow */
    x = err_get_compo(E, 2);
    /* e_INV has two components, 1: function name 2: noninvertible x */
    if (typ(x) != t_INTMOD) pari_err(0, E); /* unexpected type, rethrow */
    pari_CATCH_reset();
    return x; /* leave ! */
    ...
} pari_TRY {
    main branch
}
pari_ENDCATCH
```



## 4.10 A complete program.

Now that the preliminaries are out of the way, the best way to learn how to use the library mode is to study a detailed example. We want to write a program which computes the gcd of two integers, together with the Bezout coefficients. We shall use the standard quadratic algorithm which is not optimal but is not too far from the one used in the PARI function **bezout**.

Let  $x, y$  two integers and initially  $\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , so that

$$\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To apply the ordinary Euclidean algorithm to the right hand side, multiply the system from the left by  $\begin{pmatrix} 0 & 1 \\ 1 & -q \end{pmatrix}$ , with  $q = \text{floor}(x/y)$ . Iterate until  $y = 0$  in the right hand side, then the first line of the system reads

$$s_x x + s_y y = \text{gcd}(x, y).$$

In practice, there is no need to update  $s_y$  and  $t_y$  since  $\text{gcd}(x, y)$  and  $s_x$  are enough to recover  $s_y$ . The following program is now straightforward. A couple of new functions appear in there, whose description can be found in the technical reference manual in Chapter 5, but whose meaning should be clear from their name and the context.

This program can be found in `examples/extgcd.c` together with a proper `Makefile`. You may ignore the first comment

```
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
```

which instruments the program so that `gp2c-run extgcd.c` can import the `extgcd()` routine into an instance of the `gp` interpreter (under the name `gcdex`). See the `gp2c` manual for details.

```

#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
    pari_sp av = avma;
    GEN ux = gen_1, vx = gen_0, a = A, b = B;
    if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
    if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
    if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
    while (!gequal0(b))
    {
        GEN r, q = dvmdii(a, b, &r), v = vx;
        vx = subii(ux, mulii(q, vx));
        ux = v; a = b; b = r;
    }
    *U = ux;
    *V = diviexact( subii(a, mulii(A,ux)), B );
    gerepileall(av, 3, &a, U, V); return a;
}

int
main()
{
    GEN x, y, d, u, v;
    pari_init(1000000,2);
    printf("x = "); x = gp_read_stream(stdin);
    printf("y = "); y = gp_read_stream(stdin);
    d = extgcd(x, y, &u, &v);
    pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
    pari_close();
    return 0;
}

```

For simplicity, the inner loop does not include any garbage collection, hence memory use is quadratic in the size of the inputs instead of linear. Here is a better version of that loop:

```

    pari_sp av = avma;
    ...
    while (!gequal0(b))
    {
        GEN r, q = dvmdii(a, b, &r), v = vx;
        vx = subii(ux, mulii(q, vx));
        ux = v; a = b; b = r;
        if (gc_needed(av,1))
            gerepileall(av, 4, &a, &b, &ux, &vx);
    }

```

}



## Chapter 5:

### Technical Reference Guide: the basics

In the following chapters, we describe all public low-level functions of the PARI library. These include specialized functions for handling all the PARI types. Simple higher level functions, such as arithmetic or transcendental functions, are described in Chapter 3 of the GP user's manual; we will eventually see more general or flexible versions in the chapters to come. A general introduction to the major concepts of PARI programming can be found in Chapter 4, which you should really read first.

We shall now study specialized functions, more efficient than the library wrappers, but sloppier on argument checking and damage control; besides speed, their main advantage is to give finer control about the inner workings of generic routines, offering more options to the programmer.

**Important advice.** Generic routines eventually call lower level functions. Optimize your algorithms first, not overhead and conversion costs between PARI routines. For generic operations, use generic routines first; do not waste time looking for the most specialized one available unless you identify a genuine bottleneck, or you need some special behavior the generic routine does not offer. The PARI source code is part of the documentation; look for inspiration there.

The type `long` denotes a `BITS_IN_LONG`-bit signed long integer (32 or 64 bits). The type `ulong` is defined as `unsigned long`. The word *stack* always refer to the PARI stack, allocated through an initial `pari_init` call. Refer to Chapters 1–2 and 4 for general background.

We shall often refer to the notion of *shallow* function, which means that some components of the result may point to components of the input, which is more efficient than a *deep* copy (full recursive copy of the object tree). Such outputs are not suitable for `gerepileupto` and particular care must be taken when garbage collecting objects which have been input to shallow functions: corresponding outputs also become invalid and should no longer be accessed.

A function is *not stack clean* if it leaves intermediate data on the stack besides its output, for efficiency reasons.

## 5.1 Initializing the library.

The following functions enable you to start using the PARI functions in a program, and cleanup without exiting the whole program.

### 5.1.1 General purpose.

`void pari_init(size_t size, ulong maxprime)` initialize the library, with a stack of `size` bytes and a prime table up to the maximum of `maxprime` and  $2^{16}$ . Unless otherwise mentioned, no PARI function will function properly before such an initialization.

`void pari_close(void)` stop using the library (assuming it was initialized with `pari_init`) and frees all allocated objects.

### 5.1.2 Technical functions.

`void pari_init_opts(size_t size, ulong maxprime, ulong opts)` as `pari_init`, more flexible. `opts` is a mask of flags among the following:

`INIT_JMPm`: install PARI error handler. When an exception is raised, the program is terminated with `exit(1)`.

`INIT_SIGm`: install PARI signal handler.

`INIT_DFTm`: initialize the `GP_DATA` environment structure. This one *must* be enabled once. If you close pari, then restart it, you need not reinitialize `GP_DATA`; if you do not, then old values are restored.

`INIT_noPRIMEm`: do not compute the prime table (ignore the `maxprime` argument). The user *must* call `pari_init_primes` later.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_init` to initialize the multi-thread engine. If this flag is set, `pari_mt_init()` will need to be called manually. See `examples/pari-mt.c` for an example.

`INIT_noINTGMPm`: do not install PARI-specific GMP memory functions. This option is ignored when the GMP library is not in use. You may install PARI-specific GMP memory functions later by calling

```
void pari_kernel_init(void)
```

and restore the previous values using

```
void pari_kernel_close(void)
```

This option should not be used without a thorough understanding of the problem you are trying to solve. The GMP memory functions are global variables used by the GMP library. If your program is linked with two libraries that require these variables to be set to different values, conflict ensues. To avoid a conflict, the proper solution is to record their values with `mp_get_memory_functions` and to call `mp_set_memory_functions` to restore the expected values each time the code switches from using one library to the other. Here is an example:

```
void *(*pari_alloc_ptr) (size_t);
void *(*pari_realloc_ptr) (void *, size_t, size_t);
void (*pari_free_ptr) (void *, size_t);
void *(*otherlib_alloc_ptr) (size_t);
void *(*otherlib_realloc_ptr) (void *, size_t, size_t);
void (*otherlib_free_ptr) (void *, size_t);

void init(void)
{
    pari_init(8000000, 500000);
    mp_get_memory_functions(&pari_alloc_ptr,&pari_realloc_ptr,
                          &pari_free_ptr);

    otherlib_init();
    mp_get_memory_functions(&otherlib_alloc_ptr,&otherlib_realloc_ptr,
                          &otherlib_free_ptr);
}

void function_that_use_pari(void)
{
```

```

    mp_set_memory_functions(pari_alloc_ptr, pari_realloc_ptr,
                           pari_free_ptr);
    /*use PARI functions*/
}
void function_that_use_otherlib(void)
{
    mp_set_memory_functions(otherlib_alloc_ptr, otherlib_realloc_ptr,
                           otherlib_free_ptr);
    /*use OTHERLIB functions*/
}

```

`void pari_close_opts(ulong init_opts)` as `pari_close`, for a library initialized with a mask of options using `pari_init_opts`. `opts` is a mask of flags among

`INIT_SIGm`: restore `SIG_DFL` default action for signals tampered with by PARI signal handler.

`INIT_DFTm`: frees the `GP_DATA` environment structure.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_close` to close the multi-thread engine. `INIT_noINTGMPm`: do not restore GMP memory functions.

`void pari_sig_init(void (*f)(int))` install the signal handler `f` (see `signal(2)`): the signals `SIGBUS`, `SIGFPE`, `SIGINT`, `SIGBREAK`, `SIGPIPE` and `SIGSEGV` are concerned.

`void pari_init_primes(ulong maxprime)` Initialize the PARI primes. This function is called by `pari_init(..., maxprime)`. It is provided for users calling `pari_init_opts` with the flag `INIT_noPRIMEm`.

`void pari_sighandler(int signum)` the actual signal handler that PARI uses. This can be used as argument to `pari_sig_init` or `signal(2)`.

`void pari_stackcheck_init(void *stackbase)` controls the system stack exhaustion checking code in the GP interpreter. This should be used when the system stack base address change or when the address seen by `pari_init` is too far from the base address. If `stackbase` is `NULL`, disable the check, else set the base address to `stackbase`. It is normally used this way

```

int thread_start (...)
{
    long first_item_on_the_stack;
    ...
    pari_stackcheck_init(&first_item_on_the_stack);
}

```

`int pari_daemon(void)` forks a PARI daemon, detaching from the main process group. The function returns 1 in the parent, and 0 in the forked son.

`void paristack_setsize(size_t rsize, size_t vsize)` sets the default `parisize` to `rsize` and the default `parisizemax` to `vsize`, and reallocate the stack to match these value, destroying its content. Generally used just after `pari_init`.

`void paristack_resize(ulong newsize)` changes the current stack size to `newsize` (double it if `newsize` is 0). The new size is clipped to be at least the current stack size and at most `parisizemax`. The stack content is not affected by this operation.

`void parivstack_reset(void)` resets the current stack to its default size `parisize`. This is used to recover memory after a computation that enlarged the stack. This function destroys the content of the enlarged stack (between the old and the new bottom of the stack). Before calling this function, you must ensure that `avma` lies within the new smaller stack.

`void paristack_newsize(ulong newsize)` (*does not return*). Library version of  
`default(parisize, "newsize")`

Set the default `parisize` to `newsize`, or double `parisize` if `newsize` is equal to 0, then call `cb_pari_err_recover(-1)`.

`void parivstack_resize(ulong newsize)` (*does not return*). Library version of  
`default(parisizemax, "newsize")`

Set the default `parisizemax` to `newsize` and call `cb_pari_err_recover(-1)`.

### 5.1.3 Notions specific to the GP interpreter.

An **entree** is the generic object attached to an identifier (a name) in GP's interpreter, be it a built-in or user function, or a variable. For a function, it has at least the following fields:

`char *name`: the name under which the interpreter knows us.

`void *value`: a pointer to the C function to call.

`long menu`: a small integer  $\geq 1$  (to which group of function help do we belong, for the `?n` help menu).

`char *code`: the prototype code.

`char *help`: the help text for the function.

A routine in GP is described to the analyzer by an **entree** structure. Built-in PARI routines are grouped in *modules*, which are arrays of **entree** structs, the last of which satisfy `name = NULL` (sentinel). There are currently four modules in PARI/GP:

- general functions (`functions_basic`, known to `libpari`),
- gp-specific functions (`functions_gp`),

and two modules of obsolete functions. The function `pari_init` initializes the interpreter and declares all symbols in `functions_basic`. You may declare further functions on a case by case basis or as a whole module using

`void pari_add_function(entree *ep)` adds a single routine to the table of symbols in the interpreter. It assumes `pari_init` has been called.

`void pari_add_module(entree *mod)` adds all the routines in module `mod` to the table of symbols in the interpreter. It assumes `pari_init` has been called.

For instance, `gp` implements a number of private routines, which it adds to the default set via the calls

```
pari_add_module(functions_gp);
```

A GP `default` is likewise attached to a helper routine, that is run when the value is consulted, or changed by `default0` or `setdefault`. Such routines are grouped in the module `functions_default`.



`void pari_add_defaults_module(entree *mod)` adds all the defaults in module `mod` to the interpreter. It assumes that `pari_init` has been called. From this point on, all defaults in module `mod` are known to `setdefault` and friends.

#### 5.1.4 Public callbacks.

The `gp` calculator associates elaborate functions (for instance the break loop handler) to the following callbacks, and so can you:

`void (*cb_pari_ask_confirm)(const char *s)` initialized to `NULL`. Called with argument `s` whenever PARI wants confirmation for action `s`, for instance in `secure` mode.

`void (*cb_pari_init_histfile)(void)` initialized to `NULL`. Called when the `histfile` default is changed. The intent is for that callback to read the file content, append it to history in memory, then dump the expanded history to the new `histfile`.

`int (*cb_pari_is_interactive)(void)`; initialized to `NULL`.

`void (*cb_pari_quit)(long)` initialized to a no-op. Called when `gp` must evaluate the `quit` command.

`void (*cb_pari_start_output)(void)` initialized to `NULL`.

`int (*cb_pari_handle_exception)(long)` initialized to `NULL`. If not `NULL`, this routine is called with argument `-1` on `SIGINT`, and argument `err` on error `err`. If it returns a nonzero value, the error or signal handler returns, in effect further ignoring the error or signal, otherwise it raises a fatal error. A possible simple-minded handler, used by the `gp` interpreter, is

`int gp_handle_exception(long err)` if the `breakloop` default is enabled (set to 1) and `cb_pari_break_loop` is not `NULL`, we call this routine with `err` argument and return the result.

`int (*cb_pari_err_handle)(GEN)` If not `NULL`, this routine is called with a `t_ERROR` argument from `pari_err`. If it returns a nonzero value, the error returns, in effect further ignoring the error, otherwise it raises a fatal error.

The default behavior is to print a descriptive error message (display the error), then return 0, thereby raising a fatal error. This differs from `cb_pari_handle_exception` in that the function is not called on `SIGINT` (which do not generate a `t_ERROR`), only from `pari_err`. Use `cb_pari_sigint` if you need to handle `SIGINT` as well.

The following function can be used by `cb_pari_err_handle` to display the error message.

`const char* closure_func_err()` return a statically allocated string holding the name of the function that triggered the error. Return `NULL` if the error was not caused by a function.

`int (*cb_pari_break_loop)(int)` initialized to `NULL`.

`void (*cb_pari_sigint)(void)`. Function called when we receive `SIGINT`. By default, raises

`pari_err(e_MISC, "user interrupt");`

A possible simple-minded variant, used by the `gp` interpreter, is

`void gp_sigint_fun(void)`

`void (*cb_pari_pre_recover)(long)` initialized to `NULL`. If not `NULL`, this routine is called just before PARI cleans up from an error. It is not required to return. The error number is passed as argument.

`void (*cb_pari_err_recover)(long)` initialized to `pari_exit()`. This callback must not return. It is called after PARI has cleaned-up from an error. The error number is passed as argument, unless the PARI stack has been destroyed, in which case it is called with argument `-1`.

`int (*cb_pari_whatnow)(PariOUT *out, const char *s, int flag)` initialized to `NULL`. If not `NULL`, must check whether `s` existed in older versions of `pari` (the `gp` callback checks against `pari-1.39.15`). All output must be done via `out` methods.

- `flag = 0`: should print verbosely the answer, including help text if available.
- `flag = 1`: must return 0 if the function did not change, and a nonzero result otherwise. May print a help message.

### 5.1.5 Configuration variables.

`pari_library_path`: If set, It should be a path to the `libpari` library. It is used by the function `gpinstall` to locate the PARI library when searching for symbols. This should only be useful on Windows.

### 5.1.6 Utility functions.

`void pari_ask_confirm(const char *s)` raise an error if the callback `cb_pari_ask_confirm` is `NULL`. Otherwise calls

```
cb_pari_ask_confirm(s);
```

`char* gp_filter(const char *s)` pre-processor for the GP parser: filter out whitespace and GP comments from `s`. The returned string is allocated on the PARI stack and must not be freed.

GEN `pari_compile_str(const char *s)` low-level form of `compile_str`: assumes that `s` does not contain spaces or GP comments and returns the closure attached to the GP expression `s`. Note that GP metacommands are not recognized.

`int gp_meta(const char *s, int ismain)` low-level component of `gp_read_str`: assumes that `s` does not contain spaces or GP comments and try to interpret `s` as a GP metacommand (e.g. starting by `\` or `?`). If successful, execute the metacommand and return 1; otherwise return 0. The `ismain` parameter modifies the way `\r` commands are handled: if nonzero, act as if the file contents were entered via standard input (i.e. call `switchin` and divert `pari_infile`); otherwise, simply call `gp_read_file`.

`void pari_hit_return(void)` wait for the use to enter `\n` via standard input.

`void gp_load_gprc(void)` read and execute the user's GPRC file.

`void pari_center(const char *s)` print `s`, centered.

`void pari_print_version(void)` print verbose version information.

`long pari_community(void)` return the index of the support section `n` the help.

`const char* gp_format_time(long t)` format a delay of `t` ms suitable for `gp` output, with `timer` set. The string is allocated in the PARI stack via `stack_malloc`.

`const char* gp_format_prompt(const char *p)` format a prompt `p` suitable for `gp` prompting (includes colors and protecting ANSI escape sequences for readline).

`void pari_alarm(long s)` set an alarm after `s` seconds (raise an `e_ALARM` exception).

`void gp_help(const char *s, long flag)` print help for *s*, depending on the value of *flag*:

- `h_REGULAR`, basic help (?);
- `h_LONG`, extended help (??);
- `h_APROPOS`, a propos help (??).

`const char ** gphelp_keyword_list(void)` return a NULL-terminated array of strings, containing keywords known to `gphelp` besides GP functions (e.g. `modulus` or `operator`). Used by the online help system and the contextual completion engine.

`void gp_echo_and_log(const char *p, const char *s)` given a prompt *p* and attached input command *s*, update logfile and possibly print on standard output if `echo` is set and we are not in interactive mode. The callback `cb_pari_is_interactive` must be set to a sensible value.

`void gp_alarm_handler(int sig)` the SIGALRM handler set by the `gp` interpreter.

`void print_fun_list(char **list, long n)` print all elements of *list* in columns, pausing (hit return) every *n* lines. *list* is NULL terminated.

### 5.1.7 Saving and restoring the GP context.

`void gp_context_save(struct gp_context* rec)` save the current GP context.

`void gp_context_restore(struct gp_context* rec)` restore a GP context. The new context must be an ancestor of the current context.

### 5.1.8 GP history.

These functions allow to control the GP history (the `%` operator).

`void pari_add_hist(GEN x, long t, long r)` adds *x* as the last history entry; *t* (resp. *r*) is the cpu (resp. real) time used to compute it.

`GEN pari_get_hist(long p)`, if *p* > 0 returns entry of index *p* (i.e. `%p`), else returns entry of index *n* + *p* where *n* is the index of the last entry (used for `%`, `%'`, `%''`, etc.).

`long pari_get_histtime(long p)` as `pari_get_hist`, returning the cpu time used to compute the history entry, instead of the entry itself.

`long pari_get_histrtime(long p)` as `pari_get_hist`, returning the real time used to compute the history entry, instead of the entry itself.

`GEN pari_histtime(long p)` return the vector [*cpu*, *real*] where *cpu* and *real* are as above.

`ulong pari_nb_hist(void)` return the index of the last entry.

## 5.2 Handling GENs.

Almost all these functions are either macros or inlined. Unless mentioned otherwise, they do not evaluate their arguments twice. Most of them are specific to a set of types, although no consistency checks are made: e.g. one may access the `sign` of a `t_PADIC`, but the result is meaningless.

### 5.2.1 Allocation.

GEN `cgetg(long l, long t)` allocates (the root of) a GEN of type `t` and length `l`. Sets `z[0]`.

GEN `cgeti(long l)` allocates a `t_INT` of length `l` (including the 2 codewords). Sets `z[0]` only.

GEN `cgetr(long l)` allocates a `t_REAL` of length `l` (including the 2 codewords). Sets `z[0]` only.

GEN `cgetc(long prec)` allocates a `t_COMPLEX` whose real and imaginary parts are `t_REALs` of length `prec`.

GEN `cgetg_copy(GEN x, long *lx)` fast version of `cgetg`: allocate a GEN with the same type and length as `x`, setting `*lx` to `lg(x)` as a side-effect. (Only sets the first codeword.) This is a little faster than `cgetg` since we may reuse the bitmask in `x[0]` instead of recomputing it, and we do not need to check that the length does not overflow the possibilities of the implementation (since an object with that length already exists). Note that `cgetg` with arguments known at compile time, as in

```
cgetg(3, t_INTMOD)
```

will be even faster since the compiler will directly perform all computations and checks.

GEN `vec trunc_init(long l)` perform `cgetg(1,t_VEC)`, then set the length to `l` and return the result. This is used to implement vectors whose final length is easily bounded at creation time, that we intend to fill gradually using:

`void vec trunc_append(GEN x, GEN y)` assuming `x` was allocated using `vec trunc_init`, appends `y` as the last element of `x`, which grows in the process. The function is shallow: we append `y`, not a copy; it is equivalent to

```
long lx = lg(x); gel(x, lx) = y; setlg(x, lx+1);
```

Beware that the maximal size of `x` (the `l` argument to `vec trunc_init`) is unknown, hence unchecked, and stack corruption will occur if we append more than `l - 1` elements to `x`. Use the safer (but slower) `shallowconcat` when `l` is not easy to bound in advance.

An other possibility is simply to allocate using `cgetg(1, t)` then fill the components as they become available: this time the downside is that we do not obtain a correct GEN until the vector is complete. Almost no PARI function will be able to operate on it.

`void vec trunc_append_batch(GEN x, GEN y)` successively apply

```
vec trunc_append(x, gel(y, i))
```

for all elements of the vector `y`.

GEN `col trunc_init(long l)` as `vec trunc_init` but perform `cgetg(1,t_COL)`.

GEN `vec small trunc_init(long l)`

`void vec small trunc_append(GEN x, long t)` analog to the above for a `t_VECSMALL` container.

### 5.2.2 Length conversions.

These routines convert a nonnegative length to different units. Their behavior is undefined at negative integers.

`long ndec2nlong(long x)` converts a number of decimal digits to a number of words. Returns  $1 + \text{floor}(x \times \text{BITS\_IN\_LONG} \log_2 10)$ .

`long ndec2prec(long x)` converts a number of decimal digits to a number of codewords. This is equal to  $2 + \text{ndec2nlong}(x)$ .

`long ndec2nbits(long x)` converts a number of decimal digits to a number of bits.

`long prec2ndec(long x)` converts a number of codewords to a number of decimal digits.

`long nbits2nlong(long x)` converts a number of bits to a number of words. Returns the smallest word count containing  $x$  bits, i.e.  $\text{ceil}(x/\text{BITS\_IN\_LONG})$ .

`long nbits2ndec(long x)` converts a number of bits to a number of decimal digits.

`long nbits2lg(long x)` converts a number of bits to a length in code words. Currently an alias for `nbits2nlong`.

`long nbits2prec(long x)` converts a number of bits to a number of codewords. This is equal to  $2 + \text{nbits2nlong}(x)$ .

`long nbits2extraprec(long x)` converts a number of bits to the mantissa length of a `t_REAL` in codewords. This is currently an alias to `nbits2nlong`.

`long nchar2nlong(long x)` converts a number of bytes to number of words. Returns the smallest word count containing  $x$  bytes, i.e.  $\text{ceil}(x/\text{sizeof}(\text{long}))$ .

`long prec2nbits(long x)` converts a `t_REAL` length into a number of significant bits; returns  $(x - 2)\text{BITS\_IN\_LONG}$ .

`double prec2nbits_mul(long x, double y)` returns  $\text{prec2nbits}(x) \times y$ .

`long bit_accuracy(long x)` converts a length into a number of significant bits; currently an alias for `prec2nbits`.

`double bit_accuracy_mul(long x, double y)` returns  $\text{bit\_accuracy}(x) \times y$ .

`long realprec(GEN x)` length of a `t_REAL` in words; currently an alias for `lg`.

`long bit_prec(GEN x)` length of a `t_REAL` in bits.

`long precdbl(long prec)` given a length in words corresponding to a `t_REAL` precision, return the length corresponding to doubling the precision. Due to the presence of 2 code words, this is  $2(\text{prec} - 2) + 2$ .

### 5.2.3 Read type-dependent information.

`long typ(GEN x)` returns the type number of  $x$ . The header files included through `pari.h` define symbolic constants for the GEN types: `t_INT` etc. Never use their actual numerical values. E.g to determine whether  $x$  is a `t_INT`, simply check

```
if (typ(x) == t_INT) { }
```

The types are internally ordered and this simplifies the implementation of commutative binary operations (e.g addition, gcd). Avoid using the ordering directly, as it may change in the future; use type grouping functions instead (Section 5.2.6).

`const char* type_name(long t)` given a type number  $t$  this routine returns a string containing its symbolic name. E.g `type_name(t_INT)` returns `"t_INT"`. The return value is read-only.

`long lg(GEN x)` returns the length of  $x$  in `BITS_IN_LONG`-bit words.

`long lgfint(GEN x)` returns the effective length of the `t_INT`  $x$  in `BITS_IN_LONG`-bit words.

`long signe(GEN x)` returns the sign ( $-1$ ,  $0$  or  $1$ ) of  $x$ . Can be used for `t_INT`, `t_REAL`, `t_POL` and `t_SER` (for the last two types, only  $0$  or  $1$  are possible).

`long gsigne(GEN x)` returns the sign of a real number  $x$ , valid for `t_INT`, `t_REAL` as `signe`, but also for `t_FRAC` and `t_QUAD` of positive discriminants. Raise a type error if `typ(x)` is not among those.

`long expi(GEN x)` returns the binary exponent of the real number equal to the `t_INT`  $x$ . This is a special case of `gexpo`.

`long expo(GEN x)` returns the binary exponent of the `t_REAL`  $x$ .

`long mpexpo(GEN x)` returns the binary exponent of the `t_INT` or `t_REAL`  $x$ .

`long gexpo(GEN x)` same as `expo`, but also valid when  $x$  is not a `t_REAL` (returns the largest exponent found among the components of  $x$ ). When  $x$  is an exact  $0$ , this returns `-HIGHEXPOBIT`, which is lower than any valid exponent.

`long gexpo_safe(GEN x)` same as `gexpo`, but returns a value strictly less than `-HIGHEXPOBIT` when the exponent is not defined (e.g. for a `t_PADIC` or `t_INTMOD` component).

`long valp(GEN x)` returns the  $p$ -adic valuation (for a `t_PADIC`) or  $X$ -adic valuation (for a `t_SER`, taken with respect to the main variable) of  $x$ .

`long precp(GEN x)` returns the precision of the `t_PADIC`  $x$ .

`long varn(GEN x)` returns the variable number of the `t_POL` or `t_SER`  $x$  (between  $0$  and `MAXVARN`).

`long gvar(GEN x)` returns the main variable number when any variable at all occurs in the composite object  $x$  (the smallest variable number which occurs), and `NO_VARIABLE` otherwise.

`long gvar2(GEN x)` returns the variable number for the ring over which  $x$  is defined, e.g. if  $x \in \mathbb{Z}[a][b]$  return (the variable number for)  $a$ . Return `NO_VARIABLE` if  $x$  has no variable or is not defined over a polynomial ring.

`long degpol(GEN x)` is a simple macro returning `lg(x) - 3`. This is the degree of the `t_POL`  $x$  with respect to its main variable, *if* its leading coefficient is nonzero (a rational  $0$  is impossible, but an inexact  $0$  is allowed, as well as an exact modular  $0$ , e.g. `Mod(0,2)`). If  $x$  has no coefficients (rational  $0$  polynomial), its length is  $2$  and we return the expected  $-1$ .

`long lgpol(GEN x)` is equal to `degpol(x) + 1`. Used to loop over the coefficients of a `t_POL` in the following situation:

```
GEN xd = x + 2;
long i, l = lgpol(x);
for (i = 0; i < l; i++) foo( xd[i] ).
```

`long precision(GEN x)` If `x` is of type `t_REAL`, returns the precision of `x`, namely

- if `x` is not zero: the length of `x` in `BITS_IN_LONG`-bit words;
- if `x` is numerically equal to 0, of exponent `e`: the absolute accuracy `nbits2prec(e)` if  $e < 0$  and `LOWDEFAULTPREC` if  $e \geq 0$ .

If `x` is of type `t_COMPLEX`, returns the minimum of the precisions of the real and imaginary part. Otherwise, returns 0 (which stands for infinite precision). In all cases, the precision is either 0 or can be used as a `prec` parameter in transcendental functions.

`long lgcols(GEN x)` is equal to `lg(gel(x,1))`. This is the length of the columns of a `t_MAT` with at least one column.

`long nbrows(GEN x)` is equal to `lg(gel(x,1))-1`. This is the number of rows of a `t_MAT` with at least one column.

`long gprecision(GEN x)` as `precision` for scalars. Returns the lowest precision encountered among the components otherwise.

`long sizedigit(GEN x)` returns 0 if `x` is exactly 0. Otherwise, returns `gexpo(x)` multiplied by  $\log_{10}(2)$ . This gives a crude estimate for the maximal number of decimal digits of the components of `x`.

**5.2.4 Eval type-dependent information.** These routines convert type-dependent information to bitmask to fill the codewords of `GEN` objects (see Section 4.5). E.g for a `t_REAL` `z`:

```
z[1] = evalsigne(-1) | evalexpo(2)
```

Compatible components of a codeword for a given type can be OR-ed as above.

`ulong evaltyp(long x)` convert type `x` to bitmask (first codeword of all `GENs`)

`long evallg(long x)` convert length `x` to bitmask (first codeword of all `GENs`). Raise overflow error if `x` is so large that the corresponding length cannot be represented

`long _evallg(long x)` as `evallg` *without* the overflow check.

`ulong evalvarn(long x)` convert variable number `x` to bitmask (second codeword of `t_POL` and `t_SER`)

`long evalsigne(long x)` convert sign `x` (in  $-1, 0, 1$ ) to bitmask (second codeword of `t_INT`, `t_REAL`, `t_POL`, `t_SER`)

`long evalprec(long x)` convert  $p$ -adic ( $X$ -adic) precision `x` to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if `x` is so large that the corresponding precision cannot be represented.

`long _evalprec(long x)` same as `evalprec` *without* the overflow check.

`long evalvalp(long x)` convert  $p$ -adic ( $X$ -adic) valuation  $x$  to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if  $x$  is so large that the corresponding valuation cannot be represented.

`long _evalvalp(long x)` same as `evalvalp` *without* the overflow check.

`long evalexpo(long x)` convert exponent  $x$  to bitmask (second codeword of `t_REAL`). Raise overflow error if  $x$  is so large that the corresponding exponent cannot be represented

`long _evalexpo(long x)` same as `evalexpo` *without* the overflow check.

`long evallgefint(long x)` convert effective length  $x$  to bitmask (second codeword `t_INT`). This should be less or equal than the length of the `t_INT`, hence there is no overflow check for the effective length.

**5.2.5 Set type-dependent information.** Use these functions and macros with extreme care since usually the corresponding information is set otherwise, and the components and further codeword fields (which are left unchanged) may not be compatible with the new information.

`void settyp(GEN x, long s)` sets the type number of  $x$  to  $s$ .

`void setlg(GEN x, long s)` sets the length of  $x$  to  $s$ . This is an efficient way of truncating vectors, matrices or polynomials.

`void setlgefint(GEN x, long s)` sets the effective length of the `t_INT`  $x$  to  $s$ . The number  $s$  must be less than or equal to the length of  $x$ .

`void setsigne(GEN x, long s)` sets the sign of  $x$  to  $s$ . If  $x$  is a `t_INT` or `t_REAL`,  $s$  must be equal to  $-1$ ,  $0$  or  $1$ , and if  $x$  is a `t_POL` or `t_SER`,  $s$  must be equal to  $0$  or  $1$ . No sanity check is made; in particular, setting the sign of a  $0$  `t_INT` to  $\pm 1$  creates an invalid object.

`void togglesign(GEN x)` sets the sign  $s$  of  $x$  to  $-s$ , in place.

`void togglesign_safe(GEN *x)` sets the  $s$  sign of  $*x$  to  $-s$ , in place, unless  $*x$  is one of the integer universal constants in which case replace  $*x$  by its negation (e.g. replace `gen_1` by `gen_m1`).

`void setabssign(GEN x)` sets the sign  $s$  of  $x$  to  $|s|$ , in place.

`void affectsign(GEN x, GEN y)` shortcut for `setsigne(y, signe(x))`. No sanity check is made; in particular, setting the sign of a  $0$  `t_INT` to  $\pm 1$  creates an invalid object.

`void affectsign_safe(GEN x, GEN *y)` sets the sign of  $*y$  to that of  $x$ , in place, unless  $*y$  is one of the integer universal constants in which case replace  $*y$  by its negation if needed (e.g. replace `gen_1` by `gen_m1` if  $x$  is negative). No other sanity check is made; in particular, setting the sign of a  $0$  `t_INT` to  $\pm 1$  creates an invalid object.

`void normalize_frac(GEN z)` assuming  $z$  is of the form `mkfrac(a,b)` with  $b \neq 0$ , make sure that  $b > 0$  by changing the sign of  $a$  in place if needed (use `togglesign`).

`void setexpo(GEN x, long s)` sets the binary exponent of the `t_REAL`  $x$  to  $s$ . The value  $s$  must be a 24-bit signed number.

`void setvalp(GEN x, long s)` sets the  $p$ -adic or  $X$ -adic valuation of  $x$  to  $s$ , if  $x$  is a `t_PADIC` or a `t_SER`, respectively.

`void setprec(GEN x, long s)` sets the  $p$ -adic precision of the `t_PADIC`  $x$  to  $s$ .

`void setvarn(GEN x, long s)` sets the variable number of the `t_POL` or `t_SER`  $x$  to  $s$  (where  $0 \leq s \leq \text{MAXVARN}$ ).



**5.2.6 Type groups.** In the following functions, `t` denotes the type of a GEN. They used to be implemented as macros, which could evaluate their argument twice; *no longer*: it is not inefficient to write

```

    is_intreal_t(typ(x))

int is_recursive_t(long t) true iff t is a recursive type (the nonrecursive types are t_INT,
t_REAL, t_STR, t_VECSMALL). Somewhat contrary to intuition, t_LIST is also nonrecursive, ; see
the Developer's guide for details.

int is_intreal_t(long t) true iff t is t_INT or t_REAL.

int is_rational_t(long t) true iff t is t_INT or t_FRAC.

int is_real_t(long t) true iff t is t_INT or t_REAL or t_FRAC.

int is_qfb_t(long t) true iff t is t_QFB.

int is_vec_t(long t) true iff t is t_VEC or t_COL.

int is_matvec_t(long t) true iff t is t_MAT, t_VEC or t_COL.

int is_scalar_t(long t) true iff t is a scalar, i.e a t_INT, a t_REAL, a t_INTMOD, a t_FRAC, a
t_COMPLEX, a t_PADIC, a t_QUAD, or a t_POLMOD.

int is_extscalar_t(long t) true iff t is a scalar (see is_scalar_t) or t is t_POL.

int is_const_t(long t) true iff t is a scalar which is not t_POLMOD.

int is_noncalc_t(long t) true if generic operations (gadd, gmul) do not make sense for t: cor-
responds to types t_LIST, t_STR, t_VECSMALL, t_CLOSURE

```

**5.2.7 Accessors and components.** The first two functions return GEN components as copies on the stack:

GEN `compo`(GEN `x`, long `n`) creates a copy of the `n`-th true component (i.e. not counting the codewords) of the object `x`.

GEN `truecoeff`(GEN `x`, long `n`) creates a copy of the coefficient of degree `n` of `x` if `x` is a scalar, `t_POL` or `t_SER`, and otherwise of the `n`-th component of `x`.

On the contrary, the following routines return the address of a GEN component. No copy is made on the stack:

GEN `constant_coeff`(GEN `x`) returns the address of the constant coefficient of `t_POL` `x`. By convention, a 0 polynomial (whose `sign` is 0) has `gen_0` constant term.

GEN `leading_coeff`(GEN `x`) returns the address of the leading coefficient of `t_POL` `x`, i.e. the coefficient of largest index stored in the array representing `x`. This may be an inexact 0. By convention, return `gen_0` if the coefficient array is empty.

GEN `gel`(GEN `x`, long `i`) returns the address of the `x[i]` entry of `x`. (`el` stands for element.)

GEN `gcoeff`(GEN `x`, long `i`, long `j`) returns the address of the `x[i,j]` entry of `t_MAT` `x`, i.e. the coefficient at row `i` and column `j`.

GEN `gmael`(GEN `x`, long `i`, long `j`) returns the address of the `x[i][j]` entry of `x`. (`mael` stands for multidimensional array element.)

GEN `gmael2`(GEN `A`, long `x1`, long `x2`) is an alias for `gmael`. Similar macros `gmael3`, `gmael4`, `gmael5` are available.

## 5.3 Global numerical constants.

These are defined in the various public PARI headers.

### 5.3.1 Constants related to word size.

`long BITS_IN_LONG = 2TWOPOTBITS_IN_LONG`: number of bits in a `long` (32 or 64).

`long BITS_IN_HALFULONG`: `BITS_IN_LONG` divided by 2.

`long LONG_MAX`: the largest positive `long`.

`ulong ULONG_MAX`: the largest `ulong`.

`long DEFAULTPREC`: the length (`lg`) of a `t_REAL` with 64 bits of accuracy

`long MEDDEFAULTPREC`: the length (`lg`) of a `t_REAL` with 128 bits of accuracy

`long BIGDEFAULTPREC`: the length (`lg`) of a `t_REAL` with 192 bits of accuracy

`ulong HIGHBIT`: the largest power of 2 fitting in an `ulong`.

`ulong LOWMASK`: bitmask yielding the least significant bits.

`ulong HIGHMASK`: bitmask yielding the most significant bits.

The last two are used to implement the following convenience macros, returning half the bits of their operand:

`ulong LOWWORD(ulong a)` returns least significant bits.

`ulong HIGHWORD(ulong a)` returns most significant bits.

Finally

`long divsBIL(long n)` returns the Euclidean quotient of  $n$  by `BITS_IN_LONG` (with nonnegative remainder).

`long remdBIL(n)` returns the (nonnegative) Euclidean remainder of  $n$  by `BITS_IN_LONG`

`long dvmdsBIL(long n, long *r)`

`ulong dvmduBIL(ulong n, ulong *r)` sets  $r$  to `remdBIL(n)` and returns `divsBIL(n)`.

### 5.3.2 Masks used to implement the GEN type.

These constants are used by higher level macros, like `typ` or `lg`:

`EXP0numBITS`, `LGnumBITS`, `SIGNnumBITS`, `TYPnumBITS`, `VALPnumBITS`, `VARNnumBITS`: number of bits used to encode `expo`, `lg`, `signe`, `typ`, `valp`, `varn`.

`PRECPSHIFT`, `SIGNSHIFT`, `TYPSHIFT`, `VARNSHIFT`: shifts used to recover or encode `precp`, `varn`, `typ`, `signe`

`CLONEBIT`, `EXPOBITS`, `LGBITS`, `PRECPBITS`, `SIGNBITS`, `TYPBITS`, `VALPBITS`, `VARNBITS`: bitmasks used to extract `isclone`, `expo`, `lg`, `precp`, `signe`, `typ`, `valp`, `varn` from `GEN` codewords.

`MAXVARN`: the largest possible variable number.

`NO_VARIABLE`: sentinel returned by `gvar(x)` when  $x$  does not contain any polynomial; has a lower priority than any valid variable number.

`HIGHEXPOBIT`: a power of 2, one more than the largest possible exponent for a `t_REAL`.

`HIGHVALPBIT`: a power of 2, one more than the largest possible valuation for a `t_PADIC` or a `t_SER`.

### 5.3.3 $\log 2$ , $\pi$ .

These are double approximations to useful constants:

M\_PI:  $\pi$ .

M\_LN2:  $\log 2$ .

LOG10\_2:  $\log 2 / \log 10$ .

LOG2\_10:  $\log 10 / \log 2$ .

## 5.4 Iterating over small primes, low-level interface.

One of the methods used by the high-level prime iterator (see Section 4.8.2), is a precomputed table. Its direct use is deprecated, but documented here.

After `pari_init(size, maxprime)`, a “prime table” is initialized with the successive *differences* of primes up to (possibly just a little beyond) `maxprime`. The prime table occupies roughly `maxprime / log(maxprime)` bytes in memory, so be sensible when choosing `maxprime`; it is 500000 by default under `gp` and there is no real benefit in choosing a much larger value: the high-level iterator provide *fast* access to primes up to the *square* of `maxprime`. In any case, the implementation requires that `maxprime < 2BITS_IN_LONG - 2048`, whatever memory is available.

PARI currently guarantees that the first 6547 primes, up to and including 65557, are present in the table, even if you set `maxprime` to zero. in the `pari_init` call.

Some convenience functions:

`ulong maxprime()` the largest prime computable using our prime table.

`ulong maxprimeN()` the index  $N$  of the largest prime computable using the prime table. I.e.,  $p_N = \text{maxprime}()$ .

`void maxprime_check(ulong B)` raise an error if `maxprime()` is  $< B$ .

After the following initializations (the names  $p$  and  $ptr$  are arbitrary of course)

```
byteptr ptr = diffptr;
ulong p = 0;
```

calling the macro `NEXT_PRIME_VIADIFF_CHECK(p, ptr)` repeatedly will assign the successive prime numbers to  $p$ . Overrunning the prime table boundary will raise the error `e_MAXPRIME`, which just prints the error message:

```
*** not enough precomputed primes, need primelimit ~c
```

(for some numerical value  $c$ ), then the macro aborts the computation. The alternative macro `NEXT_PRIME_VIADIFF` operates in the same way, but will omit that check, and is slightly faster. It should be used in the following way:

```
byteptr ptr = diffptr;
ulong p = 0;

if (maxprime() < goal) pari_err_MAXPRIME(goal); /* not enough primes */
while (p <= goal) /* run through all primes up to goal */
{
    NEXT_PRIME_VIADIFF(p, ptr);
}
```

```

    ...
}

```

Here, we use the general error handling function `pari_err` (see Section 4.7.3), with the codeword `e_MAXPRIME`, raising the “not enough primes” error. This could be rewritten as

```

maxprime_check(goal);
while (p <= goal) /* run through all primes up to goal */
{
    NEXT_PRIME_VIADIFF(p, ptr);
    ...
}

```

`byteptr initprimes(ulong maxprime, long *L, ulong *lastp)` computes a (malloc’ed) “prime table”, in fact a table of all prime differences for  $p < \text{maxprime}$  (and possibly a little beyond). Set  $L$  to the table length (argument to `malloc`), and  $lastp$  to the last prime in the table.

`void initprimetable(ulong maxprime)` computes a prime table (of all prime differences for  $p < \text{maxprime}$ ) and assign it to the global variable `diffptr`. Don’t change `diffptr` directly, call this function instead. This calls `initprimes` and updates internal data recording the table size.

`ulong init_primepointer_geq(ulong a, byteptr *pd)` returns the smallest prime  $p \geq a$ , and sets  $*pd$  to the proper offset of `diffptr` so that `NEXT_PRIME_VIADIFF(p, *pd)` correctly returns `unextprime(p + 1)`.

`ulong init_primepointer_gt(ulong a, byteptr *pd)` returns the smallest prime  $p > a$ .

`ulong init_primepointer_leq(ulong a, byteptr *pd)` returns the largest prime  $p \leq a$ .

`ulong init_primepointer_lt(ulong a, byteptr *pd)` returns the largest prime  $p < a$ .

## 5.5 Handling the PARI stack.

### 5.5.1 Allocating memory on the stack.

`GEN cgetg(long n, long t)` allocates memory on the stack for an object of length  $n$  and type  $t$ , and initializes its first codeword.

`GEN cgeti(long n)` allocates memory on the stack for a `t_INT` of length  $n$ , and initializes its first codeword. Identical to `cgetg(n, t_INT)`.

`GEN cgetr(long n)` allocates memory on the stack for a `t_REAL` of length  $n$ , and initializes its first codeword. Identical to `cgetg(n, t_REAL)`.

`GEN cgetc(long n)` allocates memory on the stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REALs` of length  $n$ .

`GEN cgetp(GEN x)` creates space sufficient to hold the `t_PADIC`  $x$ , and sets the prime  $p$  and the  $p$ -adic precision to those of  $x$ , but does not copy (the  $p$ -adic unit or zero representative and the modulus of)  $x$ .

`GEN new_chunk(size_t n)` allocates a `GEN` with  $n$  components, *without* filling the required code words. This is the low-level constructor underlying `cgetg`, which calls `new_chunk` then sets the first code word. It works by simply returning the address  $((\text{GEN})\text{avma}) - n$ , after checking that it is larger than  $(\text{GEN})\text{bot}$ .

`void new_chunk_resize(size_t x)` this function is called by `new_chunk` when the PARI stack overflows. There is no need to call it manually. It will either extend the stack or report an `e_STACK` error.

`char* stack_malloc(size_t n)` allocates memory on the stack for  $n$  chars (*not*  $n$  GENs). This is faster than using `malloc`, and easier to use in most situations when temporary storage is needed. In particular there is no need to `free` individually all variables thus allocated: a simple `set_avma(oldavma)` might be enough. On the other hand, beware that this is not permanent independent storage, but part of the stack. The memory is aligned on `sizeof(long)` bytes boundaries.

`char* stack_malloc_align(size_t n, long k)` as `stack_malloc`, but the memory is aligned on  $k$  bytes boundaries. The number  $k$  must be a multiple of the `sizeof(long)`.

`char* stack_calloc(size_t n)` as `stack_malloc`, setting the memory to zero.

`char* stack_calloc_align(size_t n, long k)` as `stack_malloc_align`, setting the memory to zero.

Objects allocated through these last three functions cannot be `gerepile`'d, since they are not yet valid GENs: their codewords must be filled first.

`GEN cgetalloc(long t, size_t l)`, same as `cgetg(t, l)`, except that the result is allocated using `pari_malloc` instead of the PARI stack. The resulting GEN is now impervious to garbage collecting routines, but should be freed using `pari_free`.

### 5.5.2 Stack-independent binary objects.

`GENbin* copy_bin(GEN x)` copies  $x$  into a malloc'ed structure suitable for stack-independent binary transmission or storage. The object obtained is architecture independent provided, `sizeof(long)` remains the same on all PARI instances involved, as well as the multiprecision kernel (either native or GMP).

`GENbin* copy_bin_canon(GEN x)` as `copy_bin`, ensuring furthermore that the binary object is independent of the multiprecision kernel. Slower than `copy_bin`.

`GEN bin_copy(GENbin *p)` assuming  $p$  was created by `copy_bin(x)` (not necessarily by the same PARI instance: transmission or external storage may be involved), restores  $x$  on the PARI stack.

The routine `bin_copy` transparently encapsulate the following functions:

`GEN GENbinbase(GENbin *p)` the GEN data actually stored in  $p$ . All addresses are stored as offsets with respect to a common reference point, so the resulting GEN is unusable unless it is a nonrecursive type; private low-level routines must be called first to restore absolute addresses.

`void shiftaddress(GEN x, long dec)` converts relative addresses to absolute ones.

`void shiftaddress_canon(GEN x, long dec)` converts relative addresses to absolute ones, and converts leaves from a canonical form to the one specific to the multiprecision kernel in use. The `GENbin` type stores whether leaves are stored in canonical form, so `bin_copy` can call the right variant.

Objects containing closures are harder to e.g. copy and save to disk, since closures contain pointers to libpari functions that will not be valid in another gp instance: there is little chance for them to be loaded at the exact same address in memory. Such objects must be saved along with a linking table.

`GEN copybin_unlink(GEN C)` returns a linking table allowing to safely store and transmit `t_CLOSURE` objects in  $C$ . If  $C = \text{NULL}$  return a linking table corresponding to the content of all gp variables.  $C$  may then be dumped to disk in binary form, for instance.

`void bincopy_relink(GEN C, GEN V)` given a binary object  $C$ , as dumped by `writebin` and read back into a session, and a linking table  $V$ , restore all closures contained in  $C$  (function pointers are translated to their current value).

**5.5.3 Garbage collection.** See Section 4.3 for a detailed explanation and many examples.

`void set_avma(ulong av)` reset `avma` to `av`. You may think of this as a simple `avma = av` statement, but PARI developers modify this statement in special code branches to detect garbage collecting issues (by invalidating the PARI stack below `av`).

`ulong get_avma(void)` return `avma`. Useful for languages that do not provide access to TLS variables.

`GEN gc_NULL(pari_sp av)` reset `avma` to `av` and return `NULL`.

The following 6 functions reset `avma` to `av` and return  $x$ :

`int gc_bool(pari_sp av, int x)`

`double gc_double(pari_sp av, double x)`

`int gc_int(pari_sp av, int x)`

`long gc_long(pari_sp av, long x)`

`ulong gc_ulong(pari_sp av, ulong x)` This allows for instance to return `gc_ulong(av, itou(z))`, whereas

```

    pari_sp av = avma;
    GEN z = ...
    set_avma(av);
    return itou(z);

```

should be frowned upon since `set_avma(av)` conceptually destroys everything from the reference point on, including  $z$ .

`GEN gc_const(pari_sp av, GEN x)` assumes that  $x$  is either not on the stack (clone, universal constant such as `gen_0`) or was defined before `av`.

`GEN gc_all(pari_sp av, int n, ...)`. Assumes that  $1 \leq n \leq 10$ ; This is similar to `gerepileall`, expecting  $n$  further `GEN*` arguments: the stack is cleaned and the corresponding `GEN` are copied to the stack starting from `av` (in this order: the first argument comes first), and the first such `GEN` is returned. To be used in the following scenario:

```

GEN f(..., GEN *py)
{
    pari_sp av = avma;
    GEN x = ..., y = ...
    *py = y; return gc_all(av, 2, &x, py);
}

```

This function returns  $x$ , and the user also recovers  $y$  as a side effect. Note that we can later use `cgiv(y)` to recover the memory used by  $y$  while still keeping  $x$ .

`void cgiv(GEN x)` frees object `x`, assuming it is the last created on the stack.

`GEN gerepile(pari_sp p, pari_sp q, GEN x)` general garbage collector for the stack.

`void gerepileall(pari_sp av, int n, ...)` cleans up the stack from `av` on (i.e from `avma` to `av`), preserving the `n` objects which follow in the argument list (of type `GEN*`). For instance, `gerepileall(av, 2, &x, &y)` preserves `x` and `y`.

`void gerepileallsp(pari_sp av, pari_sp ltop, int n, ...)` cleans up the stack between `av` and `ltop`, updating the `n` elements which follow `n` in the argument list (of type `GEN*`). Check that the elements of `g` have no component between `av` and `ltop`, and assumes that no garbage is present between `avma` and `ltop`. Analogous to (but faster than) `gerepileall` otherwise.

`GEN gerepilecopy(pari_sp av, GEN x)` cleans up the stack from `av` on, preserving the object `x`. Special case of `gerepileall` (case `n = 1`), except that the routine returns the preserved `GEN` instead of updating its address through a pointer.

`void gerepilemany(pari_sp av, GEN* g[], int n)` alternative interface to `gerepileall`. The preserved `GENs` are the elements of the array `g` of length `n`: `g[0]`, `g[1]`, ..., `g[n-1]`. Obsolete: no more efficient than `gerepileall`, error-prone, and clumsy (need to declare an extra `GEN *g`).

`void gerepilemanysp(pari_sp av, pari_sp ltop, GEN* g[], int n)` alternative interface to `gerepileallsp`. Obsolete.

`void gerepilecoeffs(pari_sp av, GEN x, int n)` cleans up the stack from `av` on, preserving `x[0]`, ..., `x[n-1]` (which are `GENs`).

`void gerepilecoeffssp(pari_sp av, pari_sp ltop, GEN x, int n)` cleans up the stack from `av` to `ltop`, preserving `x[0]`, ..., `x[n-1]` (which are `GENs`). Same assumptions as in `gerepilemanysp`, of which this is a variant. For instance

```
z = cgetg(3, t_COMPLEX);
av = avma; garbage(); ltop = avma;
z[1] = fun1();
z[2] = fun2();
gerepilecoeffssp(av, ltop, z + 1, 2);
return z;
```

cleans up the garbage between `av` and `ltop`, and connects `z` and its two components. This is marginally more efficient than the standard

```
av = avma; garbage(); ltop = avma;
z = cgetg(3, t_COMPLEX);
z[1] = fun1();
z[2] = fun2(); return gerepile(av, ltop, z);
```

`GEN gerepileupto(pari_sp av, GEN q)` analogous to (but faster than) `gerepilecopy`. Assumes that `q` is connected and that its root was created before any component. If `q` is not on the stack, this is equivalent to `set_avma(av)`; in particular, sentinels which are not even proper `GENs` such as `q = NULL` are allowed.

`GEN gerepileuptoint(pari_sp av, GEN q)` analogous to (but faster than) `gerepileupto`. Assumes further that `q` is a `t_INT`. The length and effective length of the resulting `t_INT` are equal.

`GEN gerepileuptoleaf(pari_sp av, GEN q)` analogous to (but faster than) `gerepileupto`. Assumes further that `q` is a leaf, i.e a nonrecursive type (`is_recursive_t(typ(q))` is nonzero).

Contrary to `gerepileuptoint` and `gerepileupto`, `gerepileuptoleaf` leaves length and effective length of a `t_INT` unchanged.

#### 5.5.4 Garbage collection: advanced use.

`void stackdummy(pari_sp av, pari_sp ltop)` inhibits the memory area between `av` *included* and `ltop` *excluded* with respect to `gerepile`, in order to avoid a call to `gerepile(av, ltop, ...)`. The stack space is not reclaimed though.

More precisely, this routine assumes that `av` is recorded earlier than `ltop`, then marks the specified stack segment as a nonrecursive type of the correct length. Thus `gerepile` will not inspect the zone, at most copy it. To be used in the following situation:

```
av0 = avma; z = cgetg(t_VEC, 3);
gel(z,1) = HUGE(); av = avma; garbage(); ltop = avma;
gel(z,2) = HUGE(); stackdummy(av, ltop);
```

Compared to the orthodox

```
gel(z,2) = gerepile(av, ltop, gel(z,2));
```

or even more wasteful

```
z = gerepilecopy(av0, z);
```

we temporarily lose  $(av - ltop)$  words but save a costly `gerepile`. In principle, a garbage collection higher up the call chain should reclaim this later anyway.

Without the `stackdummy`, if the `[av, ltop]` zone is arbitrary (not even valid GENs as could happen after direct truncation via `setlg`), we would leave dangerous data in the middle of `z`, which would be a problem for a later

```
gerepile(..., ... , z);
```

And even if it were made of valid GENs, inhibiting the area makes sure `gerepile` will not inspect their components, saving time.

Another natural use in low-level routines is to “shorten” an existing GEN `z` to its first  $n - 1$  components:

```
setlg(z, n);
stackdummy((pari_sp)(z + lg(z)), (pari_sp)(z + n));
```

or to its last  $n$  components:

```
long L = lg(z) - n, tz = typ(z);
stackdummy((pari_sp)(z + L), (pari_sp)z);
z += L; z[0] = evaltyp(tz) | evallg(L);
```

The first scenario (safe shortening an existing GEN) is in fact so common, that we provide a function for this:

`void fixlg(GEN z, long ly)` a safe variant of `setlg(z, ly)`. If `ly` is larger than `lg(z)` do nothing. Otherwise, shorten `z` in place, using `stackdummy` to avoid later `gerepile` problems.

`GEN gcopy_avma(GEN x, pari_sp *AVMA)` return a copy of `x` as from `gcopy`, except that we pretend that initially `avma` is `*AVMA`, and that `*AVMA` is updated accordingly (so that the total size



of  $x$  is the difference between the two successive values of `*AVMA`). It is not necessary for `*AVMA` to initially point on the stack: `gclone` is implemented using this mechanism.

`GEN icopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume  $x$  is a `t_INT` and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

### 5.5.5 Debugging the PARI stack.

`int chk_gerepileupto(GEN x)` returns 1 if  $x$  is suitable for `gerepileupto`, and 0 otherwise. In the latter case, print a warning explaining the problem.

`void dbg_gerepile(pari_sp ltop)` outputs the list of all objects on the stack between `avma` and `ltop`, i.e. the ones that would be inspected in a call to `gerepile(..., ltop, ...)`.

`void dbg_gerepileupto(GEN q)` outputs the list of all objects on the stack that would be inspected in a call to `gerepileupto(..., q)`.

### 5.5.6 Copies.

`GEN gcopy(GEN x)` creates a new copy of  $x$  on the stack.

`GEN gcopy_lg(GEN x, long l)` creates a new copy of  $x$  on the stack, pretending that `lg(x)` is  $l$ , which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `gcopy(x)`.

`int isonstack(GEN x)` true iff  $x$  belongs to the stack.

`void copyifstack(GEN x, GEN y)` sets  $y = gcopy(x)$  if  $x$  belongs to the stack, and  $y = x$  otherwise. This macro evaluates its arguments once, contrary to

```
y = isonstack(x)? gcopy(x): x;
```

`void icopyifstack(GEN x, GEN y)` as `copyifstack` assuming  $x$  is a `t_INT`.

### 5.5.7 Simplify.

`GEN simplify(GEN x)` you should not need that function in library mode. One rather uses:

`GEN simplify_shallow(GEN x)` shallow, faster, version of `simplify`.

## 5.6 The PARI heap.

### 5.6.1 Introduction.

It is implemented as a doubly-linked list of `malloc`'ed blocks of memory, equipped with reference counts. Each block has type `GEN` but need not be a valid `GEN`: it is a chunk of data preceded by a hidden header (meaning that we allocate  $x$  and return  $x + \text{headersize}$ ). A *clone*, created by `gclone`, is a block which is a valid `GEN` and whose *clone bit* is set.

### 5.6.2 Public interface.

**GEN newblock(size\_t n)** allocates a block of  $n$  words (not bytes).

**void killblock(GEN x)** deletes the block  $x$  created by **newblock**. Fatal error if  $x$  not a block.

**GEN gclone(GEN x)** creates a new permanent copy of  $x$  on the heap (allocated using **newblock**). The *clone bit* of the result is set.

**GEN gcloneref(GEN x)** if  $x$  is not a clone, clone it and return the result; otherwise, increase the clone reference count and return  $x$ .

**void guncclone(GEN x)** deletes a clone. Deletion at first only decreases the reference count by 1. If the count remains positive, no further action is taken; if the count becomes zero, then the clone is actually deleted. In the current implementation, this is an alias for **killblock**, but it is cleaner to kill clones (valid GENs) using this function, and other blocks using **killblock**.

**void gunccloneNULL(GEN x)** same as **guncclone**, first checking whether  $x$  is NULL (and doing nothing in this case).

**void guncclone\_deep(GEN x)** is only useful in the context of the GP interpreter which may replace arbitrary components of container types (**t\_VEC**, **t\_COL**, **t\_MAT**, **t\_LIST**) by clones. If  $x$  is such a container, the function recursively deletes all clones among the components of  $x$ , then unclones  $x$ . Useless in library mode: simply use **guncclone**.

**void gunccloneNULL\_deep(GEN x)** same as **guncclone\_deep**, first checking whether  $x$  is NULL (and doing nothing in this case).

**void traverseheap(void(\*f)(GEN, void\*), void \*data)** this applies  $f(x, data)$  to each object  $x$  on the PARI heap, most recent first. Mostly for debugging purposes.

**GEN getheap()** a simple wrapper around **traverseheap**. Returns a two-component row vector giving the number of objects on the heap and the amount of memory they occupy in long words.

**GEN cgetg\_block(long x, long y)** as **cgetg(x,y)**, creating the return value as a block, not on the PARI stack.

**GEN cgetr\_block(long prec)** as **cgetr(prec)**, creating the return value as a block, not on the PARI stack.

**5.6.3 Implementation note.** The hidden block header is manipulated using the following private functions:

**void\* bl\_base(GEN x)** returns the pointer that was actually allocated by **malloc** (can be freed).

**long bl\_refc(GEN x)** the reference count of  $x$ : the number of pointers to this block. Decrement in **killblock**, incremented by the private function **void gclone\_refc(GEN x)**; block is freed when the reference count reaches 0.

**long bl\_num(GEN x)** the index of this block in the list of all blocks allocated so far (including freed blocks). Uniquely identifies a block until  $2^{\text{BITS\_IN\_LONG}}$  blocks have been allocated and this wraps around.

**GEN bl\_next(GEN x)** the block *after*  $x$  in the linked list of blocks (NULL if  $x$  is the last block allocated not yet killed).

**GEN bl\_prev(GEN x)** the block allocated *before*  $x$  (never NULL).

We documented the last four routines as functions for clarity (and type checking) but they are actually macros yielding valid lvalues. It is allowed to write **bl\_refc(x)++** for instance.

## 5.7 Handling user and temp variables.

Low-level implementation of user / temporary variables is liable to change. We describe it nevertheless for completeness. Currently variables are implemented by a single array of values divided in 3 zones: `0-nvar` (user variables), `max_avail-MAXVARN` (temporary variables), and `nvar+1-max_avail-1` (pool of free variable numbers).

### 5.7.1 Low-level.

`void pari_var_init()`: a small part of `pari_init`. Resets variable counters `nvar` and `max_avail`, notwithstanding existing variables! In effect, this even deletes `x`. Don't use it.

`void pari_var_close(void)` attached destructor, called by `pari_close`.

`long pari_var_next()`: returns `nvar`, the number of the next user variable we can create.

`long pari_var_next_temp()` returns `max_avail`, the number of the next temp variable we can create.

`long pari_var_create(entree *ep)` low-level initialization of an `EpVAR`. Return the attached (new) variable number.

`GEN vars_sort_inplace(GEN z)` given a `t_VECSMALL` `z` of variable numbers, sort `z` in place according to variable priorities (highest priority comes first).

`GEN vars_to_RgXV(GEN h)` given a `t_VECSMALL` `z` of variable numbers, return the `t_VEC` of `pol_x(z[i])`.

### 5.7.2 User variables.

`long fetch_user_var(char *s)` returns a user variable whose name is `s`, creating it is needed (and using an existing variable otherwise). Returns its variable number.

`GEN fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered `v`. Return `NULL` for a temporary variable.

`entree* is_entry(const char *s)` returns the `entree*` attached to an identifier `s` (variable or function), from the interpreter hashtables. Return `NULL` if the identifier is unknown.

### 5.7.3 Temporary variables.

`long fetch_var(void)` returns the number of a new temporary variable (decreasing `max_avail`).

`long delete_var(void)` delete latest temp variable created and return the number of previous one.

`void name_var(long n, char *s)` rename temporary variable number `n` to `s`; mostly useful for nicer printout. Error when trying to rename a user variable.

## 5.8 Adding functions to PARI.

**5.8.1 Nota Bene.** As mentioned in the `COPYING` file, modified versions of the PARI package can be distributed under the conditions of the GNU General Public License. If you do modify PARI, however, it is certainly for a good reason, and we would like to know about it, so that everyone can benefit from your changes. There is then a good chance that your improvements are incorporated into the next release.

We classify changes to PARI into four rough classes, where changes of the first three types are almost certain to be accepted. The first type includes all improvements to the documentation, in a broad sense. This includes correcting typos or inaccuracies of course, but also items which are not really covered in this document, e.g. if you happen to write a tutorial, or pieces of code exemplifying fine points unduly omitted in the present manual.

The second type is to expand or modify the configuration routines and skeleton files (the `Configure` script and anything in the `config/` subdirectory) so that compilation is possible (or easier, or more efficient) on an operating system previously not catered for. This includes discovering and removing idiosyncrasies in the code that would hinder its portability.

The third type is to modify existing (mathematical) code, either to correct bugs, to add new functionality to existing functions, or to improve their efficiency.

Finally the last type is to add new functions to PARI. We explain here how to do this, so that in particular the new function can be called from `gp`.

**5.8.2 Coding guidelines.** Code your function in a file of its own, using as a guide other functions in the PARI sources. One important thing to remember is to clean the stack before exiting your main function, since otherwise successive calls to the function clutters the stack with unnecessary garbage, and stack overflow occurs sooner. Also, if it returns a `GEN` and you want it to be accessible to `gp`, you have to make sure this `GEN` is suitable for `gerepileupto` (see Section 4.3).

If error messages or warnings are to be generated in your function, use `pari_err` and `pari_warn` respectively. Recall that `pari_err` does not return but ends with a `longjmp` statement. As well, instead of explicit `printf` / `fprintf` statements, use the following encapsulated variants:

`void pari_putc(char c):` write character `c` to the output stream.

`void pari_puts(char *s):` write `s` to the output stream.

`void pari_printf(const char *fmt, ...):` write following arguments to the output stream, according to the conversion specifications in format `fmt` (see `printf`).

`void err_printf(const char *fmt, ...):` as `pari_printf`, writing to PARI's current error stream.

`void err_flush(void)` flush error stream.

Declare all public functions in an appropriate header file, if you want to access them from C. The other functions should be declared `static` in your file.

Your function is now ready to be used in library mode after compilation and creation of the library. If possible, compile it as a shared library (see the `Makefile` coming with the `extgcd` example in the distribution). It is however still inaccessible from `gp`.

**5.8.3 GP prototypes, parser codes.** A *GP prototype* is a character string describing all the GP parser needs to know about the function prototype. It contains a sequence of the following atoms:

- Return type: **GEN** by default (must be valid for **gerepileupto**), otherwise the following can appear as the *first* char of the code string:

```

i      return int
l      return long
u      return ulong
v      return void
m      return a GEN which is not gerepile-safe.
```

The **m** code is used for member functions, to avoid unnecessary copies. A copy opcode is generated by the compiler if the result needs to be kept safe for later use.

- Mandatory arguments, appearing in the same order as the input arguments they describe:

```

G      GEN
&      *GEN
L      long (we implicitly typecast int to long)
U      ulong
V      loop variable
n      variable, expects a variable number (a long, not an *entree)
W      a GEN which is a lvalue to be modified in place (for t_LIST)
r      raw input (treated as a string without quotes). Quoted args are copied as strings
        Stops at first unquoted ')' or ', '. Special chars can be quoted using '\ '
        Example: aa"b\n)"c yields the string "aab\n)c"
s      expanded string. Example: Pi"x"2 yields "3.142x2"
        Unquoted components can be of any PARI type, converted to string following
        current output format
I      closure whose value is ignored, as in for loops,
        to be processed by void closure_evalvoid(GEN C)
E      closure whose value is used, as in sum loops,
        to be processed by void closure_evalgen(GEN C)
J      implicit function of arity 1, as in parsum loops,
        to be processed by void closure_callgen1(GEN C)
```

A *closure* is a GP function in compiled (bytecode) form. It can be efficiently evaluated using the `closure_evalxxx` functions.

- Automatic arguments:

```

f      Fake *long. C function requires a pointer but we do not use the resulting long
b      current real precision in bits
p      current real precision in words
P      series precision (default seriesprecision, global variable precdl for the library)
C      lexical context (internal, for eval, see localvars_read_str)
```

- Syntax requirements, used by functions like **for**, **sum**, etc.:
  - = separator = required at this point (between two arguments)

- Optional arguments and default values:

```

E*     any number of expressions, possibly 0 (see E)
s*     any number of strings, possibly 0 (see s)
```

*Dxxx* argument can be omitted and has a default value

The **E\*** code reads all remaining arguments in closure context and passes them as a single **t\_VEC**. The **s\*** code reads all remaining arguments in *string context* and passes the list of strings as a single **t\_VEC**. The automatic concatenation rules in string context are implemented so that adjacent strings are read as different arguments, as if they had been comma-separated. For instance, if the remaining argument sequence is: "**xx**" 1, "**yy**", the **s\*** atom sends [**a**, **b**, **c**], where *a*, *b*, *c* are GENs of type **t\_STR** (content "**xx**"), **t\_INT** (equal to 1) and **t\_STR** (content "**yy**").

The format to indicate a default value (atom starts with a **D**) is "**Dvalue,type,**", where *type* is the code for any mandatory atom (previous group), *value* is any valid GP expression which is converted according to *type*, and the ending comma is mandatory. For instance **D0,L**, stands for "this optional argument is converted to a **long**, and is 0 by default". So if the user-given argument reads 1 + 3 at this point, **4L** is sent to the function; and **0L** if the argument is omitted. The following special notations are available:

<b>DG</b>	optional <b>GEN</b> , send <b>NULL</b> if argument omitted.
<b>D&amp;</b>	optional <b>*GEN</b> , send <b>NULL</b> if argument omitted. The argument must be prefixed by <b>&amp;</b> .
<b>DI</b> , <b>DE</b>	optional closure, send <b>NULL</b> if argument omitted.
<b>DP</b>	optional <b>long</b> , send <b>prec dl</b> if argument omitted.
<b>DV</b>	optional <b>*entree</b> , send <b>NULL</b> if argument omitted.
<b>Dn</b>	optional variable number, <b>-1</b> if omitted.
<b>Dr</b>	optional raw string, send <b>NULL</b> if argument omitted.
<b>Ds</b>	optional <b>char *</b> , send <b>NULL</b> if argument omitted.

**Hardcoded limit.** C functions using more than 20 arguments are not supported. Use vectors if you really need that many parameters.

When the function is called under **gp**, the prototype is scanned and each time an atom corresponding to a mandatory argument is met, a user-given argument is read (**gp** outputs an error message if the argument was missing). Each time an optional atom is met, a default value is inserted if the user omits the argument. The "automatic" atoms fill in the argument list transparently, supplying the current value of the corresponding variable (or a dummy pointer).

For instance, here is how you would code the following prototypes, which do not involve default values:

<b>GEN f(GEN x, GEN y, long prec)</b>	----> " <b>GGp</b> "
<b>void f(GEN x, GEN y, long prec)</b>	----> " <b>vGGp</b> "
<b>void f(GEN x, long y, long prec)</b>	----> " <b>vGLp</b> "
<b>long f(GEN x)</b>	----> " <b>lG</b> "
<b>int f(long x)</b>	----> " <b>iL</b> "

If you want more examples, **gp** gives you easy access to the parser codes attached to all GP functions: just type **\h function**. You can then compare with the C prototypes as they stand in **paridecl.h**.

**Remark.** If you need to implement complicated control statements (probably for some improved summation functions), you need to know how the parser implements closures and lexicals and how the evaluator lets you deal with them, in particular the `push_lex` and `pop_lex` functions. Check their descriptions and adapt the source code in `language/sumiter.c` and `language/intnum.c`.

#### 5.8.4 Integration with `gp` as a shared module.

In this section we assume that your Operating System is supported by `install`. You have written a function in C following the guidelines in Section 5.8.2; in case the function returns a `GEN`, it must satisfy `gerepileupto` assumptions (see Section 4.3).

You then succeeded in building it as part of a shared library and want to finally tell `gp` about your function. First, find a name for it. It does not have to match the one used in library mode, but consistency is nice. It has to be a valid GP identifier, i.e. use only alphabetic characters, digits and the underscore character (`_`), the first character being alphabetic.

Then figure out the correct parser code corresponding to the function prototype (as explained in Section 5.8.3) and write a GP script like the following:

```
install(libname, code, gpname, library)
addhelp(gpname, "some help text")
```

The `addhelp` part is not mandatory, but very useful if you want others to use your module. `libname` is how the function is named in the library, usually the same name as one visible from C.

Read that file from your `gp` session, for instance from your preferences file (or `gprc`), and that's it. You can now use the new function `gpname` under `gp`, and we would very much like to hear about it!

**Example.** A complete description could look like this:

```
{
  install(bnfinit0, "GD0,L,DGp", ClassGroupInit, "libpari.so");
  addhelp(ClassGroupInit, "ClassGroupInit(P,{flag=0},{data=[]}):
    compute the necessary data for ...");
}
```

which means we have a function `ClassGroupInit` under `gp`, which calls the library function `bnfinit0`. The function has one mandatory argument, and possibly two more (two '`D`' in the code), plus the current real precision. More precisely, the first argument is a `GEN`, the second one is converted to a `long` using `itos` (0 is passed if it is omitted), and the third one is also a `GEN`, but we pass `NULL` if no argument was supplied by the user. This matches the C prototype (from `paridecl.h`):

```
GEN bnfinit0(GEN P, long flag, GEN data, long prec)
```

This function is in fact coded in `basemath/buch2.c`, and is in this case completely identical to the GP function `bnfinit` but `gp` does not need to know about this, only that it can be found somewhere in the shared library `libpari.so`.

**Important note.** You see in this example that it is the function's responsibility to correctly interpret its operands: `data = NULL` is interpreted *by the function* as an empty vector. Note that since `NULL` is never a valid GEN pointer, this trick always enables you to distinguish between a default value and actual input: the user could explicitly supply an empty vector!

### 5.8.5 Library interface for `install`.

There is a corresponding library interface for this `install` functionality, letting you expand the GP parser/evaluator available in the library with new functions from your C source code. Functions such as `gp_read_str` may then evaluate a GP expression sequence involving calls to these new function!

```
entree * install(void *f, const char *gpname, const char *code)
```

where `f` is the (address of the) function (cast to `void*`), `gpname` is the name by which you want to access your function from within your GP expressions, and `code` is as above.

### 5.8.6 Integration by patching `gp`.

If `install` is not available, and installing Linux or a BSD operating system is not an option (why?), you have to hardcode your function in the `gp` binary. Here is what needs to be done:

- Fetch the complete sources of the PARI distribution.
- Drop the function source code module in an appropriate directory (a priori `src/modules`), and declare all public functions in `src/headers/paridecl.h`.
- Choose a help section and add a file `src/functions/section/gpname` containing the following, keeping the notation above:

```
Function:  gpname
Section:   section
C-Name:    libname
Prototype: code
Help:      some help text
```

(If the help text does not fit on a single line, continuation lines must start by a whitespace character.) Two GP2C-related fields (`Description` and `Wrapper`) are also available to improve the code GP2C generates when compiling scripts involving your function. See the GP2C documentation for details.

- Launch `Configure`, which should pick up your C files and build an appropriate `Makefile`. At this point you can recompile `gp`, which will first rebuild the functions database.

**Example.** We reuse the `ClassGroupInit` / `bnfinit0` from the preceding section. Since the C source code is already part of PARI, we only need to add a file

```
functions/number_fields/ClassGroupInit
```

containing the following:

```
Function: ClassGroupInit
Section: number_fields
C-Name: bnfinit0
Prototype: GD0,L,DGp
Help: ClassGroupInit(P,{flag=0},{tech=[]}): this routine does ...
```

and recompile `gp`.



## 5.9 Globals related to PARI configuration.

### 5.9.1 PARI version numbers.

`paricfg_version_code` encodes in a single `long`, the Major and minor version numbers as well as the patchlevel.

`long PARI_VERSION(long M, long m, long p)` produces the version code attached to release  $M.m.p$ . Each code identifies a unique PARI release, and corresponds to the natural total order on the set of releases (bigger code number means more recent release).

`PARI_VERSION_SHIFT` is the number of bits used to store each of the integers  $M, m, p$  in the version code.

`paricfg_vcsversion` is a version string related to the revision control system used to handle your sources, if any. For instance `git-commit hash` if compiled from a git repository.

The two character strings `paricfg_version` and `paricfg_buildinfo`, correspond to the first two lines printed by `gp` just before the Copyright message. The character string `paricfg_compileddate` is the date of compilation which appears on the next line. The character string `paricfg_mt_engine` is the name of the threading engine on the next line.

In the string `paricfg_buildinfo`, the substring `"%s"` needs to be substituted by the output of the function `pari_kernel_version`.

```
const char * pari_kernel_version(void)
```

`GEN pari_version()` returns the version number as a PARI object, a `t_VEC` with three `t_INT` and one `t_STR` components.

### 5.9.2 Miscellaneous.

`paricfg_datadir`: character string. The location of PARI's `datadir`.

`paricfg_gphelp`: character string. The name of an external help command for ?? (such as the `gphelp` script)



## Chapter 6:

### Arithmetic kernel: Level 0 and 1

#### 6.1 Level 0 kernel (operations on ulongs).

**6.1.1 Micro-kernel.** The Level 0 kernel simulates basic operations of the 68020 processor on which PARI was originally implemented. They need “global” `ulong` variables `overflow` (which will contain only 0 or 1) and `hiremainder` to function properly. A routine using one of these lowest-level functions where the description mentions either `hiremainder` or `overflow` must declare the corresponding

```
LOCAL_HIREMAINDER; /* provides 'hiremainder' */
LOCAL_OVERFLOW;    /* provides 'overflow' */
```

in a declaration block. Variables `hiremainder` and `overflow` then become available in the enclosing block. For instance a loop over the powers of an `ulong p` protected from overflows could read

```
while (pk < lim)
{
    LOCAL_HIREMAINDER;
    ...
    pk = mulll(pk, p); if (hiremainder) break;
}
```

For most architectures, the functions mentioned below are really chunks of inlined assembler code, and the above ‘global’ variables are actually local register values.

`ulong addll(ulong x, ulong y)` adds `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong addllx(ulong x, ulong y)` adds `overflow` to the sum of the `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong subll(ulong x, ulong y)` subtracts `x` and `y`, returns the lower `BITS_IN_LONG` bits and put the carry (borrow) bit into `overflow`.

`ulong subllx(ulong x, ulong y)` subtracts `overflow` from the difference of `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry (borrow) bit into `overflow`.

`int bfffo(ulong x)` returns the number of leading zero bits in `x`. That is, the number of bit positions by which it would have to be shifted left until its leftmost bit first becomes equal to 1, which can be between 0 and `BITS_IN_LONG - 1` for nonzero `x`. When `x` is 0, the result is undefined.

`ulong mulll(ulong x, ulong y)` multiplies `x` by `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong addmul(ulong x, ulong y)` adds `hiremainder` to the product of `x` and `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong divll(ulong x, ulong y)` returns the quotient of  $(\text{hiremainder} * 2^{\text{BITS\_IN\_LONG}}) + x$  by  $y$  and stores the remainder into `hiremainder`. An error occurs if the quotient cannot be represented by an `ulong`, i.e. if initially  $\text{hiremainder} \geq y$ .

`long hammingl(ulong x)` returns the Hamming weight of  $x$ , i.e. the number of nonzero bits in its binary expansion.

**Obsolete routines.** Those functions are awkward and no longer used; they are only provided for backward compatibility:

`ulong shiftl(ulong x, ulong y)` returns  $x$  shifted left by  $y$  bits, i.e.  $x \ll y$ , where we assume that  $0 \leq y \leq \text{BITS\_IN\_LONG}$ . The global variable `hiremainder` receives the bits that were shifted out, i.e.  $x \gg (\text{BITS\_IN\_LONG} - y)$ .

`ulong shiftr(ulong x, ulong y)` returns  $x$  shifted right by  $y$  bits, i.e.  $x \gg y$ , where we assume that  $0 \leq y \leq \text{BITS\_IN\_LONG}$ . The global variable `hiremainder` receives the bits that were shifted out, i.e.  $x \ll (\text{BITS\_IN\_LONG} - y)$ .

**6.1.2 Modular kernel.** The following routines are not part of the level 0 kernel per se, but implement modular operations on words in terms of the above. They are written so that no overflow may occur. Let  $m \geq 1$  be the modulus; all operands representing classes modulo  $m$  are assumed to belong to  $[0, m - 1]$ . The result may be wrong for a number of reasons otherwise: it may not be reduced, overflow can occur, etc.

`int odd(ulong x)` returns 1 if  $x$  is odd, and 0 otherwise.

`int both_odd(ulong x, ulong y)` returns 1 if  $x$  and  $y$  are both odd, and 0 otherwise.

`ulong invmod2BIL(ulong x)` returns the smallest positive representative of  $x^{-1} \bmod 2^{\text{BITS\_IN\_LONG}}$ , assuming  $x$  is odd.

`ulong Fl_add(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of  $x + y$  modulo  $m$ .

`ulong Fl_neg(ulong x, ulong m)` returns the smallest nonnegative representative of  $-x$  modulo  $m$ .

`ulong Fl_sub(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of  $x - y$  modulo  $m$ .

`long Fl_center(ulong x, ulong m, ulong mo2)` returns the representative in  $] - m/2, m/2]$  of  $x$  modulo  $m$ . Assume  $0 \leq x < m$  and  $\text{mo2} = m \gg 1$ .

`ulong Fl_mul(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of  $xy$  modulo  $m$ .

`ulong Fl_double(ulong x, ulong m)` returns  $2x$  modulo  $m$ .

`ulong Fl_triple(ulong x, ulong m)` returns  $3x$  modulo  $m$ .

`ulong Fl_half(ulong x, ulong m)` returns  $z$  such that  $2z = x$  modulo  $m$  assuming such  $z$  exists.

`ulong Fl_sqr(ulong x, ulong m)` returns the smallest nonnegative representative of  $x^2$  modulo  $m$ .

`ulong Fl_inv(ulong x, ulong m)` returns the smallest positive representative of  $x^{-1}$  modulo  $m$ . If  $x$  is not invertible mod  $m$ , raise an exception.

`ulong Fl_invsafe(ulong x, ulong m)` returns the smallest positive representative of  $x^{-1}$  modulo  $m$ . If  $x$  is not invertible mod  $m$ , return 0 (which is ambiguous if  $m = 1$ ).

`ulong Fl_invgen(ulong x, ulong m, ulong *pg)` set `*pg` to  $g = \gcd(x, m)$  and return  $u$  in  $(\mathbf{Z}/m\mathbf{Z})^*$  such that  $xu = g$  modulo  $m$ . We have  $g = 1$  if and only if  $x$  is invertible, and in this case  $u$  is its inverse.

`ulong Fl_div(ulong x, ulong y, ulong m)` returns the smallest nonnegative representative of  $xy^{-1}$  modulo  $m$ . If  $y$  is not invertible mod  $m$ , raise an exception.

`ulong Fl_powu(ulong x, ulong n, ulong m)` returns the smallest nonnegative representative of  $x^n$  modulo  $m$ .

`GEN Fl_powers(ulong x, long n, ulong p)` returns  $[x^0, \dots, x^n]$  modulo  $m$ , as a `t_VECSMALL`.

`ulong Fl_sqrt(ulong x, ulong p)` returns the square root of  $x$  modulo  $p$  (smallest nonnegative representative). Assumes  $p$  to be prime, and  $x$  to be a square modulo  $p$ .

`ulong Fl_sqrtl(ulong x, ulong l, ulong p)` returns a  $l$ -th root of  $x$  modulo  $p$ . Assumes  $p$  to be prime and  $p \equiv 1 \pmod{l}$ , and  $x$  to be a  $l$ -th power modulo  $p$ .

`ulong Fl_sqrtn(ulong a, ulong n, ulong p, ulong *zn)` returns `ULONG_MAX` if  $a$  is not an  $n$ -th power residue mod  $p$ . Otherwise, returns an  $n$ -th root of  $a$ ; if `zn` is not `NULL` set it to a primitive  $m$ -th root of 1,  $m = \gcd(p-1, n)$  allowing to compute all  $m$  solutions in  $\mathbf{F}_p$  of the equation  $x^n = a$ .

`ulong Fl_log(ulong a, ulong g, ulong ord, ulong p)` Let  $g$  such that  $g^{\text{ord}} \equiv 1 \pmod{p}$ . Return an integer  $e$  such that  $a^e \equiv g \pmod{p}$ . If  $e$  does not exist, the result is undefined.

`ulong Fl_order(ulong a, ulong o, ulong p)` returns the order of the  $\mathbf{F}_p$   $a$ . It is assumed that  $o$  is a multiple of the order of  $a$ , 0 being allowed (no nontrivial information).

`ulong random_Fl(ulong p)` returns a pseudo-random integer uniformly distributed in  $0, 1, \dots, p-1$ .

`ulong nonsquare_Fl(ulong p)` return a quadratic nonresidue modulo  $p$ , assuming  $p$  is an odd prime. If  $p$  is 3 mod 4, return  $p-1$ , else return the smallest (prime) nonresidue.

`ulong pgener_Fl(ulong p)` returns the smallest primitive root modulo  $p$ , assuming  $p$  is prime.

`ulong pgener_Zl(ulong p)` returns the smallest primitive root modulo  $p^k$ ,  $k > 1$ , assuming  $p$  is an odd prime.

`ulong pgener_Fl_local(ulong p, GEN L)`, see `gener_Fp_local`,  $L$  is an `Flv`.

`ulong factorial_Fl(long n, ulong p)` return  $n! \bmod p$ .

### 6.1.3 Modular kernel with “precomputed inverse”.

This is based on an algorithm by T. Grandlund and N. Möller in “Improved division by invariant integers” <http://gmplib.org/~tege/division-paper.pdf>.

In the following, we set  $B = \text{BITS\_IN\_LONG}$ .

`ulong get_Fl_red(ulong p)` returns a pseudoinverse  $pi$  for  $p$ . Namely an integer  $0 < pi < B$  such that, given  $0 \leq x < B^2$  (by two long words), we can compute the Euclidean quotient and remainder of  $x$  modulo  $p$  by performing 2 multiplications and some additions. Precisely, once we set  $q = 2^k p$  for the unique  $k$  such that  $B/2 \leq q < B$ , the pseudoinverse  $pi$  is equal to the Euclidean quotient of  $B^2 - qB + B - 1$  by  $q$ . In particular  $(pi + B)/B^2$  is very close to  $1/q$ .

Note that this algorithm is generally less efficient than ordinary quotient and remainders (`divll` or even `/` and `%`) when  $0 \leq x < B$  and  $p \leq B^{1/2}$  are small. High level functions below allow setting  $pi = 0$  to cater for this possibility and avoid calling `get_Fl_red` for arguments where the standard algorithm is preferable.

`ulong divll_pre(ulong x, ulong p, ulong pi)` as `divll`, where  $pi$  is the pseudoinverse of  $p$ .

`ulong remll_pre(ulong u1, ulong u0, ulong p, ulong pi)` returns the Euclidean remainder of  $u_1 2^B + u_0$  modulo  $p$ , assuming  $pi$  is the pseudoinverse of  $p$ . This function is faster if  $u_1 < p$ .

`ulong remlll_pre(ulong u2, ulong u1, ulong u0, ulong p, ulong pi)` returns the Euclidean remainder of  $u_2 2^{2B} + u_1 2^B + u_0$  modulo  $p$ , assuming  $pi$  is the pseudoinverse of  $p$ .

`ulong Fl_sqr_pre(ulong x, ulong p, ulong pi)` returns  $x^2$  modulo  $p$ , assuming  $pi$  is the pseudoinverse of  $p$ .

`ulong Fl_mul_pre(ulong x, ulong y, ulong p, ulong pi)` returns  $xy$  modulo  $p$ , assuming  $pi$  is the pseudoinverse of  $p$ .

`ulong Fl_addmul_pre(ulong a, ulong b, ulong c, ulong p, ulong pi)` returns  $a + bc$  modulo  $p$ , assuming  $pi$  is the pseudoinverse of  $p$ .

`ulong Fl_addmulmul_pre(ulong a, ulong b, ulong c, ulong d, ulong p, ulong pi)` returns  $ab + cd$  modulo  $p$ , assuming  $pi$  is the pseudoinverse of  $p$ .

`ulong Fl_powu_pre(ulong x, ulong n, ulong p, ulong pi)` returns  $x^n$  modulo  $p$ , assuming  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we either use ordinary divisions if  $p < B^{1/2}$  is small and call `get_Fl_red` ourselves otherwise.

`GEN Fl_powers_pre(ulong x, long n, ulong p, ulong pi)` returns the vector  $(t\_VECSMALL)(x^0, \dots, x^n)$ , assuming  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we either use ordinary divisions if  $p < B^{1/2}$  is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_log_pre(ulong a, ulong g, ulong ord, ulong p, ulong pi)` as `Fl_log`, assuming  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we either use ordinary divisions if  $p < B^{1/2}$  is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrt_pre(ulong x, ulong p, ulong pi)` returns a square root of  $x$  modulo  $p$ , see `Fl_sqrt`. We assume  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we either use ordinary divisions if  $p < B^{1/2}$  is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrtl_pre(ulong x, ulong l, ulong p, ulong pi)` returns a  $l$ -th root of  $x$  modulo  $p$ , assuming  $p$  prime,  $p \equiv 1 \pmod{l}$ , and  $x$  to be a  $l$ -th power modulo  $p$ . We assume  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we either use ordinary divisions if  $p < B^{1/2}$  is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrtn_pre(ulong x, ulong n, ulong p, ulong pi, ulong *zn)` See `Fl_sqrtn`, assuming  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we either use ordinary divisions if  $p < B^{1/2}$  is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_2gener_pre(ulong p, ulong pi)` return a generator of the 2-Sylow subgroup of  $\mathbf{F}_p^*$ , to be used in `Fl_sqrt_pre_i`. We assume  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we either use ordinary divisions if  $p < B^{1/2}$  is small and call `get_Fl_red` ourselves otherwise.

`ulong Fl_sqrt_pre_i(ulong x, ulong s2, ulong p, ulong pi)` as `Fl_sqrt_pre` where  $s2$  is the element returned by `Fl_2gener_pre`. We assume  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we either use ordinary divisions if  $p < B^{1/2}$  is small and call `get_Fl_red` ourselves otherwise.

#### 6.1.4 Switching between Fl\_xxx and standard operators.

Even though the Fl\_xxx routines are efficient, they are slower than ordinary long operations, using the standard +, %, etc. operators. The following macro is used to choose in a portable way the most efficient functions for given operands:

int SMALL\_ULONG(ulong p) true if  $2p^2 < 2^{\text{BITS\_IN\_LONG}}$ . In that case, it is possible to use ordinary operators efficiently. If  $p < 2^{\text{BITS\_IN\_LONG}}$ , one may still use the Fl\_xxx routines. Otherwise, one must use generic routines. For instance, the scalar product of the GENs  $x$  and  $y$  mod  $p$  could be computed as follows.

```
long i, l = lg(x);
if (lgefint(p) > 3)
{ /* arbitrary */
    GEN s = gen_0;
    for (i = 1; i < l; i++) s = addii(s, mulii(gel(x,i), gel(y,i)));
    return modii(s, p).
}
else
{
    ulong s = 0, pp =itou(p);
    x = ZV_to_Flv(x, pp);
    y = ZV_to_Flv(y, pp);
    if (SMALL_ULONG(pp))
    { /* very small */
        for (i = 1; i < l; i++)
        {
            s += x[i] * y[i];
            if (s & HIGHBIT) s %= pp;
        }
        s %= pp;
    }
    else
    { /* small */
        for (i = 1; i < l; i++)
            s = Fl_add(s, Fl_mul(x[i], y[i], pp), pp);
    }
    return utoi(s);
}
```

In effect, we have three versions of the same code: very small, small, and arbitrary inputs. The very small and arbitrary variants use lazy reduction and reduce only when it becomes necessary: when overflow might occur (very small), and at the very end (very small, arbitrary).

## 6.2 Level 1 kernel (operations on longs, integers and reals).

**Note.** Some functions consist of an elementary operation, immediately followed by an assignment statement. They will be introduced as in the following example:

`GEN gadd[z](GEN x, GEN y[, GEN z])` followed by the explicit description of the function

`GEN gadd(GEN x, GEN y)`

which creates its result on the stack, returning a GEN pointer to it, and the parts in brackets indicate that there exists also a function

`void gaddz(GEN x, GEN y, GEN z)`

which assigns its result to the pre-existing object `z`, leaving the stack unchanged. These assignment variants are kept for backward compatibility but are inefficient: don't use them.

### 6.2.1 Creation.

`GEN cgeti(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its first codeword. Identical to `cgetg(n, t_INT)`.

`GEN cgetipos(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to 1.

`GEN cgetineg(long n)` allocates memory on the PARI stack for a negative `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to  $-1$ .

`GEN cgetr(long n)` allocates memory on the PARI stack for a `t_REAL` of length `n`, and initializes its first codeword. Identical to `cgetg(n, t_REAL)`.

`GEN cgetc(long n)` allocates memory on the PARI stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REALs` of length `n`.

`GEN real_1(long prec)` create a `t_REAL` equal to 1 to `prec` words of accuracy.

`GEN real_1_bit(long bitprec)` create a `t_REAL` equal to 1 to `bitprec` bits of accuracy.

`GEN real_m1(long prec)` create a `t_REAL` equal to  $-1$  to `prec` words of accuracy.

`GEN real_0_bit(long bit)` create a `t_REAL` equal to 0 with exponent  $-\text{bit}$ .

`GEN real_0(long prec)` is a shorthand for

`real_0_bit( -prec2nbits(prec) )`

`GEN int2n(long n)` creates a `t_INT` equal to  $1 \ll n$  (i.e  $2^n$  if  $n \geq 0$ , and 0 otherwise).

`GEN int2u(ulong n)` creates a `t_INT` equal to  $2^n$ .

`GEN int2um1(long n)` creates a `t_INT` equal to  $2^n - 1$ .

`GEN real2n(long n, long prec)` create a `t_REAL` equal to  $2^n$  to `prec` words of accuracy.

`GEN real_m2n(long n, long prec)` create a `t_REAL` equal to  $-2^n$  to `prec` words of accuracy.

`GEN strtol(char *s)` convert the character string `s` to a nonnegative `t_INT`. Decimal numbers, hexadecimal numbers prefixed by `0x` and binary numbers prefixed by `0b` are allowed. The string `s` consists exclusively of digits: no leading sign, no whitespace. Leading zeroes are discarded.

`GEN strtod(char *s, long prec)` convert the character string `s` to a nonnegative `t_REAL` of precision `prec`. The string `s` consists exclusively of digits and optional decimal point and exponent (`e` or `E`): no leading sign, no whitespace. Leading zeroes are discarded.



**6.2.2 Assignment.** In this section, the  $z$  argument in the  $z$ -functions must be of type  $t\_INT$  or  $t\_REAL$ .

`void mpaff(GEN x, GEN z)` assigns  $x$  into  $z$  (where  $x$  and  $z$  are  $t\_INT$  or  $t\_REAL$ ). Assumes that  $lg(z) > 2$ .

`void affii(GEN x, GEN z)` assigns the  $t\_INT$   $x$  into the  $t\_INT$   $z$ .

`void affir(GEN x, GEN z)` assigns the  $t\_INT$   $x$  into the  $t\_REAL$   $z$ . Assumes that  $lg(z) > 2$ .

`void affiz(GEN x, GEN z)` assigns  $t\_INT$   $x$  into  $t\_INT$  or  $t\_REAL$   $z$ . Assumes that  $lg(z) > 2$ .

`void affsi(long s, GEN z)` assigns the `long`  $s$  into the  $t\_INT$   $z$ . Assumes that  $lg(z) > 2$ .

`void affsr(long s, GEN z)` assigns the `long`  $s$  into the  $t\_REAL$   $z$ . Assumes that  $lg(z) > 2$ .

`void affsz(long s, GEN z)` assigns the `long`  $s$  into the  $t\_INT$  or  $t\_REAL$   $z$ . Assumes that  $lg(z) > 2$ .

`void affui(ulong u, GEN z)` assigns the `ulong`  $u$  into the  $t\_INT$   $z$ . Assumes that  $lg(z) > 2$ .

`void affur(ulong u, GEN z)` assigns the `ulong`  $u$  into the  $t\_REAL$   $z$ . Assumes that  $lg(z) > 2$ .

`void affrr(GEN x, GEN z)` assigns the  $t\_REAL$   $x$  into the  $t\_REAL$   $z$ .

`void affgr(GEN x, GEN z)` assigns the scalar  $x$  into the  $t\_REAL$   $z$ , if possible.

The function `affrs` and `affri` do not exist. So don't use them.

`void affrr_fixlg(GEN y, GEN z)` a variant of `affrr`. First shorten  $z$  so that it is no longer than  $y$ , then assigns  $y$  to  $z$ . This is used in the following scenario: room is reserved for the result but, due to cancellation, fewer words of accuracy are available than had been anticipated; instead of appending meaningless 0s to the mantissa, we store what was actually computed.

Note that shortening  $z$  is not quite straightforward, since `setlg(z, ly)` would leave garbage on the stack, which `gerepile` might later inspect. It is done using

`void fixlg(GEN z, long ly)` see `stackdummy` and the examples that follow.

### 6.2.3 Copy.

`GEN icopy(GEN x)` copy relevant words of the  $t\_INT$   $x$  on the stack: the length and effective length of the copy are equal.

`GEN rcopy(GEN x)` copy the  $t\_REAL$   $x$  on the stack.

`GEN leafcopy(GEN x)` copy the leaf  $x$  on the stack (works in particular for  $t\_INT$ s and  $t\_REAL$ s). Contrary to `icopy`, `leafcopy` preserves the original length of a  $t\_INT$ . The obsolete form `GEN mpcopy(GEN x)` is still provided for backward compatibility.

This function also works on recursive types, copying them as if they were leaves, i.e. making a shallow copy in that case: the components of the copy point to the same data as the component of the source; see also `shallowcopy`.

`GEN leafcopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume  $x$  is a leaf and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

`GEN icopyspec(GEN x, long nx)` copy the `nx` words  $x[2], \dots, x[nx+1]$  to make up a new  $t\_INT$ . Set the sign to 1.

### 6.2.4 Conversions.

GEN `itor(GEN x, long prec)` converts the `t_INT` `x` to a `t_REAL` of length `prec` and return the latter. Assumes that `prec > 2`.

`long itos(GEN x)` converts the `t_INT` `x` to a `long` if possible, otherwise raise an exception. We consider the conversion to be possible if and only if  $|x| \leq \text{LONG\_MAX}$ , i.e.  $|x| < 2^{63}$  on a 64-bit architecture. Since the range is symmetric, the output of `itos` can safely be negated.

`long itos_or_0(GEN x)` converts the `t_INT` `x` to a `long` if possible, otherwise return 0.

`int is_bigint(GEN n)` true if `itos(n)` would give an error.

`ulong itou(GEN x)` converts the `t_INT`  $|x|$  to an `ulong` if possible, otherwise raise an exception. The conversion is possible if and only if `lgefint(x) ≤ 3`.

`long itou_or_0(GEN x)` converts the `t_INT`  $|x|$  to an `ulong` if possible, otherwise return 0.

GEN `stoi(long s)` creates the `t_INT` corresponding to the `long s`.

GEN `stor(long s, long prec)` converts the `long s` into a `t_REAL` of length `prec` and return the latter. Assumes that `prec > 2`.

GEN `utoi(ulong s)` converts the `ulong s` into a `t_INT` and return the latter.

GEN `utoipos(ulong s)` converts the *nonzero* `ulong s` into a `t_INT` and return the latter.

GEN `utoineg(ulong s)` converts the *nonzero* `ulong s` into the `t_INT`  $-s$  and return the latter.

GEN `utor(ulong s, long prec)` converts the `ulong s` into a `t_REAL` of length `prec` and return the latter. Assumes that `prec > 2`.

GEN `rtor(GEN x, long prec)` converts the `t_REAL` `x` to a `t_REAL` of length `prec` and return the latter. If `prec < lg(x)`, round properly. If `prec > lg(x)`, pad with zeroes. Assumes that `prec > 2`.

The following function is also available as a special case of `mkintn`:

GEN `uu32toi(ulong a, ulong b)` returns the GEN equal to  $2^{32}a + b$ , *assuming* that  $a, b < 2^{32}$ . This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

GEN `uu32toineg(ulong a, ulong b)` returns the GEN equal to  $-(2^{32}a + b)$ , *assuming* that  $a, b < 2^{32}$  and that one of  $a$  or  $b$  is positive. This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

GEN `uutoi(ulong a, ulong b)` returns the GEN equal to  $2^{\text{BITS\_IN\_LONG}}a + b$ .

GEN `uutoineg(ulong a, ulong b)` returns the GEN equal to  $-(2^{\text{BITS\_IN\_LONG}}a + b)$ .

**6.2.5 Integer parts.** The following four functions implement the conversion from `t_REAL` to `t_INT` using standard rounding modes. Contrary to usual semantics (complement the mantissa with an infinite number of 0), they will raise an error *precision loss in truncation* if the `t_REAL` represents a range containing more than one integer.

`GEN ceilr(GEN x)` smallest integer larger or equal to the `t_REAL`  $x$  (i.e. the `ceil` function).

`GEN floorr(GEN x)` largest integer smaller or equal to the `t_REAL`  $x$  (i.e. the `floor` function).

`GEN roundr(GEN x)` rounds the `t_REAL`  $x$  to the nearest integer (towards  $+\infty$  in case of tie).

`GEN truncr(GEN x)` truncates the `t_REAL`  $x$  (not the same as `floorr` if  $x$  is negative).

The following four function are analogous, but can also treat the trivial case when the argument is a `t_INT`:

`GEN mpceil(GEN x)` as `ceilr` except that  $x$  may be a `t_INT`.

`GEN mpfloor(GEN x)` as `floorr` except that  $x$  may be a `t_INT`.

`GEN mpround(GEN x)` as `roundr` except that  $x$  may be a `t_INT`.

`GEN mptrunc(GEN x)` as `truncr` except that  $x$  may be a `t_INT`.

`GEN diviiround(GEN x, GEN y)` if  $x$  and  $y$  are `t_INT`s, returns the quotient  $x/y$  of  $x$  and  $y$ , rounded to the nearest integer. If  $x/y$  falls exactly halfway between two consecutive integers, then it is rounded towards  $+\infty$  (as for `roundr`).

`GEN ceil_safe(GEN x)`,  $x$  being a real number (not necessarily a `t_REAL`) returns the smallest integer which is larger than any possible incarnation of  $x$ . (Recall that a `t_REAL` represents an interval of possible values.) Note that `gceil` raises an exception if the input accuracy is too low compared to its magnitude.

`GEN floor_safe(GEN x)`,  $x$  being a real number (not necessarily a `t_REAL`) returns the largest integer which is smaller than any possible incarnation of  $x$ . (Recall that a `t_REAL` represents an interval of possible values.) Note that `gfloor` raises an exception if the input accuracy is too low compared to its magnitude.

`GEN trunc_safe(GEN x)`,  $x$  being a real number (not necessarily a `t_REAL`) returns the integer with the largest absolute value, which is closer to 0 than any possible incarnation of  $x$ . (Recall that a `t_REAL` represents an interval of possible values.)

`GEN roundr_safe(GEN x)` rounds the `t_REAL`  $x$  to the nearest integer (towards  $+\infty$ ). Complement the mantissa with an infinite number of 0 before rounding, hence never raise an exception.

## 6.2.6 2-adic valuations and shifts.

`long vals(long s)` 2-adic valuation of the `long`  $s$ . Returns  $-1$  if  $s$  is equal to 0.

`long vali(GEN x)` 2-adic valuation of the `t_INT`  $x$ . Returns  $-1$  if  $x$  is equal to 0.

`GEN mpshift(GEN x, long n)` shifts the `t_INT` or `t_REAL`  $x$  by  $n$ . If  $n$  is positive, this is a left shift, i.e. multiplication by  $2^n$ . If  $n$  is negative, it is a right shift by  $-n$ , which amounts to the truncation of the quotient of  $x$  by  $2^{-n}$ .

`GEN shifti(GEN x, long n)` shifts the `t_INT`  $x$  by  $n$ .

`GEN shiftr(GEN x, long n)` shifts the `t_REAL`  $x$  by  $n$ .

`void shiftr_inplace(GEN x, long n)` shifts the `t_REAL`  $x$  by  $n$ , in place.

`GEN trunc2nr(GEN x, long n)` given a `t_REAL`  $x$ , returns `truncr(shiftr(x,n))`, but faster, without leaving garbage on the stack and never raising a *precision loss in truncation* error. Called by `gtrunc2n`.

`GEN mantissa2nr(GEN x, long n)` given a `t_REAL`  $x$ , returns the mantissa of  $x2^n$  (disregards the exponent of  $x$ ). Equivalent to

`trunc2nr(x, n-expo(x)+bit_prec(x)-1)`

`GEN mantissa_real(GEN z, long *e)` returns the mantissa  $m$  of  $z$ , and sets `*e` to the exponent `bit_accuracy(lg(z)) - 1 - expo(z)`, so that  $z = m/2^e$ .

**Low-level.** In the following two functions,  $s$ (ource) and  $t$ (arget) need not be valid GENs (in practice, they usually point to some part of a `t_REAL` mantissa): they are considered as arrays of words representing some mantissa, and we shift globally  $s$  by  $n > 0$  bits, storing the result in  $t$ . We assume that  $m \leq M$  and only access  $s[m], s[m+1], \dots s[M]$  (read) and likewise for  $t$  (write); we may have  $s = t$  but more general overlaps are not allowed. The word  $f$  is concatenated to  $s$  to supply extra bits.

`void shift_left(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

$s[m], s[m+1], \dots s[M], f$

left by  $n$  bits.

`void shift_right(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

$f, s[m], s[m+1], \dots s[M]$

right by  $n$  bits.

### 6.2.7 From `t_INT` to bits or digits in base $2^k$ and back.

`GEN binary_zv(GEN x)` given a `t_INT`  $x$ , return a `t_VECSMALL` of bits, from most significant to least significant.

`GEN binary_2k(GEN x, long k)` given a `t_INT`  $x$ , and  $k > 0$ , return a `t_VEC` of digits of  $x$  in base  $2^k$ , as `t_INT`s, from most significant to least significant.

`GEN binary_2k_nv(GEN x, long k)` given a `t_INT`  $x$ , and  $0 < k < \text{BITS\_IN\_LONG}$ , return a `t_VECSMALL` of digits of  $x$  in base  $2^k$ , as `ulong`s, from most significant to least significant.

`GEN bits_to_int(GEN x, long l)` given a vector  $x$  of  $l$  bits (as a `t_VECSMALL` or even a pointer to a part of a larger vector, so not a proper GEN), return the integer  $\sum_{i=1}^l x[i]2^{l-i}$ , as a `t_INT`.

`ulong bits_to_u(GEN v, long l)` same as `bits_to_int`, where  $l < \text{BITS\_IN\_LONG}$ , so we can return an `ulong`.

`GEN fromdigitsu(GEN x, GEN B)` given a `t_VECSMALL`  $x$  of length  $l$  and a `t_INT`  $B$ , return the integer  $\sum_{i=1}^l x[i]B^{i-1}$ , as a `t_INT`, where the  $x[i]$  are seen as unsigned integers.

`GEN fromdigits_2k(GEN x, long k)` converse of `binary_2k`; given a `t_VEC`  $x$  of length  $l$  and a positive `long`  $k$ , where each  $x[i]$  is a `t_INT` with  $0 \leq x[i] < 2^k$ , return the integer  $\sum_{i=1}^l x[i]2^{k(l-i)}$ , as a `t_INT`.

`GEN nv_fromdigits_2k(GEN x, long k)` as `fromdigits_2k`, but with  $x$  being a `t_VECSMALL` and each  $x[i]$  being a `ulong` with  $0 \leq x[i] < 2^{\min\{k, \text{BITS\_IN\_LONG}\}}$ . Here  $k$  may be any positive `long`, and the  $x[i]$  are regarded as  $k$ -bit integers by truncating or extending with zeroes.

**6.2.8 Integer valuation.** For integers  $x$  and  $p$ , such that  $x \neq 0$  and  $|p| > 1$ , we define  $v_p(x)$  to be the largest integer exponent  $e$  such that  $p^e$  divides  $x$ . If  $p$  is prime, this is the ordinary valuation of  $x$  at  $p$ .

`long Z_pvalrem(GEN x, GEN p, GEN *r)` applied to `t_INTs`  $x \neq 0$  and  $p$ ,  $|p| > 1$ , returns  $e := v_p(x)$ . The quotient  $x/p^e$  is returned in `*r`. If  $|p|$  is a prime, `*r` is the prime-to- $p$  part of  $x$ .

`long Z_pval(GEN x, GEN p)` as `Z_pvalrem` but only returns  $v_p(x)$ .

`long Z_lvalrem(GEN x, ulong p, GEN *r)` as `Z_pvalrem`, except that  $p$  is an `ulong` ( $p > 1$ ).

`long Z_lvalrem_stop(GEN *x, ulong p, int *stop)` assume  $x > 0$ ; returns  $e := v_p(x)$  and replaces  $x$  by  $x/p^e$ . Set `stop` to 1 if the new value of  $x$  is  $< p^2$  (and 0 otherwise). To be used when trial dividing  $x$  by successive primes: the `stop` condition is cheaply tested while testing whether  $p$  divides  $x$  (is the quotient less than  $p$ ?), and allows to decide that  $n$  is prime if no prime  $< p$  divides  $n$ . Not memory-clean.

`long Z_lval(GEN x, ulong p)` as `Z_pval`, except that  $p$  is an `ulong` ( $p > 1$ ).

`long u_lvalrem(ulong x, ulong p, ulong *r)` as `Z_pvalrem`, except the inputs/outputs are now `ulongs`.

`long u_lvalrem_stop(ulong *n, ulong p, int *stop)` as `Z_pvalrem_stop`.

`long u_pvalrem(ulong x, GEN p, ulong *r)` as `Z_pvalrem`, except  $x$  and  $r$  are now `ulongs`.

`long u_lval(ulong x, ulong p)` as `Z_pval`, except the inputs are now `ulongs`.

`long u_pval(ulong x, GEN p)` as `Z_pval`, except  $x$  is now an `ulong`.

`long z_lval(long x, ulong p)` as `u_lval`, for signed  $x$ .

`long z_lvalrem(long x, ulong p)` as `u_lvalrem`, for signed  $x$ .

`long z_pval(long x, GEN p)` as `Z_pval`, except  $x$  is now a `long`.

`long z_pvalrem(long x, GEN p)` as `Z_pvalrem`, except  $x$  is now a `long`.

`long factorial_lval(ulong n, ulong p)` returns  $v_p(n!)$ , assuming  $p$  is prime.

The following convenience functions generalize `Z_pval` and its variants to “containers” (`ZV` and `ZX`):

`long ZV_pvalrem(GEN x, GEN p, GEN *r)`  $x$  being a `ZV` (a vector of `t_INTs`), return the min  $v$  of the valuations of its components and set `*r` to  $x/p^v$ . Infinite loop if  $x$  is the zero vector. This function is not stack clean.

`long ZV_pval(GEN x, GEN p)` as `ZV_pvalrem` but only returns the “valuation”.

`int ZV_Z_dvd(GEN x, GEN p)` returns 1 if  $p$  divides all components of  $x$  and 0 otherwise. Faster than testing `ZV_pval(x,p) >= 1`.

`long ZV_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_pvalrem`, except that  $p$  is an `ulong` ( $p > 1$ ). This function is not stack-clean.

`long ZV_lval(GEN x, ulong p)` as `ZV_pval`, except that  $p$  is an `ulong` ( $p > 1$ ).

`long ZX_pvalrem(GEN x, GEN p, GEN *r)` as `ZV_pvalrem`, for a `ZX`  $x$  (a `t_POL` with `t_INT` coefficients). This function is not stack-clean.

`long ZX_pval(GEN x, GEN p)` as `ZV_pval` for a `ZX`  $x$ .

long ZX\_lvalrem(GEN x, ulong p, GEN \*px) as ZV\_lvalrem, a ZX  $x$ . This function is not stack-clean.

long ZX\_lval(GEN x, ulong p) as ZX\_pval, except that  $p$  is an ulong ( $p > 1$ ).

**6.2.9 Generic unary operators.** Let “ $op$ ” be a unary operation among

- **neg**: negation ( $-x$ ).
- **abs**: absolute value ( $|x|$ ).
- **sqr**: square ( $x^2$ ).

The names and prototypes of the low-level functions corresponding to  $op$  are as follows. The result is of the same type as  $x$ .

GEN *opi*(GEN  $x$ ) creates the result of  $op$  applied to the  $t\_INT$   $x$ .

GEN *opr*(GEN  $x$ ) creates the result of  $op$  applied to the  $t\_REAL$   $x$ .

GEN *mpop*(GEN  $x$ ) creates the result of  $op$  applied to the  $t\_INT$  or  $t\_REAL$   $x$ .

Complete list of available functions:

GEN *absi*(GEN  $x$ ), GEN *absr*(GEN  $x$ ), GEN *mpabs*(GEN  $x$ )

GEN *negi*(GEN  $x$ ), GEN *negr*(GEN  $x$ ), GEN *mpneg*(GEN  $x$ )

GEN *sqri*(GEN  $x$ ), GEN *sqrr*(GEN  $x$ ), GEN *mpsqr*(GEN  $x$ )

GEN *absi\_shallow*(GEN  $x$ )  $x$  being a  $t\_INT$ , returns a shallow copy of  $|x|$ , in particular returns  $x$  itself when  $x \geq 0$ , and *negi*( $x$ ) otherwise.

GEN *mpabs\_shallow*(GEN  $x$ )  $x$  being a  $t\_INT$  or a  $t\_REAL$ , returns a shallow copy of  $|x|$ , in particular returns  $x$  itself when  $x \geq 0$ , and *mpneg*( $x$ ) otherwise.

Some miscellaneous routines:

GEN *sqrs*(long  $x$ ) returns  $x^2$ .

GEN *squu*(ulong  $x$ ) returns  $x^2$ .

**6.2.10 Comparison operators.**

int *cmpss*(long  $s$ , long  $t$ ) compares the long  $s$  to the  $t\_long$   $t$ .

int *cmpuu*(ulong  $u$ , ulong  $v$ ) compares the ulong  $u$  to the  $t\_ulong$   $v$ .

long *minss*(long  $x$ , long  $y$ )

ulong *minuu*(ulong  $x$ , ulong  $y$ )

double *mindd*(double  $x$ , double  $y$ ) returns the min of  $x$  and  $y$ .

long *maxss*(long  $x$ , long  $y$ )

ulong *maxuu*(ulong  $x$ , ulong  $y$ )

double *maxdd*(double  $x$ , double  $y$ ) returns the max of  $x$  and  $y$ .

int *mpcmp*(GEN  $x$ , GEN  $y$ ) compares the  $t\_INT$  or  $t\_REAL$   $x$  to the  $t\_INT$  or  $t\_REAL$   $y$ . The result is the sign of  $x - y$ .

`int cmpii(GEN x, GEN y)` compares the `t_INT` `x` to the `t_INT` `y`.  
`int cmpir(GEN x, GEN y)` compares the `t_INT` `x` to the `t_REAL` `y`.  
`int cmpis(GEN x, long s)` compares the `t_INT` `x` to the `long` `s`.  
`int cmpiu(GEN x, ulong s)` compares the `t_INT` `x` to the `ulong` `s`.  
`int cmpsi(long s, GEN x)` compares the `long` `s` to the `t_INT` `x`.  
`int cmpui(ulong s, GEN x)` compares the `ulong` `s` to the `t_INT` `x`.  
`int cmpsr(long s, GEN x)` compares the `long` `s` to the `t_REAL` `x`.  
`int cmpri(GEN x, GEN y)` compares the `t_REAL` `x` to the `t_INT` `y`.  
`int cmpr(GEN x, GEN y)` compares the `t_REAL` `x` to the `t_REAL` `y`.  
`int cmprs(GEN x, long s)` compares the `t_REAL` `x` to the `long` `s`.  
`int equalii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is 1 if  $x = y$ , 0 otherwise.  
`int equalrr(GEN x, GEN y)` compares the `t_REAL`s `x` and `y`. The result is 1 if  $x = y$ , 0 otherwise. Equality is decided according to the following rules: all real zeroes are equal, and different from a nonzero real; two nonzero reals are equal if all their digits coincide up to the length of the shortest of the two, and the remaining words in the mantissa of the longest are all 0.  
`int equalis(GEN x, long s)` compare the `t_INT` `x` and the `long` `s`. The result is 1 if  $x = y$ , 0 otherwise.  
`int equalsi(long s, GEN x)`  
`int equaliu(GEN x, ulong s)` compare the `t_INT` `x` and the `ulong` `s`. The result is 1 if  $x = y$ , 0 otherwise.  
`int equalui(ulong s, GEN x)`

The remaining comparison operators disregard the sign of their operands

`int absequaliu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `s`. The result is 1 if  $|x| = y$ , 0 otherwise. This is marginally more efficient than `equalis` even when `x` is known to be nonnegative.  
`int absequalui(ulong u, GEN x)`  
`int absmpiu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `u`.  
`int absmpui(ulong u, GEN x)`  
`int absmpii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is the sign of  $|x| - |y|$ .  
`int absequalii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is 1 if  $|x| = |y|$ , 0 otherwise.  
`int absmprr(GEN x, GEN y)` compares the `t_REAL`s `x` and `y`. The result is the sign of  $|x| - |y|$ .  
`int absrnz_equal2n(GEN x)` tests whether a nonzero `t_REAL` `x` is equal to  $\pm 2^e$  for some integer `e`.  
`int absrnz_equal1(GEN x)` tests whether a nonzero `t_REAL` `x` is equal to  $\pm 1$ .

**6.2.11 Generic binary operators.** The operators in this section have arguments of C-type GEN, long, and ulong, and only t\_INT and t\_REAL GENs are allowed. We say an argument is a real type if it is a t\_REAL GEN, and an integer type otherwise. The result is always a t\_REAL unless both x and y are integer types.

Let “*op*” be a binary operation among

- **add:** addition ( $x + y$ ).
- **sub:** subtraction ( $x - y$ ).
- **mul:** multiplication ( $x * y$ ).

• **div:** division ( $x / y$ ). In the case where x and y are both integer types, the result is the Euclidean quotient, where the remainder has the same sign as the dividend x. It is the ordinary division otherwise. A division-by-0 error occurs if y is equal to 0.

The last two generic operations are defined only when arguments have integer types; and the result is a t\_INT:

• **rem:** remainder (“ $x \% y$ ”). The result is the Euclidean remainder corresponding to div, i.e. its sign is that of the dividend x.

• **mod:** true remainder ( $x \% y$ ). The result is the true Euclidean remainder, i.e. nonnegative and less than the absolute value of y.

**Important technical note.** The rules given above fixing the output type (to t\_REAL unless both inputs are integer types) are subtly incompatible with the general rules obeyed by PARI’s generic functions, such as gmul or gdiv for instance: the latter return a result containing as much information as could be deduced from the inputs, so it is not true that if  $x$  is a t\_INT and  $y$  a t\_REAL, then gmul( $x, y$ ) is always the same as mulir( $x, y$ ). The exception is  $x = 0$ , in that case we can deduce that the result is an exact 0, so gmul returns gen\_0, while mulir returns a t\_REAL 0. Specifically, the one resulting from the conversion of gen\_0 to a t\_REAL of precision precision( $y$ ), multiplied by  $y$ ; this determines the exponent of the real 0 we obtain.

The reason for the discrepancy between the two rules is that we use the two sets of functions in different contexts: generic functions allow to write high-level code forgetting about types, letting PARI return results which are sensible and as simple as possible; type specific functions are used in kernel programming, where we do care about types and need to maintain strict consistency: it is much easier to compute the types of results when they are determined from the types of the inputs only (without taking into account further arithmetic properties, like being nonzero).

The names and prototypes of the low-level functions corresponding to *op* are as follows. In this section, the z argument in the z-functions must be of type t\_INT when no r or mp appears in the argument code (no t\_REAL operand is involved, only integer types), and of type t\_REAL otherwise.

GEN mpop[z](GEN x, GEN y[, GEN z]) applies *op* to the t\_INT or t\_REAL x and y. The function mpdivz does not exist (its semantic would change drastically depending on the type of the z argument), and neither do mprem[z] nor mpmod[z] (specific to integers).

GEN opsi[z](long s, GEN x[, GEN z]) applies *op* to the long s and the t\_INT x. These functions always return the global constant gen\_0 (not a copy) when the sign of the result is 0.

GEN opsr[z](long s, GEN x[, GEN z]) applies *op* to the long s and the t\_REAL x.

GEN opss[z](long s, long t[, GEN z]) applies *op* to the longs s and t. These functions always return the global constant gen\_0 (not a copy) when the sign of the result is 0.



GEN *opii*[z](GEN x, GEN y[, GEN z]) applies *op* to the t\_INTs x and y. These functions always return the global constant *gen\_0* (not a copy) when the sign of the result is 0.

GEN *opir*[z](GEN x, GEN y[, GEN z]) applies *op* to the t\_INT x and the t\_REAL y.

GEN *opis*[z](GEN x, long s[, GEN z]) applies *op* to the t\_INT x and the long s. These functions always return the global constant *gen\_0* (not a copy) when the sign of the result is 0.

GEN *opri*[z](GEN x, GEN y[, GEN z]) applies *op* to the t\_REAL x and the t\_INT y.

GEN *oprr*[z](GEN x, GEN y[, GEN z]) applies *op* to the t\_REALs x and y.

GEN *oprs*[z](GEN x, long s[, GEN z]) applies *op* to the t\_REAL x and the long s.

Some miscellaneous routines:

long *expu*(ulong x) assuming  $x > 0$ , returns the binary exponent of the real number equal to *x*. This is a special case of *gexpo*.

GEN *adduu*(ulong x, ulong y)

GEN *addiu*(GEN x, ulong y)

GEN *addui*(ulong x, GEN y) adds x and y.

GEN *subuu*(ulong x, ulong y)

GEN *subiu*(GEN x, ulong y)

GEN *subui*(ulong x, GEN y) subtracts x by y.

GEN *muluu*(ulong x, ulong y) multiplies x by y.

ulong *umuluu\_le*(ulong x, ulong y, ulong n) multiplies x by y. Return *xy* if  $xy \leq n$  and 0 otherwise (in particular if *xy* does not fit in an ulong).

ulong *umuluu\_or\_0*(ulong x, ulong y) multiplies x by y. Return 0 if *xy* does not fit in an ulong.

GEN *mului*(ulong x, GEN y) multiplies x by y.

GEN *muluui*(ulong x, ulong y, GEN z) return *xyz*.

GEN *muliu*(GEN x, ulong y) multiplies x by y.

void *addumului*(ulong a, ulong b, GEN x) return  $a + b|X|$ .

GEN *addmuliu*(GEN x, GEN y, ulong u) returns  $x + yu$ .

GEN *addmulii*(GEN x, GEN y, GEN z) returns  $x + yz$ .

GEN *addmulii\_inplace*(GEN x, GEN y, GEN z) returns  $x + yz$ , but returns *x* itself and not a copy if  $yz = 0$ . Not suitable for *gerepile* or *gerepileupto*.

GEN *addmuliu\_inplace*(GEN x, GEN y, ulong u) returns  $x + yu$ , but returns *x* itself and not a copy if  $yu = 0$ . Not suitable for *gerepile* or *gerepileupto*.

GEN *submuliu\_inplace*(GEN x, GEN y, ulong u) returns  $x - yu$ , but returns *x* itself and not a copy if  $yu = 0$ . Not suitable for *gerepile* or *gerepileupto*.

GEN *lincombii*(GEN u, GEN v, GEN x, GEN y) returns  $ux + vy$ .

GEN *mulsubii*(GEN y, GEN z, GEN x) returns  $yz - x$ .

GEN submulii(GEN x, GEN y, GEN z) returns  $x - yz$ .

GEN submuliu(GEN x, GEN y, ulong u) returns  $x - yu$ .

GEN mulu\_interval(ulong a, ulong b) returns  $a(a+1) \cdots b$ , assuming that  $a \leq b$ .

GEN mulu\_interval\_step(ulong a, ulong b, ulong s) returns the product of all integers in  $[a, b]$  congruent to  $a$  modulo  $s$ . Assume  $a \leq b$  and  $s > 0$ ;

GEN muls\_interval(long a, long b) returns  $a(a+1) \cdots b$ , assuming that  $a \leq b$ .

GEN invr(GEN x) returns the inverse of the nonzero  $\mathbf{t\_REAL}$   $x$ .

GEN truedivii(GEN x, GEN y) returns the true Euclidean quotient (with nonnegative remainder less than  $|y|$ ).

GEN truedivis(GEN x, long y) returns the true Euclidean quotient (with nonnegative remainder less than  $|y|$ ).

GEN truedivsi(long x, GEN y) returns the true Euclidean quotient (with nonnegative remainder less than  $|y|$ ).

GEN centermodii(GEN x, GEN y, GEN y2), given  $\mathbf{t\_INTs}$   $x, y$ , returns  $z$  congruent to  $x$  modulo  $y$ , such that  $-y/2 \leq z < y/2$ . The function requires an extra argument  $y2$ , such that  $y2 = \mathbf{shifti}(y, -1)$ . (In most cases,  $y$  is constant for many reductions and  $y2$  need only be computed once.)

GEN remi2n(GEN x, long n) returns  $x \bmod 2^n$ .

GEN addii\_sign(GEN x, long sx, GEN y, long sy) add the  $\mathbf{t\_INTs}$   $x$  and  $y$  as if their signs were  $sx$  and  $sy$ .

GEN addir\_sign(GEN x, long sx, GEN y, long sy) add the  $\mathbf{t\_INT}$   $x$  and the  $\mathbf{t\_REAL}$   $y$  as if their signs were  $sx$  and  $sy$ .

GEN addrr\_sign(GEN x, long sx, GEN y, long sy) add the  $\mathbf{t\_REALs}$   $x$  and  $y$  as if their signs were  $sx$  and  $sy$ .

GEN addsi\_sign(long x, GEN y, long sy) add  $x$  and the  $\mathbf{t\_INT}$   $y$  as if its sign was  $sy$ .

GEN addui\_sign(ulong x, GEN y, long sy) add  $x$  and the  $\mathbf{t\_INT}$   $y$  as if its sign was  $sy$ .

### 6.2.12 Exact division and divisibility.

GEN diviixact(GEN x, GEN y) returns the Euclidean quotient  $x/y$ , assuming  $y$  divides  $x$ . Uses Jebelean algorithm (Jebelean-Krandick bidirectional exact division is not implemented).

GEN diviuexact(GEN x, ulong y) returns the Euclidean quotient  $x/y$ , assuming  $y$  divides  $x$  and  $y$  is nonzero.

GEN diviuuexact(GEN x, ulong y, ulong z) returns the Euclidean quotient  $x/(yz)$ , assuming  $yz$  divides  $x$  and  $yz \neq 0$ .

The following routines return 1 (true) if  $y$  divides  $x$ , and 0 otherwise. All GEN are assumed to be  $\mathbf{t\_INTs}$ :

```
int dvdi(GEN x, GEN y), int dvdis(GEN x, long y), int dvdiu(GEN x, ulong y),
int dvdsi(long x, GEN y), int dvdui(ulong x, GEN y).
```

The following routines return 1 (true) if  $y$  divides  $x$ , and in that case assign the quotient to  $z$ ; otherwise they return 0. All GEN are assumed to be  $\mathbf{t\_INTs}$ :

`int dvdiiz(GEN x, GEN y, GEN z), int dvdisz(GEN x, long y, GEN z).`

`int dvdiuz(GEN x, ulong y, GEN z)` if  $y$  divides  $x$ , assigns the quotient  $|x|/y$  to  $z$  and returns 1 (true), otherwise returns 0 (false).

### 6.2.13 Division with integral operands and `t_REAL` result.

`GEN rdivii(GEN x, GEN y, long prec)`, assuming  $x$  and  $y$  are both of type `t_INT`, return the quotient  $x/y$  as a `t_REAL` of precision `prec`.

`GEN rdiviiz(GEN x, GEN y, GEN z)`, assuming  $x$  and  $y$  are both of type `t_INT`, and  $z$  is a `t_REAL`, assign the quotient  $x/y$  to  $z$ .

`GEN rdivis(GEN x, long y, long prec)`, assuming  $x$  is of type `t_INT`, return the quotient  $x/y$  as a `t_REAL` of precision `prec`.

`GEN rdivsi(long x, GEN y, long prec)`, assuming  $y$  is of type `t_INT`, return the quotient  $x/y$  as a `t_REAL` of precision `prec`.

`GEN rdivss(long x, long y, long prec)`, return the quotient  $x/y$  as a `t_REAL` of precision `prec`.

**6.2.14 Division with remainder.** The following functions return two objects, unless specifically asked for only one of them — a quotient and a remainder. The quotient is returned and the remainder is returned through the variable whose address is passed as the `r` argument. The term *true Euclidean remainder* refers to the nonnegative one (`mod`), and *Euclidean remainder* by itself to the one with the same sign as the dividend (`rem`). All GENs, whether returned directly or through a pointer, are created on the stack.

`GEN dvmdii(GEN x, GEN y, GEN *r)` returns the Euclidean quotient of the `t_INT`  $x$  by a `t_INT`  $y$  and puts the remainder into `*r`. If `r` is equal to `NULL`, the remainder is not created, and if `r` is equal to `ONLY_REM`, only the remainder is created and returned. In the generic case, the remainder is created after the quotient and can be disposed of individually with a `cgiv(r)`. The remainder is always of the sign of the dividend  $x$ . If the remainder is 0 set `r = gen_0`.

`void dvmdiiz(GEN x, GEN y, GEN z, GEN t)` assigns the Euclidean quotient of the `t_INT`s  $x$  and  $y$  into the `t_INT`  $z$ , and the Euclidean remainder into the `t_INT`  $t$ .

Analogous routines `dvmdis[z]`, `dvmdsi[z]`, `dvmdss[z]` are available, where `s` denotes a `long` argument. But the following routines are in general more flexible:

`long sdivss_rem(long s, long t, long *r)` computes the Euclidean quotient and remainder of the longs  $s$  and  $t$ . Puts the remainder into `*r`, and returns the quotient. The remainder is of the sign of the dividend  $s$ , and has strictly smaller absolute value than  $t$ .

`long sdivsi_rem(long s, GEN x, long *r)` computes the Euclidean quotient and remainder of the long  $s$  by the `t_INT`  $x$ . As `sdivss_rem` otherwise.

`long sdivsi(long s, GEN x)` as `sdivsi_rem`, without remainder.

`GEN divis_rem(GEN x, long s, long *r)` computes the Euclidean quotient and remainder of the `t_INT`  $x$  by the long  $s$ . As `sdivss_rem` otherwise.

`GEN absdiviu_rem(GEN x, ulong s, ulong *r)` computes the Euclidean quotient and remainder of *absolute value* of the `t_INT`  $x$  by the `ulong`  $s$ . As `sdivss_rem` otherwise.

`ulong uabsdiviu_rem(GEN n, ulong d, ulong *r)` as `absdiviu_rem`, assuming that  $|n|/d$  fits into an `ulong`.

`ulong uabsdivui_rem(ulong x, GEN y, ulong *rem)` computes the Euclidean quotient and remainder of  $x$  by  $|y|$ . As `sdivss_rem` otherwise.

`ulong udivuu_rem(ulong x, ulong y, ulong *rem)` computes the Euclidean quotient and remainder of  $x$  by  $y$ . As `sdivss_rem` otherwise.

`ulong ceildivuu(ulong x, ulong y)` return the ceiling of  $x/y$ .

`GEN divsi_rem(long s, GEN y, long *r)` computes the Euclidean quotient and remainder of the long  $s$  by the GEN  $y$ . As `sdivss_rem` otherwise.

`GEN divss_rem(long x, long y, long *r)` computes the Euclidean quotient and remainder of the long  $x$  by the long  $y$ . As `sdivss_rem` otherwise.

`GEN truedvmdii(GEN x, GEN y, GEN *r)`, as `dvmdii` but with a nonnegative remainder.

`GEN truedvmdis(GEN x, long y, GEN *z)`, as `dvmdis` but with a nonnegative remainder.

`GEN truedvmdsi(long x, GEN y, GEN *z)`, as `dvmdsi` but with a nonnegative remainder.

**6.2.15 Modulo to longs.** The following variants of `modii` do not clutter the stack:

`long smodis(GEN x, long y)` computes the true Euclidean remainder of the `t_INT`  $x$  by the long  $y$ . This is the nonnegative remainder, not the one whose sign is the sign of  $x$  as in the `div` functions.

`long smodss(long x, long y)` computes the true Euclidean remainder of the long  $x$  by a long  $y$ .

`ulong umodsu(long x, ulong y)` computes the true Euclidean remainder of the long  $x$  by a ulong  $y$ .

`ulong umodiu(GEN x, ulong y)` computes the true Euclidean remainder of the `t_INT`  $x$  by the ulong  $y$ .

`ulong umodui(ulong x, GEN y)` computes the true Euclidean remainder of the ulong  $x$  by the `t_INT`  $|y|$ .

The routine `smodsi` does not exist, since it would not always be defined: for a *negative*  $x$ , if the quotient is  $\pm 1$ , the result  $x + |y|$  would in general not fit into a long. Use either `umodui` or `modsi`.

These functions directly access the binary data and are thus much faster than the generic modulo functions:

`int mpodd(GEN x)` which is 1 if  $x$  is odd, and 0 otherwise.

`ulong Mod2(GEN x)`

`ulong Mod4(GEN x)`

`ulong Mod8(GEN x)`

`ulong Mod16(GEN x)`

`ulong Mod32(GEN x)`

`ulong Mod64(GEN x)` give the residue class of  $x$  modulo the corresponding power of 2.

`ulong umodi2n(GEN x, long n)` give the residue class of  $x$  modulo  $2^n$ ,  $0 \leq n < BITS\_IN\_LONG$ .

The following functions assume that  $x \neq 0$  and in fact disregard the sign of  $x$ . There are about 10% faster than the safer variants above:

`long mod2(GEN x)`

`long mod4(GEN x)`

`long mod8(GEN x)`

`long mod16(GEN x)`

`long mod32(GEN x)`

`long mod64(GEN x)` give the residue class of  $|x|$  modulo the corresponding power of 2, for *nonzero*  $x$ . As well,

`ulong mod2BIL(GEN x)` returns the least significant word of  $|x|$ , still assuming that  $x \neq 0$ .

### 6.2.16 Powering, Square root.

`GEN powii(GEN x, GEN n)`, assumes  $x$  and  $n$  are `t_INT`s and returns  $x^n$ .

`GEN powuu(ulong x, ulong n)`, returns  $x^n$ .

`GEN powiu(GEN x, ulong n)`, assumes  $x$  is a `t_INT` and returns  $x^n$ .

`GEN powis(GEN x, long n)`, assumes  $x$  is a `t_INT` and returns  $x^n$  (possibly a `t_FRAC` if  $n < 0$ ).

`GEN powrs(GEN x, long n)`, assumes  $x$  is a `t_REAL` and returns  $x^n$ . This is considered as a sequence of `mulrr`, possibly empty: as such the result has type `t_REAL`, even if  $n = 0$ . Note that the generic function `gpows(x,0)` would return `gen_1`, see the technical note in Section 6.2.11.

`GEN powru(GEN x, ulong n)`, assumes  $x$  is a `t_REAL` and returns  $x^n$  (always a `t_REAL`, even if  $n = 0$ ).

`GEN powersr(GEN e, long n)`. Given a `t_REAL`  $e$ , return the vector  $v$  of all  $e^i$ ,  $0 \leq i \leq n$ , where  $v[i] = e^{i-1}$ .

`GEN powrshalf(GEN x, long n)`, assumes  $x$  is a `t_REAL` and returns  $x^{n/2}$  (always a `t_REAL`, even if  $n = 0$ ).

`GEN powruhalf(GEN x, ulong n)`, assumes  $x$  is a `t_REAL` and returns  $x^{n/2}$  (always a `t_REAL`, even if  $n = 0$ ).

`GEN powrfrac(GEN x, long n, long d)`, assumes  $x$  is a `t_REAL` and returns  $x^{n/d}$  (always a `t_REAL`, even if  $n = 0$ ).

`GEN powIs(long n)` returns  $I^n \in \{1, I, -1, -I\}$  (`t_INT` for even  $n$ , `t_COMPLEX` otherwise).

`ulong upowuu(ulong x, ulong n)`, returns  $x^n$  when  $< 2^{\text{BITS\_IN\_LONG}}$ , and 0 otherwise (overflow).

`ulong upowers(ulong x, long n)`, returns  $[1, x, \dots, x^n]$  as a `t_VECSMALL`. Assume there is no overflow.

`GEN sqrtremi(GEN N, GEN *r)`, returns the integer square root  $S$  of the nonnegative `t_INT`  $N$  (rounded towards 0) and puts the remainder  $R$  into  $*r$ . Precisely,  $N = S^2 + R$  with  $0 \leq R \leq 2S$ . If  $r$  is equal to `NULL`, the remainder is not created. In the generic case, the remainder is created after the quotient and can be disposed of individually with `cgiv(R)`. If the remainder is 0 set  $R = \text{gen\_0}$ .

Uses a divide and conquer algorithm (discrete variant of Newton iteration) due to Paul Zimmermann (“Karatsuba Square Root”, INRIA Research Report 3805 (1999)).

**GEN sqrti**(GEN N), returns the integer square root  $S$  of the nonnegative  $\mathbf{t\_INT}$  N (rounded towards 0). This is identical to **sqrtremi**(N, NULL).

**long logintall**(GEN B, GEN y, GEN \*ptq) returns the floor  $e$  of  $\log_y B$ , where  $B > 0$  and  $y > 1$  are integers. If ptq is not NULL, set it to  $y^e$ . (Analogous to **logint0**, without sanity checks.)

**ulong ulogintall**(ulong B, ulong y, ulong \*ptq) as **logintall** for **ulong** arguments.

**long logint**(GEN B, GEN y) returns the floor  $e$  of  $\log_y B$ , where  $B > 0$  and  $y > 1$  are integers.

**ulong ulogint**(ulong B, ulong y) as **logint** for **ulong** arguments.

**GEN vecpowuu**(long N, ulong a) return the vector of  $n^a$ ,  $n = 1, \dots, N$ . Not memory clean.

**GEN vecpowug**(long N, GEN a, long prec) return the vector of  $n^a$ ,  $n = 1, \dots, N$ , where the powers are computed at precision **prec**. Not memory clean.

### 6.2.17 GCD, extended GCD and LCM.

**long cgcd**(long x, long y) returns the GCD of x and y.

**ulong ugcd**(ulong x, ulong y) returns the GCD of x and y.

**ulong ugcdui**(GEN x, ulong y) returns the GCD of x and y.

**ulong ugcdui**(ulong x, GEN y) returns the GCD of x and y.

**GEN coprimes\_zv**(ulong N) return a  $\mathbf{t\_VECSMALL}$  T with N entries such that  $T[i] = 1$  iff  $(i, N) = 1$  and 0 otherwise.

**long clcm**(long x, long y) returns the LCM of x and y, provided it fits into a **long**. Silently overflows otherwise.

**ulong ulcm**(ulong x, ulong y) returns the LCM of x and y, provided it fits into an **ulong**. Silently overflows otherwise.

**GEN gcdii**(GEN x, GEN y), returns the GCD of the  $\mathbf{t\_INT}$ s x and y.

**GEN lcmii**(GEN x, GEN y), returns the LCM of the  $\mathbf{t\_INT}$ s x and y.

**GEN bezout**(GEN a, GEN b, GEN \*u, GEN \*v), returns the GCD  $d$  of  $\mathbf{t\_INT}$ s a and b and sets u, v to the Bezout coefficients such that  $au + bv = d$ .

**long cbezout**(long a, long b, long \*u, long \*v), returns the GCD  $d$  of a and b and sets u, v to the Bezout coefficients such that  $au + bv = d$ .

**GEN halfgcdii**(GEN x, GEN y) assuming x and y are  $\mathbf{t\_INT}$ s, returns a 2-components  $\mathbf{t\_VEC}$   $[M, V]$  where  $M$  is a  $2 \times 2$   $\mathbf{t\_MAT}$  and  $V$  a 2-component  $\mathbf{t\_COL}$ , both with  $\mathbf{t\_INT}$  entries, such that  $M * [x, y] == V$  and such that if  $V = [a, b]$ , then  $a \geq \left\lceil \sqrt{\max(|x|, |y|)} \right\rceil > b$ .

**GEN ZV\_extgcd**(GEN A) given a vector of  $n$  integers A, returns  $[d, U]$ , where  $d$  is the GCD of the  $A[i]$  and  $U$  is a matrix in  $\mathrm{GL}_n(\mathbf{Z})$  such that  $AU = [0, \dots, 0, D]$ .

**GEN ZV\_lcm**(GEN v) given a vector  $v$  of integers returns the LCM of its entries.

**GEN ZV\_snf\_gcd**(GEN v, GEN N) given a vector  $v$  of integers and a positive integer  $N$ , return the vector whose entries are the gcds  $(v[i], N)$ . Use case: if  $v$  gives the cyclic components for some Abelian group  $G$  of finite type, then this returns the structure of the finite groupe  $G/G^N$ .

### 6.2.18 Continued fractions and convergents.

GEN `ZV_allpnqn`(GEN `x`) given  $x = [a_0, \dots, a_n]$  a continued fraction from `gboundcf`,  $n \geq 0$ , return all convergents as  $[P, Q]$ , where  $P = [p_0, \dots, p_n]$  and  $Q = [q_0, \dots, q_n]$ .

**6.2.19 Pseudo-random integers.** These routine return pseudo-random integers uniformly distributed in some interval. The all use the same underlying generator which can be seeded and restarted using `getrand` and `setrand`.

`void setrand`(GEN `seed`) reseeds the random number generator using the seed  $n$ . The seed is either a technical array output by `getrand` or a small positive integer, used to generate deterministically a suitable state array. For instance, running a randomized computation starting by `setrand(1)` twice will generate the exact same output.

GEN `getrand`(void) returns the current value of the seed used by the pseudo-random number generator `random`. Useful mainly for debugging purposes, to reproduce a specific chain of computations. The returned value is technical (reproduces an internal state array of type `t_VECSMALL`), and can only be used as an argument to `setrand`.

`ulong pari_rand`(void) returns a random  $0 \leq x < 2^{\text{BITS\_IN\_LONG}}$ .

`long random_bits`(long `k`) returns a random  $0 \leq x < 2^k$ . Assumes that  $0 \leq k \leq \text{BITS\_IN\_LONG}$ .

`ulong random_Fl`(ulong `p`) returns a pseudo-random integer in  $0, 1, \dots, p-1$ .

GEN `randomi`(GEN `n`) returns a random `t_INT` between 0 and `n` - 1.

GEN `randomr`(long `prec`) returns a random `t_REAL` in  $[0, 1[$ , with precision `prec`.

**6.2.20 Modular operations.** In this subsection, all GENs are `t_INT`.

GEN `Fp_red`(GEN `a`, GEN `m`) returns `a` modulo `m` (smallest nonnegative residue). (This is identical to `modii`).

GEN `Fp_neg`(GEN `a`, GEN `m`) returns  $-a$  modulo `m` (smallest nonnegative residue).

GEN `Fp_add`(GEN `a`, GEN `b`, GEN `m`) returns the sum of `a` and `b` modulo `m` (smallest nonnegative residue).

GEN `Fp_sub`(GEN `a`, GEN `b`, GEN `m`) returns the difference of `a` and `b` modulo `m` (smallest nonnegative residue).

GEN `Fp_center`(GEN `a`, GEN `p`, GEN `pov2`) assuming that `pov2` is `shifti(p, -1)` and that  $-p/2 < a < p$ , returns the representative of `a` in the symmetric residue system  $] -p/2, p/2]$ .

GEN `Fp_center_i`(GEN `a`, GEN `p`, GEN `pov2`) internal variant of `Fp_center`, not `gerepile`-safe: when `a` is already in the proper interval, it is returned as is, without a copy.

GEN `Fp_mul`(GEN `a`, GEN `b`, GEN `m`) returns the product of `a` by `b` modulo `m` (smallest nonnegative residue).

GEN `Fp_addmul`(GEN `x`, GEN `y`, GEN `z`, GEN `p`) returns  $x + yz$ .

GEN `Fp_mulu`(GEN `a`, ulong `b`, GEN `m`) returns the product of `a` by `b` modulo `m` (smallest nonnegative residue).

GEN `Fp_muls`(GEN `a`, long `b`, GEN `m`) returns the product of `a` by `b` modulo `m` (smallest nonnegative residue).

GEN Fp\_half(GEN x, GEN m) returns  $z$  such that  $2z = x$  modulo  $m$  assuming such  $z$  exists.

GEN Fp\_sqr(GEN a, GEN m) returns  $a^2$  modulo  $m$  (smallest nonnegative residue).

ulong Fp\_powu(GEN x, ulong n, GEN m) raises  $x$  to the  $n$ -th power modulo  $m$  (smallest nonnegative residue). Not memory-clean, but suitable for `gerepileupto`.

ulong Fp\_pows(GEN x, long n, GEN m) raises  $x$  to the  $n$ -th power modulo  $m$  (smallest nonnegative residue). A negative  $n$  is allowed. Not memory-clean, but suitable for `gerepileupto`.

GEN Fp\_pow(GEN x, GEN n, GEN m) returns  $x^n$  modulo  $m$  (smallest nonnegative residue).

GEN Fp\_pow\_init(GEN x, GEN n, long k, GEN p) Return a table  $R$  that can be used with `Fp_pow_table` to compute the powers of  $x$  up to  $n$ . The table is of size  $2^k \log_2(n)$ .

GEN Fp\_pow\_table(GEN R, GEN n, GEN p) return  $x^n$ , where  $R$  is given by `Fp_pow_init(x,m,k,p)` for some integer  $m \geq n$ .

GEN Fp\_powers(GEN x, long n, GEN m) returns  $[x^0, \dots, x^n]$  modulo  $m$  as a `t_VEC` (smallest nonnegative residue).

GEN Fp\_inv(GEN a, GEN m) returns an inverse of  $a$  modulo  $m$  (smallest nonnegative residue). Raise an error if  $a$  is not invertible.

GEN Fp\_invsafe(GEN a, GEN m) as `Fp_inv`, but return `NULL` if  $a$  is not invertible.

GEN Fp\_invgen(GEN x, GEN m, GEN \*pg) set `*pg` to  $g = \gcd(x, m)$  and return  $u$  in  $(\mathbf{Z}/m\mathbf{Z})^*$  such that  $xu = g$  modulo  $m$ . We have  $g = 1$  if and only if  $x$  is invertible, and in this case  $u$  is its inverse.

GEN FpV\_prod(GEN x, GEN p) returns the product of the components of  $x$ .

GEN FpV\_inv(GEN x, GEN m)  $x$  being a vector of `t_INTs`, return the vector of inverses of the  $x[i]$  mod  $m$ . The routine uses Montgomery's trick, and involves a single inversion mod  $m$ , plus  $3(N-1)$  multiplications for  $N$  entries. The routine is not stack-clean:  $2N$  integers mod  $m$  are left on stack, besides the  $N$  in the result.

GEN Fp\_div(GEN a, GEN b, GEN m) returns the quotient of  $a$  by  $b$  modulo  $m$  (smallest nonnegative residue). Raise an error if  $b$  is not invertible.

GEN Fp\_divu(GEN a, ulong b, GEN m) returns the quotient of  $a$  by  $b$  modulo  $m$  (smallest nonnegative residue). Raise an error if  $b$  is not invertible.

int invmod(GEN a, GEN m, GEN \*g), return 1 if  $a$  modulo  $m$  is invertible, else return 0 and set  $g = \gcd(a, m)$ .

In the following three functions the integer parameter `ord` can be given either as a positive `t_INT`  $N$ , or as its factorization matrix  $faN$ , or as a pair  $[N, faN]$ . The parameter may be omitted by setting it to `NULL` (the value is then  $p-1$ ).

GEN Fp\_log(GEN a, GEN g, GEN ord, GEN p) Let  $g$  such that  $g^{ord} \equiv 1 \pmod{p}$ . Return an integer  $e$  such that  $a^e \equiv g \pmod{p}$ . If  $e$  does not exist, the result is undefined.

GEN Fp\_order(GEN a, GEN ord, GEN p) returns the order of the Fp  $a$ . Assume that `ord` is a multiple of the order of  $a$ .

GEN Fp\_factored\_order(GEN a, GEN ord, GEN p) returns  $[o, F]$ , where  $o$  is the multiplicative order of the Fp  $a$  in  $\mathbf{F}_p^*$ , and  $F$  is the factorization of  $o$ . Assume that `ord` is a multiple of the order of  $a$ .



`int Fp_issquare(GEN x, GEN p)` returns 1 if  $x$  is a square modulo  $p$ , and 0 otherwise.

`int Fp_ispower(GEN x, GEN n, GEN p)` returns 1 if  $x$  is an  $n$ -th power modulo  $p$ , and 0 otherwise.

`GEN Fp_sqrt(GEN x, GEN p)` returns a square root of  $x$  modulo  $p$  (the smallest nonnegative residue), where  $x, p$  are `t_INTs`, and  $p$  is assumed to be prime. Return `NULL` if  $x$  is not a quadratic residue modulo  $p$ .

`GEN Fp_2gener(GEN p)` return a generator of the 2-Sylow subgroup of  $\mathbf{F}_p^*$ . To use with `Fp_sqrt_i`.

`GEN Fp_sqrt_i(GEN x, GEN s2, GEN p)` as `Fp_sqrt` where  $s2$  is the element returned by `Fp_2gener`.

`GEN Fp_sqrtn(GEN a, GEN n, GEN p, GEN *zn)` returns `NULL` if  $a$  is not an  $n$ -th power residue mod  $p$ . Otherwise, returns an  $n$ -th root of  $a$ ; if  $zn$  is not `NULL` set it to a primitive  $m$ -th root of 1,  $m = \gcd(p-1, n)$  allowing to compute all  $m$  solutions in  $\mathbf{F}_p$  of the equation  $x^n = a$ .

`GEN Zn_sqrt(GEN x, GEN n)` returns one of the square roots of  $x$  modulo  $n$  (possibly not prime), where  $x$  is a `t_INT` and  $n$  is either a `t_INT` or is given by its factorization matrix. Return `NULL` if no such square root exist.

`GEN Zn_quad_roots(GEN N, GEN B, GEN C)` solves the equation  $X^2 + BX + C$  modulo  $N$ . Return `NULL` if there are no solutions. Else returns  $[v, M]$  where  $M \mid N$  and the `FpV`  $v$  of distinct integers (reduced, implicitly modulo  $M$ ) is such that  $x$  modulo  $N$  is a solution to the equation if and only if  $x$  modulo  $M$  belongs to  $v$ . If the discriminant  $B^2 - 4C$  is coprime to  $N$ , we have  $M = N$  but in general  $M$  can be a strict divisor of  $N$ .

`long kross(long x, long y)` returns the Kronecker symbol  $(x|y)$ , i.e.  $-1, 0$  or  $1$ . If  $y$  is an odd prime, this is the Legendre symbol. (Contrary to `krouu`, `kross` also supports  $y = 0$ )

`long krouu(ulong x, ulong y)` returns the Kronecker symbol  $(x|y)$ , i.e.  $-1, 0$  or  $1$ . Assumes  $y$  is nonzero. If  $y$  is an odd prime, this is the Legendre symbol.

`long krois(GEN x, long y)` returns the Kronecker symbol  $(x|y)$  of `t_INT`  $x$  and `long`  $y$ . As `kross` otherwise.

`long kroiu(GEN x, ulong y)` returns the Kronecker symbol  $(x|y)$  of `t_INT`  $x$  and nonzero `ulong`  $y$ . As `krouu` otherwise.

`long krosi(long x, GEN y)` returns the Kronecker symbol  $(x|y)$  of `long`  $x$  and `t_INT`  $y$ . As `kross` otherwise.

`long kroui(ulong x, GEN y)` returns the Kronecker symbol  $(x|y)$  of `long`  $x$  and `t_INT`  $y$ . As `kross` otherwise.

`long kronecker(GEN x, GEN y)` returns the Kronecker symbol  $(x|y)$  of `t_INTs`  $x$  and  $y$ . As `kross` otherwise.

`GEN factorial_Fp(long n, GEN p)` return  $n! \bmod p$ .

`GEN pgener_Fp(GEN p)` returns the smallest primitive root modulo  $p$ , assuming  $p$  is prime.

`GEN pgener_Zp(GEN p)` returns the smallest primitive root modulo  $p^k$ ,  $k > 1$ , assuming  $p$  is an odd prime.

`long Zp_issquare(GEN x, GEN p)` returns 1 if the `t_INT`  $x$  is a  $p$ -adic square, 0 otherwise.

`long Zn_issquare(GEN x, GEN n)` returns 1 if `t_INT`  $x$  is a square modulo  $n$  (possibly not prime), where  $n$  is either a `t_INT` or is given by its factorization matrix. Return 0 otherwise.

`long Zn_ispower(GEN x, GEN n, GEN K, GEN *py)` returns 1 if  $x$  is a  $K$ -th power modulo  $n$  (possibly not prime), where  $n$  is either a `t_INT` or is given by its factorization matrix. Return 0 otherwise. If `py` is not `NULL`, set it to  $y$  such that  $y^K = x$  modulo  $n$ .

`GEN pgener_Fp_local(GEN p, GEN L)`,  $L$  being a vector of primes dividing  $p - 1$ , returns the smallest integer  $x > 1$  which is a generator of the  $\ell$ -Sylow of  $\mathbf{F}_p^*$  for every  $\ell$  in  $L$ . In other words,  $x^{(p-1)/\ell} \neq 1$  for all such  $\ell$ . In particular, returns `pgener_Fp(p)` if  $L$  contains all primes dividing  $p - 1$ . It is not necessary, and in fact slightly inefficient, to include  $\ell = 2$ , since 2 is treated separately in any case, i.e. the generator obtained is never a square.

`GEN rootsof1_Fp(GEN n, GEN p)` returns a primitive  $n$ -th root modulo the prime  $p$ .

`GEN rootsof1u_Fp(ulong n, GEN p)` returns a primitive  $n$ -th root modulo the prime  $p$ .

`ulong rootsof1_Fl(ulong n, ulong p)` returns a primitive  $n$ -th root modulo the prime  $p$ .

### 6.2.21 Extending functions to vector inputs.

The following functions apply  $f$  to the given arguments, recursively if they are of vector / matrix type:

`GEN map_proto_G(GEN (*f)(GEN), GEN x)` For instance, if  $x$  is a `t_VEC`, return a `t_VEC` whose components are the  $f(x[i])$ .

`GEN map_proto_lG(long (*f)(GEN), GEN x)` As above, applying the function `stoi( f() )`.

`GEN map_proto_GL(GEN (*f)(GEN, long), GEN x, long y)`

`GEN map_proto_lGL(long (*f)(GEN, long), GEN x, long y)`

In the last function,  $f$  implements an associative binary operator, which we extend naturally to an  $n$ -ary operator  $f_n$  for any  $n$ : by convention,  $f_0() = 1$ ,  $f_1(x) = x$ , and

$$f_n(x_1, \dots, x_n) = f(f_{n-1}(x_1, \dots, x_{n-1}), x_n),$$

for  $n \geq 2$ .

`GEN gassoc_proto(GEN (*f)(GEN, GEN), GEN x, GEN y)` If  $y$  is not `NULL`, return  $f(x, y)$ . Otherwise,  $x$  must be of vector type, and we return the result of  $f$  applied to its components, computed using a divide-and-conquer algorithm. More precisely, return

$$f(f(x_1, \text{NULL}), f(x_2, \text{NULL})),$$

where  $x_1, x_2$  are the two halves of  $x$ .

### 6.2.22 Miscellaneous arithmetic functions.

`long bigomegau(ulong n)` returns the number of prime divisors of  $n > 0$ , counted with multiplicity.

`ulong coreu(ulong n)`, unique squarefree integer  $d$  dividing  $n$  such that  $n/d$  is a square.

`ulong coreu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`ulong corediscs(long d, ulong *pt_f)`,  $d$  (possibly negative) being congruent to 0 or 1 modulo 4, return the fundamental discriminant  $D$  such that  $d = D * f^2$  and set `*pt_f` to  $f$  (if `*pt_f` not NULL).

`GEN coredisc2_fact(GEN fa, long s, GEN *pP, GEN *pE)` let  $d$  be an integer congruent to 0 or 1 mod 4. Return  $D = \text{coredisc}(d)$  assuming that `fa` is the factorization of  $|d|$  and  $sd > 0$  ( $s$  is the sign of  $d$ ). Set `*pP` and `*pE` to the factorization of the conductor  $f$  such that  $d = Df^2$ , where  $P$  is a `t_VEC` of primes and  $E$  a `t_VECSMALL` of exponents.

`ulong coredisc2u_fact(GEN fa, long s, GEN *pP, GEN *pE)` let  $d$  be an integer congruent to 0 or 1 mod 4 whose absolute value fits in an `ulong`. Return the absolute value of  $D = \text{corediscs}(d)$  assuming that `fa` is the factorization of  $|d|$  and  $sd > 0$  ( $s$  is the sign of  $d$  and  $D$ ). Set `*pP` and `*pE` to the factorization of the conductor  $f$  such that  $d = Df^2$ , where  $P$  is a `t_VECSMALL` of primes and  $E$  a `t_VECSMALL` of exponents.

`ulong eulerphiu(ulong n)`, Euler's totient function of  $n$ .

`ulong eulerphiu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long moebiusu(ulong n)`, Moebius  $\mu$ -function of  $n$ .

`long moebiusu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`ulong radicalu(ulong n)`, product of primes dividing  $n$ .

`GEN divisorsu(ulong n)`, returns the divisors of  $n$  in a `t_VECSMALL`, sorted by increasing order.

`GEN divisorsu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`GEN divisorsu_fact_factored(GEN fa)` where `fa` is `factoru(n)`. Return a vector  $[D, F]$ , where  $D$  is a `t_VECSMALL` containing the divisors of  $u$  and  $F[i]$  contains `factoru(D[i])`.

`GEN divisorsu_moebius(GEN P)` returns the divisors of  $n$  of the form  $\prod_{p \in S} (-p)$ ,  $S \subset P$  in a `t_VECSMALL`. The vector is not sorted but its first element is guaranteed to be 1. If  $P$  is `factoru(n)[1]`, this returns the set of  $\mu(d)d$  where  $d$  runs through the squarefree divisors of  $n$ .

`long numdivu(ulong n)`, returns the number of positive divisors of  $n > 0$ .

`long numdivu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long omegau(ulong n)` returns the number of prime divisors of  $n > 0$ .

`long maxomegau(ulong x)` return the optimal  $B$  such that  $\omega(n) \leq B$  for all  $n \leq x$ .

`long maxomegaoddu(ulong x)` return the optimal  $B$  such that  $\omega(n) \leq B$  for all odd  $n \leq x$ .

`long uissquarefree(ulong n)` returns 1 if  $n$  is square-free, and 0 otherwise.

`long uissquarefree_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long uposisfundamental(ulong x)` return 1 if  $x$  is a fundamental discriminant, and 0 otherwise.

`long unegisfundamental(ulong x)` return 1 if  $-x$  is a fundamental discriminant, and 0 otherwise.  
`long sisfundamental(long x)` return 1 if  $x$  is a fundamental discriminant, and 0 otherwise.  
`int uis_357_power(ulong x, ulong *pt, ulong *mask)` as `is_357_power` for `ulong x`.  
`int uis_357_powermod(ulong x, ulong *mask)` as `uis_357_power`, but only check for 3rd, 5th or 7th powers modulo  $211 \times 209 \times 61 \times 203 \times 117 \times 31 \times 43 \times 71$ .  
`long uisprimepower(ulong n, ulong *p)` as `isprimepower`, for `ulong n`.  
`int uislucaspsp(ulong n)` returns 1 if the `ulong n` fails Lucas compositeness test (it thus may be prime or composite), and 0 otherwise (proving that  $n$  is composite).  
`int uis2psp(ulong n)` returns 1 if the odd `ulong n` fails a strong Rabin-Miller test for the base 2 (it thus may be prime or composite), and 0 otherwise (proving that  $n$  is composite).  
`int uispsp(ulong a, ulong n)` returns 1 if the odd `ulong n` fails a strong Rabin-Miller test for the base  $1 < a < n$  (it thus may be prime or composite), and 0 otherwise (proving that  $n$  is composite).  
`ulong sumdigitsu(ulong n)` returns the sum of decimal digits of  $u$ .  
`GEN usumdiv_fact(GEN fa)`, sum of divisors of `ulong n`, where `fa` is `factoru(n)`.  
`GEN usumdivk_fact(GEN fa, ulong k)`, sum of  $k$ -th powers of divisors of `ulong n`, where `fa` is `factoru(n)`.  
`GEN hilbertii(GEN x, GEN y, GEN p)`, returns the Hilbert symbol  $(x, y)$  at the prime  $p$  (NULL for the place at infinity);  $x$  and  $y$  are `t_INTs`.  
`GEN sumdedekind(GEN h, GEN k)` returns the Dedekind sum attached to the `t_INT`  $h$  and  $k$ ,  $k > 0$ .  
`GEN sumdedekind_coprime(GEN h, GEN k)` as `sumdedekind`, except that  $h$  and  $k$  are assumed to be coprime `t_INTs`.  
`GEN u_sumdedekind_coprime(long h, long k)` Let  $k > 0$ ,  $0 \leq h < k$ ,  $(h, k) = 1$ . Returns  $[s_1, s_2]$  in a `t_VECSMALL`, such that  $s(h, k) = (s_2 + ks_1)/(12k)$ . Requires  $\max(h + k/2, k) < \text{LONG\_MAX}$  to avoid overflow, in particular  $k \leq (2/3)\text{LONG\_MAX}$  is fine.

## Chapter 7:

### Level 2 kernel

These functions deal with modular arithmetic, linear algebra and polynomials where assumptions can be made about the types of the coefficients.

#### 7.1 Naming scheme.

A function name is built in the following way:  $A_1 \dots A_n fun$  for an operation  $fun$  with  $n$  arguments of class  $A_1, \dots, A_n$ . A class name is given by a base ring followed by a number of code letters. Base rings are among

**F1**:  $\mathbf{Z}/l\mathbf{Z}$  where  $l < 2^{\text{BITS\_IN\_LONG}}$  is not necessarily prime. Implemented using **ulongs**

**Fp**:  $\mathbf{Z}/p\mathbf{Z}$  where  $p$  is a **t\_INT**, not necessarily prime. Implemented as **t\_INTs**  $z$ , preferably satisfying  $0 \leq z < p$ . More precisely, any **t\_INT** can be used as an **Fp**, but reduced inputs are treated more efficiently. Outputs from **Fpxxx** routines are reduced.

**Fq**:  $\mathbf{Z}[X]/(p, T(X))$ ,  $p$  a **t\_INT**,  $T$  a **t\_POL** with **Fp** coefficients or **NULL** (in which case no reduction modulo  $T$  is performed). Implemented as **t\_POLs**  $z$  with **Fp** coefficients,  $\deg(z) < \deg T$ , although  $z$  a **t\_INT** is allowed for elements in the prime field.

**Z**: the integers  $\mathbf{Z}$ , implemented as **t\_INTs**.

**Zp**: the  $p$ -adic integers  $\mathbf{Z}_p$ , implemented as **t\_INTs**, for arbitrary  $p$

**Z1**: the  $p$ -adic integers  $\mathbf{Z}_p$ , implemented as **t\_INTs**, for  $p < 2^{\text{BITS\_IN\_LONG}}$

**z**: the integers  $\mathbf{Z}$ , implemented using (signed) **longs**.

**Q**: the rational numbers  $\mathbf{Q}$ , implemented as **t\_INTs** and **t\_FRACs**.

**Rg**: a commutative ring, whose elements can be **gadd**-ed, **gmul**-ed, etc.

Possible letters are:

**X**: polynomial in  $X$  (**t\_POL** in a fixed variable), e.g. **FpX** means  $\mathbf{Z}/p\mathbf{Z}[X]$

**Y**: polynomial in  $Y \neq X$ . This is used to resolve ambiguities. E.g. **FpXY** means  $((\mathbf{Z}/p\mathbf{Z})[X])[Y]$ .

**V**: vector (**t\_VEC** or **t\_COL**), treated as a line vector (independently of the actual type). E.g. **ZV** means  $\mathbf{Z}^k$  for some  $k$ .

**C**: vector (**t\_VEC** or **t\_COL**), treated as a column vector (independently of the actual type). The difference with **V** is purely semantic: if the result is a vector, it will be of type **t\_COL** unless mentioned otherwise. For instance the function **ZC\_add** receives two integral vectors (**t\_COL** or **t\_VEC**, possibly different types) of the same length and returns a **t\_COL** whose entries are the sums of the input coefficients.

**M**: matrix (**t\_MAT**). E.g. **QM** means a matrix with rational entries

**T**: Trees. Either a leaf or a **t\_VEC** of trees.

**E**: point over an elliptic curve, represented as two-component vectors **[x,y]**, except for the represented by the one-component vector **[0]**. Not all curve models are supported.

**Q**: representative (**t\_POL**) of a class in a polynomial quotient ring. E.g. an **FpXQ** belongs to  $(\mathbf{Z}/p\mathbf{Z})[X]/(T(X))$ , **FpXQV** means a vector of such elements, etc.

**n**: a polynomial representative (**t\_POL**) for a truncated power series modulo  $X^n$ . E.g. an **FpXn** belongs to  $(\mathbf{Z}/p\mathbf{Z})[X]/(X^n)$ , **FpXnV** means a vector of such elements, etc.

**x**, **y**, **m**, **v**, **c**, **q**: as their uppercase counterpart, but coefficient arrays are implemented using **t\_VECSMALLs**, which coefficient understood as **ulongs**.

**x** and **y** (and **q**) are implemented by a **t\_VECSMALL** whose first coefficient is used as a code-word and the following are the coefficients, similarly to a **t\_POL**. This is known as a 'POLSMALL'.

**m** are implemented by a **t\_MAT** whose components (columns) are **t\_VECSMALLs**. This is known as a 'MATSMALL'.

**v** and **c** are regular **t\_VECSMALLs**. Difference between the two is purely semantic.

Omitting the letter means the argument is a scalar in the base ring. Standard functions *fun* are

**add**: add

**sub**: subtract

**mul**: multiply

**sqr**: square

**div**: divide (Euclidean quotient)

**rem**: Euclidean remainder

**divrem**: return Euclidean quotient, store remainder in a pointer argument. Three special values of that pointer argument modify the default behavior: **NULL** (do not store the remainder, used to implement **div**), **ONLY\_REM** (return the remainder, used to implement **rem**), **ONLY\_DIVIDES** (return the quotient if the division is exact, and **NULL** otherwise).

**gcd**: GCD

**extgcd**: return GCD, store Bezout coefficients in pointer arguments

**pow**: exponentiate

**eval**: evaluation / composition

## 7.2 Coefficient ring.

`long Rg_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the object  $x$  is defined.

Raise an error if it detects consistency problems in modular objects: incompatible rings (e.g.  $\mathbf{F}_p$  and  $\mathbf{F}_q$  for primes  $p \neq q$ ,  $\mathbf{F}_p[X]/(T)$  and  $\mathbf{F}_p[X]/(U)$  for  $T \neq U$ ). Minor discrepancies are supported if they make general sense (e.g.  $\mathbf{F}_p$  and  $\mathbf{F}_{p^k}$ , but not  $\mathbf{F}_p$  and  $\mathbf{Q}_p$ ); `t_FFELT` and `t_POLMOD` of `t_INTMODs` are considered inconsistent, even if they define the same field: if you need to use simultaneously these different finite field implementations, multiply the polynomial by a `t_FFELT` equal to 1 first.

- 0: none of the others (presumably multivariate, possibly inconsistent).
- `t_INT`: defined over  $\mathbf{Z}$ .
- `t_FRAC`: defined over  $\mathbf{Q}$ .
- `t_INTMOD`: defined over  $\mathbf{Z}/p\mathbf{Z}$ , where `*ptp` is set to  $p$ . It is not checked whether  $p$  is prime.
- `t_COMPLEX`: defined over  $\mathbf{C}$  (at least one `t_COMPLEX` with at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.
- `t_REAL`: defined over  $\mathbf{R}$  (at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.
- `t_PADIC`: defined over  $\mathbf{Q}_p$ , where `*ptp` is set to  $p$  and `*ptprec` to the  $p$ -adic accuracy.
- `t_FFELT`: defined over a finite field  $\mathbf{F}_{p^k}$ , where `*ptp` is set to the field characteristic  $p$  and `*ptpol` is set to a `t_FFELT` belonging to the field.
- `t_POL`: defined over a polynomial ring.
- other values are composite corresponding to quotients  $R[X]/(T)$ , with one primary type `t1`, describing the form of the quotient, and a secondary type `t2`, describing  $R$ . If `t` is the `RgX_type`, `t1` and `t2` are recovered using

```
void RgX_type_decode(long t, long *t1, long *t2)
```

`t1` is one of

`t_POLMOD`: at least one `t_POLMOD` component, set `*ppol` to the modulus,

`t_QUAD`: no `t_POLMOD`, at least one `t_QUAD` component, set `*ppol` to the modulus (`-.pol`) of the `t_QUAD`,

`t_COMPLEX`: no `t_POLMOD` or `t_QUAD`, at least one `t_COMPLEX` component, set `*ppol` to  $y^2 + 1$ .

and the underlying base ring  $R$  is given by `t2`, which is one of `t_INT`, `t_INTMOD` (set `*ptp`) or `t_PADIC` (set `*ptp` and `*ptprec`), with the same meaning as above.

`int RgX_type_is_composite(long t)`  $t$  as returned by `RgX_type`, return 1 if  $t$  is a composite type, and 0 otherwise.

`GEN Rg_get_0(GEN x)` returns 0 in the base ring over which  $x$  is defined, to the proper accuracy (e.g. 0, Mod(0,3), 0(5<sup>10</sup>)).

`GEN Rg_get_1(GEN x)` returns 1 in the base ring over which  $x$  is defined, to the proper accuracy (e.g. 0, Mod(0,3),

`long RgX_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomial  $x$  is defined, otherwise as `Rg_type`.

`long RgX_Rg_type(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomial  $x$  and the element  $y$  are defined, otherwise as `Rg_type`.

`long RgX_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomials  $x$  and  $y$  are defined, otherwise as `Rg_type`.

`long RgX_type3(GEN x, GEN y, GEN z, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomials  $x$ ,  $y$  and  $z$  are defined, otherwise as `Rg_type`.

`long RgM_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the matrix  $x$  is defined, otherwise as `Rg_type`.

`long RgM_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the matrices  $x$  and  $y$  are defined, otherwise as `Rg_type`.

`long RgV_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the vector  $x$  is defined, otherwise as `Rg_type`.

`long RgV_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the vectors  $x$  and  $y$  are defined, otherwise as `Rg_type`.

`long RgM_RgC_type(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the matrix  $x$  and the vector  $y$  are defined, otherwise as `Rg_type`.

## 7.3 Modular arithmetic.

These routines implement univariate polynomial arithmetic and linear algebra over finite fields, in fact over finite rings of the form  $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$ , where  $p$  is not necessarily prime and  $T \in (\mathbf{Z}/p\mathbf{Z})[X]$  is possibly reducible; and finite extensions thereof. All this can be emulated with `t_INTMOD` and `t_POLMOD` coefficients and using generic routines, at a considerable loss of efficiency. Also, specialized routines are available that have no obvious generic equivalent.

**7.3.1 FpC / FpV, FpM.** A ZV (resp. a ZM) is a `t_VEC` or `t_COL` (resp. `t_MAT`) with `t_INT` coefficients. An FpV or FpM, with respect to a given `t_INT`  $p$ , is the same with Fp coordinates; operations are understood over  $\mathbf{Z}/p\mathbf{Z}$ .

### 7.3.1.1 Conversions.

`int Rg_is_Fp(GEN z, GEN *p)`, checks if  $z$  can be mapped to  $\mathbf{Z}/p\mathbf{Z}$ : a `t_INT` or a `t_INTMOD` whose modulus is equal to  $*p$ , (if  $*p$  not NULL), in that case return 1, else 0. If a modulus is found it is put in  $*p$ , else  $*p$  is left unchanged.

`int RgV_is_FpV(GEN z, GEN *p)`,  $z$  a `t_VEC` (resp. `t_COL`), checks if it can be mapped to a FpV (resp. FpC), by checking `Rg_is_Fp` coefficientwise.

`int RgM_is_FpM(GEN z, GEN *p)`,  $z$  a `t_MAT`, checks if it can be mapped to a FpM, by checking `RgV_is_FpV` columnwise.

`GEN Rg_to_Fp(GEN z, GEN p)`,  $z$  a scalar which can be mapped to  $\mathbf{Z}/p\mathbf{Z}$ : a `t_INT`, a `t_INTMOD` whose modulus is divisible by  $p$ , a `t_FRAC` whose denominator is coprime to  $p$ , or a `t_PADIC` with underlying prime  $\ell$  satisfying  $p = \ell^n$  for some  $n$  (less than the accuracy of the input). Returns `lift(z * Mod(1,p))`, normalized.



GEN padic\_to\_Fp(GEN x, GEN p) special case of Rg\_to\_Fp, for a  $x$  a `t_PADIC`.

GEN RgV\_to\_FpV(GEN z, GEN p),  $z$  a `t_VEC` or `t_COL`, returns the `FpV` (as a `t_VEC`) obtained by applying `Rg_to_Fp` coefficientwise.

GEN RgC\_to\_FpC(GEN z, GEN p),  $z$  a `t_VEC` or `t_COL`, returns the `FpC` (as a `t_COL`) obtained by applying `Rg_to_Fp` coefficientwise.

GEN RgM\_to\_FpM(GEN z, GEN p),  $z$  a `t_MAT`, returns the `FpM` obtained by applying `RgC_to_FpC` columnwise.

GEN RgM\_Fp\_init(GEN z, GEN p, ulong \*pp), given an `RgM`  $z$ , whose entries can be mapped to  $\mathbf{F}_p$  (as per `Rg_to_Fp`), and a prime number  $p$ . This routine returns a normal form of  $z$ : either an `F2m` ( $p = 2$ ), an `F1m` ( $p$  fits into an `ulong`) or an `FpM`. In the first two cases, `pp` is set to `itou(p)`, and to 0 in the last.

The functions above are generally used as follows:

```
GEN add(GEN x, GEN y)
{
    GEN p = NULL;
    if (Rg_is_Fp(x, &p) && Rg_is_Fp(y, &p) && p)
    {
        x = Rg_to_Fp(x, p); y = Rg_to_Fp(y, p);
        z = Fp_add(x, y, p);
        return Fp_to_mod(z);
    }
    else return gadd(x, y);
}
```

GEN FpC\_red(GEN z, GEN p),  $z$  a `ZC`. Returns `lift(Col(z) * Mod(1,p))`, hence a `t_COL`.

GEN FpV\_red(GEN z, GEN p),  $z$  a `ZV`. Returns `lift(Vec(z) * Mod(1,p))`, hence a `t_VEC`.

GEN FpM\_red(GEN z, GEN p),  $z$  a `ZM`. Returns `lift(z * Mod(1,p))`, which is an `FpM`.

### 7.3.1.2 Basic operations.

GEN random\_FpC(long n, GEN p) returns a random `FpC` with  $n$  components.

GEN random\_FpV(long n, GEN p) returns a random `FpV` with  $n$  components.

GEN FpC\_center(GEN z, GEN p, GEN pov2) returns a `t_COL` whose entries are the `Fp_center` of the `gel(z,i)`.

GEN FpM\_center(GEN z, GEN p, GEN pov2) returns a matrix whose entries are the `Fp_center` of the `gcoeff(z,i,j)`.

void FpC\_center\_inplace(GEN z, GEN p, GEN pov2) in-place version of `FpC_center`, using `affii`.

void FpM\_center\_inplace(GEN z, GEN p, GEN pov2) in-place version of `FpM_center`, using `affii`.

GEN FpC\_add(GEN x, GEN y, GEN p) adds the `ZC`  $x$  and  $y$  and reduce modulo  $p$  to obtain an `FpC`.

GEN FpV\_add(GEN x, GEN y, GEN p) same as `FpC_add`, returning and `FpV`.

GEN FpM\_add(GEN x, GEN y, GEN p) adds the two ZMs x and y (assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpC\_sub(GEN x, GEN y, GEN p) subtracts the ZC y to the ZC x and reduce modulo p to obtain an FpC.

GEN FpV\_sub(GEN x, GEN y, GEN p) same as FpC\_sub, returning and FpV.

GEN FpM\_sub(GEN x, GEN y, GEN p) subtracts the two ZMs x and y (assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpC\_Fp\_mul(GEN x, GEN y, GEN p) multiplies the ZC x (seen as a column vector) by the t\_INT y and reduce modulo p to obtain an FpC.

GEN FpM\_Fp\_mul(GEN x, GEN y, GEN p) multiplies the ZM x (seen as a column vector) by the t\_INT y and reduce modulo p to obtain an FpM.

GEN FpC\_FpV\_mul(GEN x, GEN y, GEN p) multiplies the ZC x (seen as a column vector) by the ZV y (seen as a row vector, assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpM\_mul(GEN x, GEN y, GEN p) multiplies the two ZMs x and y (assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpM\_powu(GEN x, ulong n, GEN p) computes  $x^n$  where x is a square FpM.

GEN FpM\_FpC\_mul(GEN x, GEN y, GEN p) multiplies the ZM x by the ZC y (seen as a column vector, assumed to have compatible dimensions), and reduce modulo p to obtain an FpC.

GEN FpM\_FpC\_mul\_FpX(GEN x, GEN y, GEN p, long v) is a memory-clean version of

```
GEN tmp = FpM_FpC_mul(x,y,p);
return RgV_to_RgX(tmp, v);
```

GEN FpV\_FpC\_mul(GEN x, GEN y, GEN p) multiplies the ZV x (seen as a row vector) by the ZC y (seen as a column vector, assumed to have compatible dimensions), and reduce modulo p to obtain an Fp.

GEN FpV\_dotproduct(GEN x, GEN y, GEN p) scalar product of x and y (assumed to have the same length).

GEN FpV\_dotsquare(GEN x, GEN p) scalar product of x with itself. has t\_INT entries.

GEN FpV\_factorback(GEN L, GEN e, GEN p) given an FpV L and a ZV or zv e of the same length, return  $\prod_i L_i^{e_i}$  modulo p.

**7.3.1.3 Fp-linear algebra.** The implementations are not asymptotically efficient ( $O(n^3)$  standard algorithms).

GEN FpM\_deplin(GEN x, GEN p) returns a nontrivial kernel vector, or NULL if none exist.

GEN FpM\_det(GEN x, GEN p) as det

GEN FpM\_gauss(GEN a, GEN b, GEN p) as gauss, where a and b are FpM.

GEN FpM\_FpC\_gauss(GEN a, GEN b, GEN p) as gauss, where a is a FpM and b a FpC.

GEN FpM\_image(GEN x, GEN p) as image

GEN FpM\_intersect(GEN x, GEN y, GEN p) as intersect

GEN FpM\_intersect\_i(GEN x, GEN y, GEN p) internal variant of FpM\_intersect but the result is only a generating set, not necessarily an  $\mathbf{F}_p$ -basis. It is not gerepile-clean either, but suitable for gerepileupto.

GEN FpM\_inv(GEN x, GEN p) returns a left inverse of  $x$  (the inverse if  $x$  is square), or NULL if  $x$  is not invertible.

GEN FpM\_FpC\_invimage(GEN A, GEN y, GEN p) given an FpM  $A$  and an FpC  $y$ , returns an  $x$  such that  $Ax = y$ , or NULL if no such vector exist.

GEN FpM\_invimage(GEN A, GEN y, GEN p) given two FpM  $A$  and  $y$ , returns  $x$  such that  $Ax = y$ , or NULL if no such matrix exist.

GEN FpM\_ker(GEN x, GEN p) as ker

long FpM\_rank(GEN x, GEN p) as rank

GEN FpM\_indexrank(GEN x, GEN p) as indexrank

GEN FpM\_suppl(GEN x, GEN p) as suppl

GEN FpM\_hess(GEN x, GEN p) upper Hessenberg form of  $x$  over  $\mathbf{F}_p$ .

GEN FpM\_charpoly(GEN x, GEN p) characteristic polynomial of  $x$ .

#### 7.3.1.4 FqC, FqM and Fq-linear algebra.

An FqM (resp. FqC) is a matrix (resp a t\_COL) with Fq coefficients (with respect to given T, p), not necessarily reduced (i.e arbitrary t\_INTs and ZXs in the same variable as T).

GEN RgC\_to\_FqC(GEN z, GEN T, GEN p)

GEN RgM\_to\_FqM(GEN z, GEN T, GEN p)

GEN FqC\_add(GEN a, GEN b, GEN T, GEN p)

GEN FqC\_sub(GEN a, GEN b, GEN T, GEN p)

GEN FqC\_FqC\_mul(GEN a, GEN b, GEN T, GEN p)

GEN FqC\_FqV\_mul(GEN a, GEN b, GEN T, GEN p)

GEN FqM\_FqC\_gauss(GEN a, GEN b, GEN T, GEN p) as gauss, where  $b$  is a FqC.

GEN FqM\_FqC\_invimage(GEN a, GEN b, GEN T, GEN p)

GEN FqM\_FqC\_mul(GEN a, GEN b, GEN T, GEN p)

GEN FqM\_deplin(GEN x, GEN T, GEN p) returns a nontrivial kernel vector, or NULL if none exist.

GEN FqM\_det(GEN x, GEN T, GEN p) as det

GEN FqM\_gauss(GEN a, GEN b, GEN T, GEN p) as gauss, where  $b$  is a FqM.

GEN FqM\_image(GEN x, GEN T, GEN p) as image

GEN FqM\_indexrank(GEN x, GEN T, GEN p) as indexrank

GEN FqM\_inv(GEN x, GEN T, GEN p) returns the inverse of  $x$ , or NULL if  $x$  is not invertible.

GEN FqM\_invimage(GEN a, GEN b, GEN T, GEN p) as invimage

GEN FqM\_ker(GEN x, GEN T, GEN p) as ker

`GEN FqM_mul(GEN a, GEN b, GEN T, GEN p)`  
`long FqM_rank(GEN x, GEN T, GEN p)` as rank  
`GEN FqM_suppl(GEN x, GEN T, GEN p)` as suppl  
**7.3.2 Flc / Flv, Flm.** See FpV, FpM operations.  
`GEN Flv_copy(GEN x)` returns a copy of  $x$ .  
`GEN Flv_center(GEN z, ulong p, ulong ps2)`  
`GEN random_Flv(long n, ulong p)` returns a random Flv with  $n$  components.  
`GEN Flm_copy(GEN x)` returns a copy of  $x$ .  
`GEN matid_Flm(long n)` returns an Flm which is an  $n \times n$  identity matrix.  
`GEN scalar_Flm(long s, long n)` returns an Flm which is  $s$  times the  $n \times n$  identity matrix.  
`GEN Flm_center(GEN z, ulong p, ulong ps2)`  
`GEN Flm_Fl_add(GEN x, ulong y, ulong p)` returns  $x + y * \text{Id}$  ( $x$  must be square).  
`GEN Flm_Fl_sub(GEN x, ulong y, ulong p)` returns  $x - y * \text{Id}$  ( $x$  must be square).  
`GEN Flm_Flc_mul(GEN x, GEN y, ulong p)` multiplies  $x$  and  $y$  (assumed to have compatible dimensions).  
`GEN Flm_Flc_mul_pre(GEN x, GEN y, ulong p, ulong pi)` multiplies  $x$  and  $y$  (assumed to have compatible dimensions), assuming  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume  $\text{SMALL\_ULONG}(p)$ .  
`GEN Flc_Flv_mul(GEN x, GEN y, ulong p)` multiplies the column vector  $x$  by the row vector  $y$ . The result is a matrix.  
`GEN Flm_Flc_mul_pre_Flx(GEN x, GEN y, ulong p, ulong pi, long sv)` return `Flv_to_Flx(Flm_Flc_mul_pre(x, y, p, pi), sv)`.  
`GEN Flm_Fl_mul(GEN x, ulong y, ulong p)` multiplies the Flm  $x$  by  $y$ .  
`GEN Flm_Fl_mul_pre(GEN x, ulong y, ulong p, ulong pi)` multiplies the Flm  $x$  by  $y$  assuming  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume  $p < B^{1/2}$  is small.  
`GEN Flm_neg(GEN x, ulong p)` negates the Flm  $x$ .  
`void Flm_Fl_mul_inplace(GEN x, ulong y, ulong p)` replaces the Flm  $x$  by  $x * y$ .  
`GEN Flv_Fl_mul(GEN x, ulong y, ulong p)` multiplies the Flv  $x$  by  $y$ .  
`void Flv_Fl_mul_inplace(GEN x, ulong y, ulong p)` replaces the Flc  $x$  by  $x * y$ .  
`void Flv_Fl_mul_part_inplace(GEN x, ulong y, ulong p, long l)` multiplies  $x[1..l]$  by  $y$  modulo  $p$ . In place.  
`GEN Flv_Fl_div(GEN x, ulong y, ulong p)` divides the Flv  $x$  by  $y$ .  
`void Flv_Fl_div_inplace(GEN x, ulong y, ulong p)` replaces the Flv  $x$  by  $x/y$ .  
`void Flc_lincomb1_inplace(GEN X, GEN Y, ulong v, ulong q)` sets  $X \leftarrow X + vY$ , where  $X, Y$  are Flc. Memory efficient (e.g. no-op if  $v = 0$ ), and gerepile-safe.

`GEN Flv_add(GEN x, GEN y, ulong p)` adds two Flv.  
`void Flv_add_inplace(GEN x, GEN y, ulong p)` replaces  $x$  by  $x + y$ .  
`GEN Flv_neg(GEN x, ulong p)` returns  $-x$ .  
`void Flv_neg_inplace(GEN x, ulong p)` replaces  $x$  by  $-x$ .  
`GEN Flv_sub(GEN x, GEN y, ulong p)` subtracts  $y$  to  $x$ .  
`void Flv_sub_inplace(GEN x, GEN y, ulong p)` replaces  $x$  by  $x - y$ .  
`ulong Flv_dotproduct(GEN x, GEN y, ulong p)` returns the scalar product of  $x$  and  $y$   
`ulong Flv_dotproduct_pre(GEN x, GEN y, ulong p, ulong pi)` returns the scalar product of  $x$  and  $y$  assuming  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume `SMALL_ULONG(p)`.  
`GEN Flv_factorback(GEN L, GEN e, ulong p)` given an Flv  $L$  and a zv  $e$  of the same length, return  $\prod_i L_i^{e_i}$  modulo  $p$ .  
`ulong Flv_sum(GEN x, ulong p)` returns the sum of the components of  $x$ .  
`ulong Flv_prod(GEN x, ulong p)` returns the product of the components of  $x$ .  
`ulong Flv_prod_pre(GEN x, ulong p, ulong pi)` as `Flv_prod` assuming  $pi$  is the pseudoinverse of  $p$ .  
`GEN Flv_inv(GEN x, ulong p)` returns the vector of inverses of the elements of  $x$  (as a Flv). Use Montgomery's trick.  
`void Flv_inv_inplace(GEN x, ulong p)` in place variant of `Flv_inv`.  
`GEN Flv_inv_pre(GEN x, ulong p, ulong pi)` as `Flv_inv` assuming  $pi$  is the pseudoinverse of  $p$ .  
`void Flv_inv_pre_inplace(GEN x, ulong p, ulong pi)` in place variant of `Flv_inv`.  
`GEN Flc_FpV_mul(GEN x, GEN y, GEN p)` multiplies  $x$  (seen as a column vector) by  $y$  (seen as a row vector, assumed to have compatible dimensions) to obtain an Flm.  
`GEN zero_Flm(long m, long n)` creates a Flm with  $m \times n$  components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.  
`GEN zero_Flm_copy(long m, long n)` creates a Flm with  $m \times n$  components set to 0.  
`GEN zero_Flv(long n)` creates a Flv with  $n$  components set to 0.  
`GEN Flm_row(GEN A, long x0)` return  $A[i, ]$ , the  $i$ -th row of the Flm  $A$ .  
`GEN Flm_add(GEN x, GEN y, ulong p)` adds  $x$  and  $y$  (assumed to have compatible dimensions).  
`GEN Flm_sub(GEN x, GEN y, ulong p)` subtracts  $x$  and  $y$  (assumed to have compatible dimensions).  
`GEN Flm_mul(GEN x, GEN y, ulong p)` multiplies  $x$  and  $y$  (assumed to have compatible dimensions).  
`GEN Flm_mul_pre(GEN x, GEN y, ulong p, ulong pi)` multiplies  $x$  and  $y$  (assumed to have compatible dimensions), assuming  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume `SMALL_ULONG(p)`.  
`GEN Flm_powers(GEN x, ulong n, ulong p)` returns  $[x^0, \dots, x^n]$  as a t\_VEC of Flms.

GEN Flm\_powu(GEN x, ulong n, ulong p) computes  $x^n$  where  $x$  is a square Flm.

GEN Flm\_charpoly(GEN x, ulong p) return the characteristic polynomial of the square Flm  $x$ , as a Flx.

GEN Flm\_deplin(GEN x, ulong p)

ulong Flm\_det(GEN x, ulong p)

ulong Flm\_det\_sp(GEN x, ulong p), as Flm\_det, in place (destroys x).

GEN Flm\_gauss(GEN a, GEN b, ulong p) as gauss, where  $b$  is a Flm.

GEN Flm\_Flc\_gauss(GEN a, GEN b, ulong p) as gauss, where  $b$  is a Flc.

GEN Flm\_indexrank(GEN x, ulong p)

GEN Flm\_inv(GEN x, ulong p)

GEN Flm\_adjoint(GEN x, ulong p) as matadjoint.

GEN Flm\_Flc\_invimage(GEN A, GEN y, ulong p) given an Flm  $A$  and an Flc  $y$ , returns an  $x$  such that  $Ax = y$ , or NULL if no such vector exist.

GEN Flm\_invimage(GEN A, GEN y, ulong p) given two Flm  $A$  and  $y$ , returns  $x$  such that  $Ax = y$ , or NULL if no such matrix exist.

GEN Flm\_ker(GEN x, ulong p)

GEN Flm\_ker\_sp(GEN x, ulong p, long deplin), as Flm\_ker (if deplin=0) or Flm\_deplin (if deplin=1), in place (destroys x).

long Flm\_rank(GEN x, ulong p)

long Flm\_suppl(GEN x, ulong p)

GEN Flm\_image(GEN x, ulong p)

GEN Flm\_intersect(GEN x, GEN y, ulong p)

GEN Flm\_intersect\_i(GEN x, GEN y, GEN p) internal variant of Flm\_intersect but the result is only a generating set, not necessarily an  $\mathbf{F}_p$ -basis. It *is* a basis if both  $x$  and  $y$  have independent columns. It is not gerepile-clean either, but suitable for gerepileupto.

GEN Flm\_transpose(GEN x)

GEN Flm\_hess(GEN x, ulong p) upper Hessenberg form of  $x$  over  $\mathbf{F}_p$ .

**7.3.3 F2c / F2v, F2m.** An F2v  $v$  is a `t_VECSMALL` representing a vector over  $\mathbf{F}_2$ . Specifically  $z[0]$  is the usual codeword,  $z[1]$  is the number of components of  $v$  and the coefficients are given by the bits of remaining words by increasing indices.

`ulong F2v_coeff(GEN x, long i)` returns the coefficient  $i \geq 1$  of  $x$ .

`void F2v_clear(GEN x, long i)` sets the coefficient  $i \geq 1$  of  $x$  to 0.

`int F2v_equal0(GEN x)` returns 1 if all entries are 0, and return 0 otherwise.

`void F2v_flip(GEN x, long i)` adds 1 to the coefficient  $i \geq 1$  of  $x$ .

`void F2v_set(GEN x, long i)` sets the coefficient  $i \geq 1$  of  $x$  to 1.

`void F2v_copy(GEN x)` returns a copy of  $x$ .

`GEN F2v_slice(GEN x, long a, long b)` returns the F2v with entries  $x[a], \dots, x[b]$ . Assumes  $a \leq b$ .

`ulong F2m_coeff(GEN x, long i, long j)` returns the coefficient  $(i, j)$  of  $x$ .

`void F2m_clear(GEN x, long i, long j)` sets the coefficient  $(i, j)$  of  $x$  to 0.

`void F2m_flip(GEN x, long i, long j)` adds 1 to the coefficient  $(i, j)$  of  $x$ .

`void F2m_set(GEN x, long i, long j)` sets the coefficient  $(i, j)$  of  $x$  to 1.

`GEN F2m_copy(GEN x)` returns a copy of  $x$ .

`GEN F2m_transpose(GEN x)` returns the transpose of  $x$ .

`GEN F2m_row(GEN x, long j)` returns the F2v which corresponds to the  $j$ -th row of the F2m  $x$ .

`GEN F2m_rowslice(GEN x, long a, long b)` returns the F2m built from the  $a$ -th to  $b$ -th rows of the F2m  $x$ . Assumes  $a \leq b$ .

`GEN F2m_F2c_mul(GEN x, GEN y)` multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

`GEN F2m_image(GEN x)` gives a subset of the columns of  $x$  that generate the image of  $x$ .

`GEN F2m_invimage(GEN A, GEN B)`

`GEN F2m_F2c_invimage(GEN A, GEN y)`

`GEN F2m_gauss(GEN a, GEN b)` as `gauss`, where  $b$  is a F2m.

`GEN F2m_F2c_gauss(GEN a, GEN b)` as `gauss`, where  $b$  is a F2c.

`GEN F2m_indexrank(GEN x)`  $x$  being a matrix of rank  $r$ , returns a vector with two `t_VECSMALL` components  $y$  and  $z$  of length  $r$  giving a list of rows and columns respectively (starting from 1) such that the extracted matrix obtained from these two vectors using `vecextract(x, y, z)` is invertible.

`GEN F2m_mul(GEN x, GEN y)` multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

`GEN F2m_powu(GEN x, ulong n)` computes  $x^n$  where  $x$  is a square F2m.

`long F2m_rank(GEN x)` as `rank`.

`long F2m_suppl(GEN x)` as `suppl`.

`GEN matid_F2m(long n)` returns an F2m which is an  $n \times n$  identity matrix.

`GEN zero_F2v(long n)` creates a F2v with  $n$  components set to 0.

GEN `const_F2v(long n)` creates a `F2v` with `n` components set to 1.

GEN `F2v_ei(long n, long i)` creates a `F2v` with `n` components set to 0, but for the  $i$ -th one, which is set to 1 ( $i$ -th vector in the canonical basis).

GEN `zero_F2m(long m, long n)` creates a `F1m` with  $m \times n$  components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

GEN `zero_F2m_copy(long m, long n)` creates a `F2m` with  $m \times n$  components set to 0.

GEN `F2v_to_F1v(GEN x)`

GEN `F2c_to_ZC(GEN x)`

GEN `ZV_to_F2v(GEN x)`

GEN `RgV_to_F2v(GEN x)`

GEN `F2m_to_F1m(GEN x)`

GEN `F2m_to_ZM(GEN x)`

GEN `F1v_to_F2v(GEN x)`

GEN `F1m_to_F2m(GEN x)`

GEN `ZM_to_F2m(GEN x)`

GEN `RgM_to_F2m(GEN x)`

void `F2v_add_inplace(GEN x, GEN y)` replaces  $x$  by  $x + y$ . It is allowed for  $y$  to be shorter than  $x$ .

void `F2v_and_inplace(GEN x, GEN y)` replaces  $x$  by the term-by term product of  $x$  and  $y$  (which is the logical and). It is allowed for  $y$  to be shorter than  $x$ .

void `F2v_negimply_inplace(GEN x, GEN y)` replaces  $x$  by the term-by term logical **and not** of  $x$  and  $y$ . It is allowed for  $y$  to be shorter than  $x$ .

void `F2v_or_inplace(GEN x, GEN y)` replaces  $x$  by the term-by term logical **or** of  $x$  and  $y$ . It is allowed for  $y$  to be shorter than  $x$ .

int `F2v_subset(GEN x, GEN y)` return 1 if the set of indices of non-zero components of  $y$  is a subset of the set of indices of non-zero components of  $x$ , 0 otherwise.

ulong `F2v_hamming(GEN x)` returns the Hamming weight of  $x$ , that is the number of nonzero entries.

ulong `F2m_det(GEN x)`

ulong `F2m_det_sp(GEN x)`, as `F2m_det`, in place (destroys  $x$ ).

GEN `F2m_deplin(GEN x)`

ulong `F2v_dotproduct(GEN x, GEN y)` returns the scalar product of  $x$  and  $y$

GEN `F2m_inv(GEN x)`

GEN `F2m_ker(GEN x)`

GEN `F2m_ker_sp(GEN x, long deplin)`, as `F2m_ker` (if `deplin=0`) or `F2m_deplin` (if `deplin=1`), in place (destroys  $x$ ).



**7.3.4 F3c / F3v, F3m.** An F3v  $v$  is a `t_VECSMALL` representing a vector over  $\mathbf{F}_3$ . Specifically  $z[0]$  is the usual codeword,  $z[1]$  is the number of components of  $v$  and the coefficients are given by pair of adjacent bits of remaining words by increasing indices, with the coding  $00 \mapsto 0, 01 \mapsto 1, 10 \mapsto 2$  and  $11$  is undefined.

`ulong F3v_coeff(GEN x, long i)` returns the coefficient  $i \geq 1$  of  $x$ .

`void F3v_clear(GEN x, long i)` sets the coefficient  $i \geq 1$  of  $x$  to 0.

`void F3v_set(GEN x, long i, ulong n)` sets the coefficient  $i \geq 1$  of  $x$  to  $n < 3$ ,

`ulong F3m_coeff(GEN x, long i, long j)` returns the coefficient  $(i, j)$  of  $x$ .

`void F3m_set(GEN x, long i, long j, ulong n)` sets the coefficient  $(i, j)$  of  $x$  to  $n < 3$ .

`GEN F3m_copy(GEN x)` returns a copy of  $x$ .

`GEN F3m_transpose(GEN x)` returns the transpose of  $x$ .

`GEN F3m_row(GEN x, long j)` returns the F3v which corresponds to the  $j$ -th row of the F3m  $x$ .

`GEN F3m_ker(GEN x)`

`GEN F3m_ker_sp(GEN x, long deplin)`, as `F3m_ker` (if `deplin=0`) or `F2m_deplin` (if `deplin=1`), in place (destroys  $x$ ).

`GEN F3m_mul(GEN x, GEN y)` multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

`GEN zero_F3v(long n)` creates a F3v with  $n$  components set to 0.

`GEN zero_F3m_copy(long m, long n)` creates a F3m with  $m \times n$  components set to 0.

`GEN F3v_to_Flv(GEN x)`

`GEN ZV_to_F3v(GEN x)`

`GEN RgV_to_F3v(GEN x)`

`GEN F3c_to_ZC(GEN x)`

`GEN F3m_to_Flm(GEN x)`

`GEN F3m_to_ZM(GEN x)`

`GEN Flv_to_F3v(GEN x)`

`GEN Flm_to_F3m(GEN x)`

`GEN ZM_to_F3m(GEN x)`

`GEN RgM_to_F3m(GEN x)`

**7.3.5** FlxqV, FlxqC, FlxqM. See FqV, FqC, FqM operations.

GEN FlxqV\_dotproduct(GEN x, GEN y, GEN T, ulong p) as FpV\_dotproduct.

GEN FlxqV\_dotproduct\_pre(GEN x, GEN y, GEN T, ulong p, ulong pi) where  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume  $\text{SMALL\_ULONG}(p)$ .

GEN FlxM\_Flx\_add\_shallow(GEN x, GEN y, ulong p) as RgM\_Rg\_add\_shallow.

GEN FlxqC\_Flxq\_mul(GEN x, GEN y, GEN T, ulong p)

GEN FlxqM\_Flxq\_mul(GEN x, GEN y, GEN T, ulong p)

GEN FlxqM\_FlxqC\_gauss(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM\_FlxqC\_invimage(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM\_FlxqC\_mul(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM\_deplin(GEN x, GEN T, ulong p)

GEN FlxqM\_det(GEN x, GEN T, ulong p)

GEN FlxqM\_gauss(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM\_image(GEN x, GEN T, ulong p)

GEN FlxqM\_indexrank(GEN x, GEN T, ulong p)

GEN FlxqM\_inv(GEN x, GEN T, ulong p)

GEN FlxqM\_invimage(GEN a, GEN b, GEN T, ulong p)

GEN FlxqM\_ker(GEN x, GEN T, ulong p)

GEN FlxqM\_mul(GEN a, GEN b, GEN T, ulong p)

long FlxqM\_rank(GEN x, GEN T, ulong p)

GEN FlxqM\_suppl(GEN x, GEN T, ulong p)

GEN matid\_FlxqM(long n, GEN T, ulong p)

**7.3.6** FpX. Let  $p$  an understood  $t\_INT$ , to be given in the function arguments; in practice  $p$  is not assumed to be prime, but be wary. Recall than an  $Fp$  object is a  $t\_INT$ , preferably belonging to  $[0, p - 1]$ ; an  $FpX$  is a  $t\_POL$  in a fixed variable whose coefficients are  $Fp$  objects. Unless mentioned otherwise, all outputs in this section are  $FpX$ s. All operations are understood to take place in  $(\mathbf{Z}/p\mathbf{Z})[X]$ .

**7.3.6.1 Conversions.** In what follows  $p$  is always a  $t\_INT$ , not necessarily prime.

int RgX\_is\_FpX(GEN z, GEN \*p),  $z$  a  $t\_POL$ , checks if it can be mapped to a  $FpX$ , by checking  $Rg\_is\_Fp$  coefficientwise.

GEN RgX\_to\_FpX(GEN z, GEN p),  $z$  a  $t\_POL$ , returns the  $FpX$  obtained by applying  $Rg\_to\_Fp$  coefficientwise.

GEN FpX\_red(GEN z, GEN p),  $z$  a  $ZX$ , returns  $\text{lift}(z * \text{Mod}(1, p))$ , normalized.

GEN FpXV\_red(GEN z, GEN p),  $z$  a  $t\_VEC$  of  $ZX$ . Applies  $FpX\_red$  componentwise and returns the result (and we obtain a vector of  $FpX$ s).

GEN FpXT\_red(GEN z, GEN p),  $z$  a tree of  $ZX$ . Applies  $FpX\_red$  to each leaf and returns the result (and we obtain a tree of  $FpX$ s).

**7.3.6.2 Basic operations.** In what follows  $p$  is always a  $\mathbf{t\_INT}$ , not necessarily prime.

Now, except for  $p$ , the operands and outputs are all  $\mathbf{FpX}$  objects. Results are undefined on other inputs.

$\mathbf{GEN\ FpX\_add}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  adds  $x$  and  $y$ .

$\mathbf{GEN\ FpX\_neg}(\mathbf{GEN\ x}, \mathbf{GEN\ p})$  returns  $-x$ , the components are between 0 and  $p$  if this is the case for the components of  $x$ .

$\mathbf{GEN\ FpX\_renormalize}(\mathbf{GEN\ x}, \mathbf{long\ l})$ , as  $\mathbf{normalizepol}$ , where  $l = \mathbf{lg}(x)$ , in place.

$\mathbf{GEN\ FpX\_sub}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns  $x - y$ .

$\mathbf{GEN\ FpX\_halve}(\mathbf{GEN\ x}, \mathbf{GEN\ p})$  returns  $z$  such that  $2z = x$  modulo  $p$  assuming such  $z$  exists.

$\mathbf{GEN\ FpX\_mul}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns  $xy$ .

$\mathbf{GEN\ FpX\_mulspec}(\mathbf{GEN\ a}, \mathbf{GEN\ b}, \mathbf{GEN\ p}, \mathbf{long\ na}, \mathbf{long\ nb})$  see  $\mathbf{ZX\_mulspec}$

$\mathbf{GEN\ FpX\_sqr}(\mathbf{GEN\ x}, \mathbf{GEN\ p})$  returns  $x^2$ .

$\mathbf{GEN\ FpX\_powu}(\mathbf{GEN\ x}, \mathbf{ulong\ n}, \mathbf{GEN\ p})$  returns  $x^n$ .

$\mathbf{GEN\ FpX\_convol}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  return the-term by-term product of  $x$  and  $y$ .

$\mathbf{GEN\ FpX\_divrem}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p}, \mathbf{GEN\ *pr})$  returns the quotient of  $x$  by  $y$ , and sets  $\mathbf{pr}$  to the remainder.

$\mathbf{GEN\ FpX\_div}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns the quotient of  $x$  by  $y$ .

$\mathbf{GEN\ FpX\_div\_by\_X\_x}(\mathbf{GEN\ A}, \mathbf{GEN\ a}, \mathbf{GEN\ p}, \mathbf{GEN\ *r})$  returns the quotient of the  $\mathbf{FpX\ A}$  by  $(X - a)$ , and sets  $\mathbf{r}$  to the remainder  $\mathbf{A(a)}$ .

$\mathbf{GEN\ FpX\_rem}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns the remainder  $x \bmod y$ .

$\mathbf{long\ FpX\_valrem}(\mathbf{GEN\ x}, \mathbf{GEN\ t}, \mathbf{GEN\ p}, \mathbf{GEN\ *r})$  The arguments  $x$  and  $e$  being nonzero  $\mathbf{FpX}$  returns the highest exponent  $e$  such that  $\mathbf{t}^e$  divides  $x$ . The quotient  $x/\mathbf{t}^e$  is returned in  $\mathbf{*r}$ . In particular, if  $\mathbf{t}$  is irreducible, this returns the valuation at  $\mathbf{t}$  of  $x$ , and  $\mathbf{*r}$  is the prime-to- $\mathbf{t}$  part of  $x$ .

$\mathbf{GEN\ FpX\_deriv}(\mathbf{GEN\ x}, \mathbf{GEN\ p})$  returns the derivative of  $x$ . This function is not memory-clean, but nevertheless suitable for  $\mathbf{gerepileupto}$ .

$\mathbf{GEN\ FpX\_integ}(\mathbf{GEN\ x}, \mathbf{GEN\ p})$  returns the primitive of  $x$  whose constant term is 0.

$\mathbf{GEN\ FpX\_digits}(\mathbf{GEN\ x}, \mathbf{GEN\ B}, \mathbf{GEN\ p})$  returns a vector of  $\mathbf{FpX\ [c_0, \dots, c_n]}$  of degree less than the degree of  $B$  and such that  $x = \sum_{i=0}^n c_i B^i$ .

$\mathbf{GEN\ FpXV\_FpX\_fromdigits}(\mathbf{GEN\ v}, \mathbf{GEN\ B}, \mathbf{GEN\ p})$  where  $v = [c_0, \dots, c_n]$  is a vector of  $\mathbf{FpX}$ , returns  $\sum_{i=0}^n c_i B^i$ .

$\mathbf{GEN\ FpX\_translate}(\mathbf{GEN\ P}, \mathbf{GEN\ c}, \mathbf{GEN\ p})$  let  $c$  be an  $\mathbf{Fp}$  and let  $P$  be an  $\mathbf{FpX}$ ; returns the translated  $\mathbf{FpX}$  of  $P(X + c)$ .

$\mathbf{GEN\ FpX\_gcd}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns a (not necessarily monic) greatest common divisor of  $x$  and  $y$ .

$\mathbf{GEN\ FpX\_halfgcd}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns a two-by-two  $\mathbf{FpXM\ M}$  with determinant  $\pm 1$  such that the image  $(a, b)$  of  $(x, y)$  by  $M$  has the property that  $\deg a \geq \frac{\deg x}{2} > \deg b$ .

GEN FpX\_extgcd(GEN x, GEN y, GEN p, GEN \*u, GEN \*v) returns  $d = \text{GCD}(x, y)$  (not necessarily monic), and sets \*u, \*v to the Bezout coefficients such that  $*ux + *vy = d$ . If \*u is set to NULL, it is not computed which is a bit faster. This is useful when computing the inverse of  $y$  modulo  $x$ .

GEN FpX\_center(GEN z, GEN p, GEN pov2) returns the polynomial whose coefficient belong to the symmetric residue system. Assumes the coefficients already belong to  $] -p/2, p[$  and that pov2 is shifti(p, -1).

GEN FpX\_center\_i(GEN z, GEN p, GEN pov2) internal variant of FpX\_center, not gerepile-safe.

GEN FpX\_Frobenius(GEN T, GEN p) returns  $X^p \pmod{T(X)}$ .

GEN FpX\_matFrobenius(GEN T, GEN p) returns the matrix of the Frobenius automorphism  $x \mapsto x^p$  over the power basis of  $\mathbf{F}_p[X]/(T)$ .

**7.3.6.3 Mixed operations.** The following functions implement arithmetic operations between FpX and Fp operands, the result being of type FpX. The integer p need not be prime.

GEN Z\_to\_FpX(GEN x, GEN p, long v) converts a t\_INT to a scalar polynomial in variable  $v$ , reduced modulo  $p$ .

GEN FpX\_Fp\_add(GEN y, GEN x, GEN p) add the Fp x to the FpX y.

GEN FpX\_Fp\_add\_shallow(GEN y, GEN x, GEN p) add the Fp x to the FpX y, using a shallow copy (result not suitable for gerepileupto)

GEN FpX\_Fp\_sub(GEN y, GEN x, GEN p) subtract the Fp x from the FpX y.

GEN FpX\_Fp\_sub\_shallow(GEN y, GEN x, GEN p) subtract the Fp x from the FpX y, using a shallow copy (result not suitable for gerepileupto)

GEN Fp\_FpX\_sub(GEN x, GEN y, GEN p) returns  $x - y$ , where  $x$  is a t\_INT and  $y$  an FpX.

GEN FpX\_Fp\_mul(GEN x, GEN y, GEN p) multiplies the FpX x by the Fp y.

GEN FpX\_Fp\_mulspec(GEN x, GEN y, GEN p, long lx) see ZX\_mulspec

GEN FpX\_mulu(GEN x, ulong y, GEN p) multiplies the FpX x by y.

GEN FpX\_Fp\_mul\_to\_monic(GEN y, GEN x, GEN p) returns  $yx$  assuming the result is monic of the same degree as  $y$  (in particular  $x \neq 0$ ).

GEN FpX\_Fp\_div(GEN x, GEN y, GEN p) divides the FpX x by the Fp y.

GEN FpX\_divu(GEN x, ulong y, GEN p) divides the FpX x by y.

**7.3.6.4 Miscellaneous operations.**

GEN FpX\_normalize(GEN z, GEN p) divides the FpX z by its leading coefficient. If the latter is 1, z itself is returned, not a copy. If not, the inverse remains uncollected on the stack.

GEN FpX\_invBarrett(GEN T, GEN p), returns the Barrett inverse  $M$  of  $T$  defined by  $M(x)x^n \times T(1/x) \equiv 1 \pmod{x^{n-1}}$  where  $n$  is the degree of  $T$ .

GEN FpX\_rescale(GEN P, GEN h, GEN p) returns  $h^{\deg(P)} P(x/h)$ . P is an FpX and h is a nonzero Fp (the routine would work with any nonzero t\_INT but is not efficient in this case). Neither memory-clean nor suitable for gerepileupto.

GEN FpX\_eval(GEN x, GEN y, GEN p) evaluates the FpX x at the Fp y. The result is an Fp.

GEN FpX\_FpV\_multieval(GEN P, GEN v, GEN p) returns the vector  $[P(v[1]), \dots, P(v[n])]$  as a FpV.

GEN FpX\_dotproduct(GEN x, GEN y, GEN p) return the scalar product  $\sum_{i \geq 0} x_i y_i$  of the coefficients of  $x$  and  $y$ .

GEN FpXV\_FpC\_mul(GEN V, GEN W, GEN p) multiplies a nonempty line vector of FpX by a column vector of Fp of compatible dimensions. The result is an FpX.

GEN FpXV\_prod(GEN V, GEN p),  $V$  being a vector of FpX, returns their product.

GEN FpXV\_factorback(GEN L, GEN e, GEN p, long v) returns  $\prod_i L_i^{e_i}$  where  $L$  is a vector of FpXs in the variable  $v$  and  $e$  a vector of non-negative t\_INTs or a t\_VECSMALL.

GEN FpV\_roots\_to\_pol(GEN V, GEN p, long v),  $V$  being a vector of INTs, returns the monic FpX  $\prod_i (\text{pol\_x}[v] - V[i])$ .

GEN FpX\_chinese\_coprime(GEN x, GEN y, GEN Tx, GEN Ty, GEN Tz, GEN p): returns an FpX, congruent to  $x \bmod Tx$  and to  $y \bmod Ty$ . Assumes  $Tx$  and  $Ty$  are coprime, and  $Tz = Tx * Ty$  or NULL (in which case it is computed within).

GEN FpV\_polint(GEN x, GEN y, GEN p, long v) returns the FpX interpolation polynomial with value  $y[i]$  at  $x[i]$ . Assumes lengths are the same, components are t\_INTs, and the  $x[i]$  are distinct modulo  $p$ .

GEN FpV\_FpM\_polint(GEN x, GEN V, GEN p, long v) equivalent (but faster) to applying FpV\_polint( $x, \dots$ ) to all the elements of the vector  $V$  (thus, returns a FpXV).

GEN FpX\_FpXV\_multirem(GEN A, GEN P, GEN p) given a FpX  $A$  and a vector  $P$  of pairwise coprime FpX of length  $n \geq 1$ , return a vector  $B$  of the same length such that  $B[i] = A \pmod{P[i]}$  and  $B[i]$  of minimal degree for all  $1 \leq i \leq n$ .

GEN FpXV\_chinese(GEN A, GEN P, GEN p, GEN \*pM) let  $P$  be a vector of pairwise coprime FpX, let  $A$  be a vector of FpX of the same length  $n \geq 1$  and let  $M$  be the product of the elements of  $P$ . Returns a FpX of minimal degree congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If  $pM$  is not NULL, set  $*pM$  to  $M$ .

GEN FpV\_invVandermonde(GEN L, GEN d, GEN p)  $L$  being a FpV of length  $n$ , return the inverse  $M$  of the Vandermonde matrix attached to the elements of  $L$ , eventually multiplied by  $d$  if it is not NULL. If  $A$  is a FpV and  $B = MA$ , then the polynomial  $P = \sum_{i=1}^n B[i]X^{i-1}$  verifies  $P(L[i]) = dA[i]$  for  $1 \leq i \leq n$ .

int FpX\_is\_squarefree(GEN f, GEN p) returns 1 if the FpX  $f$  is squarefree, 0 otherwise.

int FpX\_is\_irred(GEN f, GEN p) returns 1 if the FpX  $f$  is irreducible, 0 otherwise. Assumes that  $p$  is prime. If  $f$  has few factors,  $\text{FpX\_nbfact}(f, p) == 1$  is much faster.

int FpX\_is\_totally\_split(GEN f, GEN p) returns 1 if the FpX  $f$  splits into a product of distinct linear factors, 0 otherwise. Assumes that  $p$  is prime. The 0 polynomial is not totally split.

long FpX\_ispower(GEN f, ulong k, GEN p, GEN \*pt) return 1 if the FpX  $f$  is a  $k$ -th power, 0 otherwise. If  $pt$  is not NULL, set it to  $g$  such that  $g^k = f$ .

GEN FpX\_factor(GEN f, GEN p), factors the FpX  $f$ . Assumes that  $p$  is prime. The returned value  $v$  is a t\_VEC with two components:  $v[1]$  is a vector of distinct irreducible (FpX) factors, and  $v[2]$  is a t\_VECSMALL of corresponding exponents. The order of the factors is deterministic (the computation is not).

GEN FpX\_factor\_squarefree(GEN f, GEN p) returns the squarefree factorization of  $f$  modulo  $p$ . This is a vector  $[u_1, \dots, u_k]$  of squarefree and pairwise coprime FpX such that  $u_k \neq 1$  and  $f = \prod u_i^i$ . The other  $u_i$  may equal 1. Shallow function.

GEN FpX\_ddf(GEN f, GEN p) assuming that  $f$  is squarefree, returns the distinct degree factorization of  $f$  modulo  $p$ . The returned value  $v$  is a t\_VEC with two components:  $F=v[1]$  is a vector of (FpX) factors, and  $E=v[2]$  is a t\_VECSMALL, such that  $f$  is equal to the product of the  $F[i]$  and each  $F[i]$  is a product of irreducible factors of degree  $E[i]$ .

long FpX\_ddf\_degree(GEN f, GEN XP, GEN p) assuming that  $f$  is squarefree and that all its factors have the same degree, return the common degree, where  $XP$  is FpX\_Frobenius( $f$ ,  $p$ ).

long FpX\_nbfact(GEN f, GEN p), assuming the FpX  $f$  is squarefree, returns the number of its irreducible factors. Assumes that  $p$  is prime.

long FpX\_nbfact\_Frobenius(GEN f, GEN XP, GEN p), as FpX\_nbfact( $f$ ,  $p$ ) but faster, where  $XP$  is FpX\_Frobenius( $f$ ,  $p$ ).

GEN FpX\_degfact(GEN f, GEN p), as FpX\_factor, but the degrees of the irreducible factors are returned instead of the factors themselves (as a t\_VECSMALL). Assumes that  $p$  is prime.

long FpX\_nbroots(GEN f, GEN p) returns the number of distinct roots in  $\mathbf{Z}/p\mathbf{Z}$  of the FpX  $f$ . Assumes that  $p$  is prime.

GEN FpX\_oneroot(GEN f, GEN p) returns one root in  $\mathbf{Z}/p\mathbf{Z}$  of the FpX  $f$ . Return NULL if no root exists. Assumes that  $p$  is prime.

GEN FpX\_oneroot\_split(GEN f, GEN p) as FpX\_oneroot. Faster when  $f$  is close to be totally split.

GEN FpX\_roots(GEN f, GEN p) returns the roots in  $\mathbf{Z}/p\mathbf{Z}$  of the FpX  $f$  (without multiplicity, as a vector of Fps). Assumes that  $p$  is prime.

GEN FpX\_roots\_mult(GEN f, long n, GEN p) returns the roots in  $\mathbf{Z}/p\mathbf{Z}$  with multiplicity at least  $n$  of the FpX  $f$  (without multiplicity, as a vector of Fps). Assumes that  $p$  is prime.

GEN FpX\_split\_part(GEN f, GEN p) returns the largest totally split squarefree factor of  $f$ .

GEN FpX\_factcyclo(ulong n, GEN p, ulong m) returns the factors of the  $n$ -th cyclotomic polynomial over Fp. if  $m = 1$  returns a single factor.

GEN random\_FpX(long d, long v, GEN p) returns a random FpX in variable  $v$ , of degree less than  $d$ .

GEN FpX\_resultant(GEN x, GEN y, GEN p) returns the resultant of  $x$  and  $y$ , both FpX. The result is a t\_INT belonging to  $[0, p-1]$ .

GEN FpX\_disc(GEN x, GEN p) returns the discriminant of the FpX  $x$ . The result is a t\_INT belonging to  $[0, p-1]$ .

GEN FpX\_FpXY\_resultant(GEN a, GEN b, GEN p), a a t\_POL of t\_INTs (say in variable  $X$ ), b a t\_POL (say in variable  $X$ ) whose coefficients are either t\_POLs in  $\mathbf{Z}[Y]$  or t\_INTs. Returns  $\text{Res}_X(a, b)$  in  $\mathbf{F}_p[Y]$  as an FpY. The function assumes that  $X$  has lower priority than  $Y$ .

GEN FpX\_Newton(GEN x, long n, GEN p) return  $\sum i = 0^{n-1} \pi_i X^i$  where  $\pi_i$  is the sum of the  $i$ th-power of the roots of  $x$  in an algebraic closure.

GEN FpX\_fromNewton(GEN x, GEN p) recover a polynomial from its Newton sums given by the coefficients of  $x$ . This function assumes that  $p$  and the accuracy of  $x$  as a FpXn is larger than the degree of the solution.

GEN FpX\_Laplace(GEN x, GEN p) return  $\sum_{i=0}^{n-1} x_i i! X^i$ .

GEN FpX\_invLaplace(GEN x, GEN p) return  $\sum_{i=0}^{n-1} x_i / i! X^i$ .

**7.3.7 FpXQ, Fq.** Let  $p$  a t\_INT and  $T$  an FpX for  $p$ , both to be given in the function arguments; an FpXQ object is an FpX whose degree is strictly less than the degree of  $T$ . An Fq is either an FpXQ or an Fp. Both represent a class in  $(\mathbf{Z}/p\mathbf{Z}[X]/(T))$ , in which all operations below take place. In addition, Fq routines also allow  $T = \text{NULL}$ , in which case no reduction mod  $T$  is performed on the result.

For efficiency, the routines in this section may leave small unused objects behind on the stack (their output is still suitable for gerepileupto). Besides  $T$  and  $p$ , arguments are either FpXQ or Fq depending on the function name. (All Fq routines accept FpXQs by definition, not the other way round.)

### 7.3.7.1 Preconditioned reduction.

For faster reduction, the modulus  $T$  can be replaced by an extended modulus in all FpXQ- and Fq-classes functions, and in FpX\_rem and FpX\_divrem. An extended modulus(FpXT, which is a tree whose leaves are FpX)In current implementation, an extended modulus is either a plain modulus (an FpX) or a pair of polynomials, one being the plain modulus  $T$  and the other being FpX\_invBarret( $T, p$ ).

GEN FpX\_get\_red(GEN eT, GEN p) returns the extended modulus eT.

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get\_FpX\_mod(GEN eT) returns the underlying modulus  $T$ .

GEN get\_FpX\_var(GEN eT) returns the variable number varn( $T$ ).

GEN get\_FpX\_degree(GEN eT) returns the degree degpol( $T$ ).

### 7.3.7.2 Conversions.

int ff\_parse\_Tp(GEN Tp, GEN \*T, GEN \*p, long red) Tp is either a prime number  $p$  or a t\_VEC with 2 entries  $T$  (an irreducible polynomial mod  $p$ ) and  $p$  (a prime number). Sets \*p and \*T to the corresponding GENs (NULL if undefined). If red is nonzero, normalize \*T as an FpX; on the other hand, to initialize a  $p$ -adic function, set red to 0 and \*T is left as is and must be a ZX to start with. Return 1 on success, and 0 on failure. This helper routine is used by GP functions such as factormod where a single user argument defines a finite field. t\_FFELT is not supported.

GEN Rg\_is\_FpXQ(GEN z, GEN \*T, GEN \*p), checks if  $z$  is a GEN which can be mapped to  $\mathbf{F}_p[X]/(T)$ : anything for which Rg\_is\_Fp return 1, a t\_POL for which RgX\_to\_FpX return 1, a t\_POLMOD whose modulus is equal to \*T if \*T is not NULL (once mapped to a FpX), or a t\_FFELT  $z$  with the same definition field as \*T if \*T is not NULL and is a t\_FFELT.

If an integer modulus is found it is put in \*p, else \*p is left unchanged. If a polynomial modulus is found it is put in \*T, if a t\_FFELT  $z$  is found,  $z$  is put in \*T, else \*T is left unchanged.

int RgX\_is\_FpXQX(GEN z, GEN \*T, GEN \*p),  $z$  a t\_POL, checks if it can be mapped to a FpXQX, by checking Rg\_is\_FpXQ coefficientwise.

GEN Rg\_to\_FpXQ(GEN z, GEN T, GEN p),  $z$  a GEN which can be mapped to  $\mathbf{F}_p[X]/(T)$ : anything Rg\_to\_Fp can be applied to, a t\_POL to which RgX\_to\_FpX can be applied to, a t\_POLMOD whose modulus is divisible by  $T$  (once mapped to a FpX), a suitable t\_RFRAC. Returns  $z$  as an FpXQ, normalized.

GEN Rg\_to\_Fq(GEN z, GEN T, GEN p), applies Rg\_to\_Fp if  $T$  is NULL and Rg\_to\_FpXQ otherwise.

GEN RgX\_to\_FpXQX(GEN z, GEN T, GEN p),  $z$  a t\_POL, returns the FpXQ obtained by applying Rg\_to\_FpXQ coefficientwise.

GEN RgX\_to\_FqX(GEN z, GEN T, GEN p): let  $z$  be a t\_POL; returns the FqX obtained by applying Rg\_to\_Fq coefficientwise.

GEN Fq\_to\_FpXQ(GEN z, GEN T, GEN p /\*unused\*/) if  $z$  is a t\_INT, convert it to a constant polynomial in the variable of  $T$ , otherwise return  $z$  (shallow function).

GEN Fq\_red(GEN x, GEN T, GEN p),  $x$  a ZX or t\_INT, reduce it to an Fq ( $T = \text{NULL}$  is allowed iff  $x$  is a t\_INT).

GEN FqX\_red(GEN x, GEN T, GEN p),  $x$  a t\_POL whose coefficients are ZXs or t\_INTs, reduce them to Fqs. (If  $T = \text{NULL}$ , as FpXX\_red( $x$ ,  $p$ )).

GEN FqV\_red(GEN x, GEN T, GEN p),  $x$  a vector of ZXs or t\_INTs, reduce them to Fqs. (If  $T = \text{NULL}$ , only reduce components mod  $p$  to FpXs or Fps.)

GEN FpXQ\_red(GEN x, GEN T, GEN p)  $x$  a t\_POL whose coefficients are t\_INTs, reduce them to FpXQs.

### 7.3.8 FpXQ.

GEN FpXQ\_add(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ\_sub(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ\_mul(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ\_sqr(GEN x, GEN T, GEN p)

GEN FpXQ\_div(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ\_inv(GEN x, GEN T, GEN p) computes the inverse of  $x$

GEN FpXQ\_invsafe(GEN x, GEN T, GEN p), as FpXQ\_inv, returning NULL if  $x$  is not invertible.

GEN FpXQ\_pow(GEN x, GEN n, GEN T, GEN p) computes  $x^n$ .

GEN FpXQ\_powu(GEN x, ulong n, GEN T, GEN p) computes  $x^n$  for small  $n$ .

In the following three functions the integer parameter `ord` can be given either as a positive t\_INT  $N$ , or as its factorization matrix  $faN$ , or as a pair  $[N, faN]$ . The parameter may be omitted by setting it to NULL (the value is then  $p^d - 1$ ,  $d = \deg T$ ).

GEN FpXQ\_log(GEN a, GEN g, GEN ord, GEN T, GEN p) Let  $g$  be of order dividing `ord` in the finite field  $\mathbf{F}_p[X]/(T)$ , return  $e$  such that  $a^e = g$ . If  $e$  does not exist, the result is undefined. Assumes that  $T$  is irreducible mod  $p$ .

GEN Fp\_FpXQ\_log(GEN a, GEN g, GEN ord, GEN T, GEN p) As FpXQ\_log,  $a$  being a Fp.

GEN FpXQ\_order(GEN a, GEN ord, GEN T, GEN p) returns the order of the FpXQ  $a$ . Assume that `ord` is a multiple of the order of  $a$ . Assume that  $T$  is irreducible mod  $p$ .



`int FpXQ_issquare(GEN x, GEN T, GEN p)` returns 1 if  $x$  is a square and 0 otherwise. Assumes that  $T$  is irreducible mod  $p$ .

`GEN FpXQ_sqrt(GEN x, GEN T, GEN p)` returns a square root of  $x$ . Return NULL if  $x$  is not a square.

`GEN FpXQ_sqrtn(GEN x, GEN n, GEN T, GEN p, GEN *zn)` Let  $T$  be irreducible mod  $p$  and  $q = p^{\deg T}$ ; returns NULL if  $a$  is not an  $n$ -th power residue mod  $p$ . Otherwise, returns an  $n$ -th root of  $a$ ; if  $zn$  is not NULL set it to a primitive  $m$ -th root of 1 in  $\mathbf{F}_q$ ,  $m = \gcd(q - 1, n)$  allowing to compute all  $m$  solutions in  $\mathbf{F}_q$  of the equation  $x^n = a$ .

### 7.3.9 Fq.

`GEN Fq_add(GEN x, GEN y, GEN T/*unused*/, GEN p)`

`GEN Fq_sub(GEN x, GEN y, GEN T/*unused*/, GEN p)`

`GEN Fq_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN Fq_Fp_mul(GEN x, GEN y, GEN T, GEN p)` multiplies the Fq  $x$  by the `t_INT`  $y$ .

`GEN Fq_mulu(GEN x, ulong y, GEN T, GEN p)` multiplies the Fq  $x$  by the scalar  $y$ .

`GEN Fq_half(GEN x, GEN T, GEN p)` returns  $z$  such that  $2z = x$  assuming such  $z$  exists.

`GEN Fq_sqr(GEN x, GEN T, GEN p)`

`GEN Fq_neg(GEN x, GEN T, GEN p)`

`GEN Fq_neg_inv(GEN x, GEN T, GEN p)` computes  $-x^{-1}$

`GEN Fq_inv(GEN x, GEN pol, GEN p)` computes  $x^{-1}$ , raising an error if  $x$  is not invertible.

`GEN Fq_invsafe(GEN x, GEN pol, GEN p)` as `Fq_inv`, but returns NULL if  $x$  is not invertible.

`GEN Fq_div(GEN x, GEN y, GEN T, GEN p)`

`GEN FqV_inv(GEN x, GEN T, GEN p)`  $x$  being a vector of Fqs, return the vector of inverses of the  $x[i]$ . The routine uses Montgomery's trick, and involves a single inversion, plus  $3(N - 1)$  multiplications for  $N$  entries. The routine is not stack-clean:  $2N$  FpXQ are left on stack, besides the  $N$  in the result.

`GEN FqV_factorback(GEN L, GEN e, GEN T, GEN p)` given an FqV  $L$  and a ZV or zv  $e$  of the same length, return  $\prod_i L_i^{e_i}$  modulo  $p$ .

`GEN Fq_pow(GEN x, GEN n, GEN pol, GEN p)` returns  $x^n$ .

`GEN Fq_powu(GEN x, ulong n, GEN pol, GEN p)` returns  $x^n$  for small  $n$ .

`GEN Fq_log(GEN a, GEN g, GEN ord, GEN T, GEN p)` as `Fp_log` or `FpXQ_log`.

`int Fq_issquare(GEN x, GEN T, GEN p)` returns 1 if  $x$  is a square and 0 otherwise. Assumes that  $T$  is irreducible mod  $p$  and that  $p$  is prime;  $T = \text{NULL}$  is forbidden unless  $x$  is an Fp.

`long Fq_ispower(GEN x, GEN n, GEN T, GEN p)` returns 1 if  $x$  is a  $n$ -th power and 0 otherwise. Assumes that  $T$  is irreducible mod  $p$  and that  $p$  is prime;  $T = \text{NULL}$  is forbidden unless  $x$  is an Fp.

`GEN Fq_sqrt(GEN x, GEN T, GEN p)` returns a square root of  $x$ . Return NULL if  $x$  is not a square.

`GEN Fq_sqrtn(GEN a, GEN n, GEN T, GEN p, GEN *zn)` as `FpXQ_sqrtn`.

`GEN FpXQ_charpoly(GEN x, GEN T, GEN p)` returns the characteristic polynomial of  $x$   
`GEN FpXQ_minpoly(GEN x, GEN T, GEN p)` returns the minimal polynomial of  $x$   
`GEN FpXQ_norm(GEN x, GEN T, GEN p)` returns the norm of  $x$   
`GEN FpXQ_trace(GEN x, GEN T, GEN p)` returns the trace of  $x$   
`GEN FpXQ_conjvec(GEN x, GEN T, GEN p)` returns the vector of conjugates  $[x, x^p, x^{p^2}, \dots, x^{p^{n-1}}]$  where  $n$  is the degree of  $T$ .  
`GEN gener_FpXQ(GEN T, GEN p, GEN *po)` returns a primitive root modulo  $(T, p)$ .  $T$  is an FpX assumed to be irreducible modulo the prime  $p$ . If  $po$  is not NULL it is set to  $[o, fa]$ , where  $o$  is the order of the multiplicative group of the finite field, and  $fa$  is its factorization.  
`GEN gener_FpXQ_local(GEN T, GEN p, GEN L)`,  $L$  being a vector of primes dividing  $p^{\deg T} - 1$ , returns an element of  $G := \mathbf{F}_p[x]/(T)$  which is a generator of the  $\ell$ -Sylow of  $G$  for every  $\ell$  in  $L$ . It is not necessary, and in fact slightly inefficient, to include  $\ell = 2$ , since 2 is treated separately in any case, i.e. the generator obtained is never a square if  $p$  is odd.  
`GEN gener_Fq_local(GEN T, GEN p, GEN L)` as `pgener_Fp_local(p, L)` if  $T$  is NULL, or `gener_FpXQ_local` (otherwise).  
`GEN FpXQ_powers(GEN x, long n, GEN T, GEN p)` returns  $[x^0, \dots, x^n]$  as a `t_VEC` of FpXQs.  
`GEN FpXQ_matrix_pow(GEN x, long m, long n, GEN T, GEN p)`, as `FpXQ_powers(x, n-1, T, p)`, but returns the powers as a  $m \times n$  matrix. Usually, we have  $m = n = \deg T$ .  
`GEN FpXQ_autpow(GEN a, ulong n, GEN T, GEN p)` computes  $\sigma^n(X)$  assuming  $a = \sigma(X)$  where  $\sigma$  is an automorphism of the algebra  $\mathbf{F}_p[X]/T(X)$ .  
`GEN FpXQ_autsum(GEN a, ulong n, GEN T, GEN p)`  $a$  being a two-component vector,  $\sigma$  being the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$ , returns the vector  $[\sigma^n(X), b\sigma(b) \dots \sigma^{n-1}(b)]$  where  $b = a[2]$ .  
`GEN FpXQ_auttrace(GEN a, ulong n, GEN T, GEN p)`  $a$  being a two-component vector,  $\sigma$  being the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$ , returns the vector  $[\sigma^n(X), b + \sigma(b) + \dots + \sigma^{n-1}(b)]$  where  $b = a[2]$ .  
`GEN FpXQ_outpowers(GEN S, long n, GEN T, GEN p)` returns  $[x, S(x), S(S(x)), \dots, S^{(n)}(x)]$  as a `t_VEC` of FpXQs.  
`GEN FpXQM_autsum(GEN a, long n, GEN T, GEN p)`  $\sigma$  being the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$ , returns the vector  $[\sigma^n(X), b\sigma(b) \dots \sigma^{n-1}(b)]$  where  $b = a[2]$  is a square matrix.  
`GEN FpX_FpXQ_eval(GEN f, GEN x, GEN T, GEN p)` returns  $f(x)$ .  
`GEN FpX_FpXQV_eval(GEN f, GEN V, GEN T, GEN p)` returns  $f(x)$ , assuming that  $V$  was computed by `FpXQ_powers(x, n, T, p)`.  
`GEN FpXC_FpXQ_eval(GEN C, GEN x, GEN T, GEN p)` applies `FpX_FpXQV_eval` to all elements of the vector  $C$  and returns a `t_COL`.  
`GEN FpXC_FpXQV_eval(GEN C, GEN V, GEN T, GEN p)` applies `FpX_FpXQV_eval` to all elements of the vector  $C$  and returns a `t_COL`.  
`GEN FpXM_FpXQV_eval(GEN M, GEN V, GEN T, GEN p)` applies `FpX_FpXQV_eval` to all elements of the matrix  $M$ .

**7.3.10 FpXn.** Let  $p$  a  $\mathbf{t\_INT}$  and  $T$  an  $\mathbf{FpX}$  for  $p$ , both to be given in the function arguments; an  $\mathbf{FpXn}$  object is an  $\mathbf{FpX}$  whose degree is strictly less than  $n$ . They represent a class in  $(\mathbf{Z}/p\mathbf{Z})[X]/(X^n)$ , in which all operations below take place. They can be seen as truncated power series.

$\mathbf{GEN\ FpXn\_mul}(\mathbf{GEN\ x, GEN\ y, long\ n, GEN\ p})$  return  $xy \pmod{X^n}$ .

$\mathbf{GEN\ FpXn\_sqr}(\mathbf{GEN\ x, long\ n, GEN\ p})$  return  $x^2 \pmod{X^n}$ .

$\mathbf{GEN\ FpXn\_div}(\mathbf{GEN\ x, GEN\ y, long\ n, GEN\ p})$  return  $x/y \pmod{X^n}$ .

$\mathbf{GEN\ FpXn\_inv}(\mathbf{GEN\ x, long\ n, GEN\ p})$  return  $1/x \pmod{X^n}$ .

$\mathbf{GEN\ FpXn\_exp}(\mathbf{GEN\ f, long\ n, GEN\ p})$  return  $\exp(f)$  as a composition of formal power series. It is required that the valuation of  $f$  is positive and that  $p > n$ .

$\mathbf{GEN\ FpXn\_expint}(\mathbf{GEN\ f, long\ n, GEN\ p})$  return  $\exp(F)$  where  $F$  is the primitive of  $f$  that vanishes at 0. It is required that  $p > n$ .

**7.3.11 FpXC, FpXM.**

$\mathbf{GEN\ FpXC\_center}(\mathbf{GEN\ C, GEN\ p, GEN\ pov2})$

$\mathbf{GEN\ FpXM\_center}(\mathbf{GEN\ M, GEN\ p, GEN\ pov2})$

**7.3.12 FpXX, FpXY.** Contrary to what the name implies, an  $\mathbf{FpXX}$  is a  $\mathbf{t\_POL}$  whose coefficients are either  $\mathbf{t\_INTs}$  or  $\mathbf{FpXs}$ . This reduces memory overhead at the expense of consistency. The prefix  $\mathbf{FpXY}$  is an alias for  $\mathbf{FpXX}$  when variables matters.

$\mathbf{GEN\ FpXX\_red}(\mathbf{GEN\ z, GEN\ p})$ ,  $z$  a  $\mathbf{t\_POL}$  whose coefficients are either  $\mathbf{ZXs}$  or  $\mathbf{t\_INTs}$ . Returns the  $\mathbf{t\_POL}$  equal to  $z$  with all components reduced modulo  $p$ .

$\mathbf{GEN\ FpXX\_renormalize}(\mathbf{GEN\ x, long\ l})$ , as  $\mathbf{normalizopol}$ , where  $l = \lg(x)$ , in place.

$\mathbf{GEN\ FpXX\_add}(\mathbf{GEN\ x, GEN\ y, GEN\ p})$  adds  $x$  and  $y$ .

$\mathbf{GEN\ FpXX\_sub}(\mathbf{GEN\ x, GEN\ y, GEN\ p})$  returns  $x - y$ .

$\mathbf{GEN\ FpXX\_neg}(\mathbf{GEN\ x, GEN\ p})$  returns  $-x$ .

$\mathbf{GEN\ FpXX\_Fp\_mul}(\mathbf{GEN\ x, GEN\ y, GEN\ p})$  multiplies the  $\mathbf{FpXX\ x}$  by the  $\mathbf{Fp\ y}$ .

$\mathbf{GEN\ FpXX\_FpX\_mul}(\mathbf{GEN\ x, GEN\ y, GEN\ p})$  multiplies the coefficients of the  $\mathbf{FpXX\ x}$  by the  $\mathbf{FpX\ y}$ .

$\mathbf{GEN\ FpXX\_mulu}(\mathbf{GEN\ x, GEN\ y, GEN\ p})$  multiplies the  $\mathbf{FpXX\ x}$  by the scalar  $y$ .

$\mathbf{GEN\ FpXX\_halve}(\mathbf{GEN\ x, GEN\ p})$  returns  $z$  such that  $2z = x$  assuming such  $z$  exists.

$\mathbf{GEN\ FpXX\_deriv}(\mathbf{GEN\ P, GEN\ p})$  differentiates  $P$  with respect to the main variable.

$\mathbf{GEN\ FpXX\_integ}(\mathbf{GEN\ P, GEN\ p})$  returns the primitive of  $P$  with respect to the main variable whose constant term is 0.

$\mathbf{GEN\ FpXY\_eval}(\mathbf{GEN\ Q, GEN\ y, GEN\ x, GEN\ p})$   $Q$  being an  $\mathbf{FpXY}$ , i.e. a  $\mathbf{t\_POL}$  with  $\mathbf{Fp}$  or  $\mathbf{FpX}$  coefficients representing an element of  $\mathbf{F}_p[X][Y]$ . Returns the  $\mathbf{Fp\ Q(x, y)}$ .

$\mathbf{GEN\ FpXY\_evalx}(\mathbf{GEN\ Q, GEN\ x, GEN\ p})$   $Q$  being an  $\mathbf{FpXY}$ , returns the  $\mathbf{FpX\ Q(x, Y)}$ , where  $Y$  is the main variable of  $Q$ .

$\mathbf{GEN\ FpXY\_evaly}(\mathbf{GEN\ Q, GEN\ y, GEN\ p, long\ vx})$   $Q$  an  $\mathbf{FpXY}$ , returns the  $\mathbf{FpX\ Q(X, y)}$ , where  $X$  is the second variable of  $Q$ , and  $vx$  is the variable number of  $X$ .

GEN FpXY\_FpXQ\_evaly(GEN Q, GEN y, GEN T, GEN p, long vx)  $Q$  an FpXY and  $y$  being an FpXQ, returns the FpXQX  $Q(X, y)$ , where  $X$  is the second variable of  $Q$ , and  $vx$  is the variable number of  $X$ .

GEN FpXY\_Fq\_evaly(GEN Q, GEN y, GEN T, GEN p, long vx)  $Q$  an FpXY and  $y$  being an Fq, returns the FqX  $Q(X, y)$ , where  $X$  is the second variable of  $Q$ , and  $vx$  is the variable number of  $X$ .

GEN FpXY\_FpXQ\_evalx(GEN Q, GEN x, ulong p)  $Q$  an FpXY and  $x$  being an FpXQ, returns the FpXQX  $Q(x, Y)$ , where  $Y$  is the first variable of  $Q$ .

GEN FpXY\_FpXQV\_evalx(GEN Q, GEN V, ulong p)  $Q$  an FpXY and  $x$  being an FpXQ, returns the FpXQX  $Q(x, Y)$ , where  $Y$  is the first variable of  $Q$ , assuming that  $V$  was computed by FpXQ\_powers( $x, n, T, p$ ).

GEN FpXYQQ\_pow(GEN x, GEN n, GEN S, GEN T, GEN p),  $x$  being a FpXY,  $T$  being a FpX and  $S$  being a FpY, return  $x^n \pmod{S, T, p}$ .

**7.3.13 FpXQX, FqX.** Contrary to what the name implies, an FpXQX is a  $t\_POL$  whose coefficients are Fqs. So the only difference between FqX and FpXQX routines is that  $T = \text{NULL}$  is not allowed in the latter. (It was thought more useful to allow  $t\_INT$  components than to enforce strict consistency, which would not imply any efficiency gain.)

#### 7.3.13.1 Basic operations.

GEN FqX\_add(GEN x, GEN y, GEN T, GEN p)

GEN FqX\_Fq\_add(GEN x, GEN y, GEN T, GEN p) adds the Fq  $y$  to the FqX  $x$ .

GEN FqX\_Fq\_sub(GEN x, GEN y, GEN T, GEN p) subtracts the Fq  $y$  to the FqX  $x$ .

GEN FqX\_neg(GEN x, GEN T, GEN p)

GEN FqX\_sub(GEN x, GEN y, GEN T, GEN p)

GEN FqX\_mul(GEN x, GEN y, GEN T, GEN p)

GEN FqX\_Fq\_mul(GEN x, GEN y, GEN T, GEN p) multiplies the FqX  $x$  by the Fq  $y$ .

GEN FqX\_mulu(GEN x, ulong y, GEN T, GEN p) multiplies the FqX  $x$  by the scalar  $y$ .

GEN FqX\_halve(GEN x, GEN T, GEN p) returns  $z$  such that  $2z = x$  assuming such  $z$  exists.

GEN FqX\_Fp\_mul(GEN x, GEN y, GEN T, GEN p) multiplies the FqX  $x$  by the  $t\_INT$   $y$ .

GEN FqX\_Fq\_mul\_to\_monic(GEN x, GEN y, GEN T, GEN p) returns  $xy$  assuming the result is monic of the same degree as  $x$  (in particular  $y \neq 0$ ).

GEN FpXQX\_normalize(GEN z, GEN T, GEN p)

GEN FqX\_normalize(GEN z, GEN T, GEN p) divides the FqX  $z$  by its leading term. The leading coefficient becomes 1 as a  $t\_INT$ .

GEN FqX\_sqr(GEN x, GEN T, GEN p)

GEN FqX\_powu(GEN x, ulong n, GEN T, GEN p)

GEN FqX\_divrem(GEN x, GEN y, GEN T, GEN p, GEN \*z)

GEN FqX\_div(GEN x, GEN y, GEN T, GEN p)

GEN FqX\_div\_by\_X\_x(GEN a, GEN x, GEN T, GEN p, GEN \*r)

`GEN FqX_rem(GEN x, GEN y, GEN T, GEN p)`  
`GEN FqX_deriv(GEN x, GEN T, GEN p)` returns the derivative of  $x$ . (This function is suitable for `gerepilupto` but not `memory-clean`.)  
`GEN FqX_integ(GEN x, GEN T, GEN p)` returns the primitive of  $x$ . whose constant term is 0.  
`GEN FqX_translate(GEN P, GEN c, GEN T, GEN p)` let  $c$  be an Fq defined modulo  $(p, T)$ , and let  $P$  be an FqX; returns the translated FqX of  $P(X + c)$ .  
`GEN FqX_gcd(GEN P, GEN Q, GEN T, GEN p)` returns a (not necessarily monic) greatest common divisor of  $x$  and  $y$ .  
`GEN FqX_extgcd(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv)` returns  $d = \text{GCD}(x, y)$  (not necessarily monic), and sets  $*u, *v$  to the Bezout coefficients such that  $*ux + *vy = d$ .  
`GEN FqX_halfgcd(GEN x, GEN y, GEN T, GEN p)` returns a two-by-two FqXM  $M$  with determinant  $\pm 1$  such that the image  $(a, b)$  of  $(x, y)$  by  $M$  has the property that  $\deg a \geq \frac{\deg x}{2} > \deg b$ .  
`GEN FqX_eval(GEN x, GEN y, GEN T, GEN p)` evaluates the FqX  $x$  at the Fq  $y$ . The result is an Fq.  
`GEN FqXY_eval(GEN Q, GEN y, GEN x, GEN T, GEN p)`  $Q$  an FqXY, i.e. a `t_POL` with Fq or FqX coefficients representing an element of  $\mathbf{F}_q[X][Y]$ . Returns the Fq  $Q(x, y)$ .  
`GEN FqXY_evalx(GEN Q, GEN x, GEN T, GEN p)`  $Q$  being an FqXY, returns the FqX  $Q(x, Y)$ , where  $Y$  is the main variable of  $Q$ .  
`GEN random_FpXQX(long d, long v, GEN T, GEN p)` returns a random FpXQX in variable  $v$ , of degree less than  $d$ .  
`GEN FpXQX_renormalize(GEN x, long lx)`  
`GEN FpXQX_red(GEN z, GEN T, GEN p)`  $z$  a `t_POL` whose coefficients are ZXXs or `t_INTs`, reduce them to FpXQs.  
`GEN FpXQXV_red(GEN z, GEN T, GEN p)`,  $z$  a `t_VEC` of ZXX. Applies `FpX_red` componentwise and returns the result (and we obtain a vector of FpXQXs).  
`GEN FpXQXT_red(GEN z, GEN T, GEN p)`,  $z$  a tree of ZXX. Applies `FpX_red` to each leaf and returns the result (and we obtain a tree of FpXQXs).  
`GEN FpXQX_mul(GEN x, GEN y, GEN T, GEN p)`  
`GEN Kronecker_to_FpXQX(GEN z, GEN T, GEN p)`. Let  $n = \deg T$  and let  $P(X, Y) \in \mathbf{Z}[X, Y]$  lift a polynomial in  $K[Y]$ , where  $K := \mathbf{F}_p[X]/(T)$  and  $\deg_X P < 2n - 1$  — such as would result from multiplying minimal degree lifts of two polynomials in  $K[Y]$ . Let  $z = P(t, t^{2*n-1})$  be a Kronecker form of  $P$  (see `RgXX_to_Kronecker`), this function returns  $Q \in \mathbf{Z}[X, t]$  such that  $Q$  is congruent to  $P(X, t) \bmod (p, T(X))$ ,  $\deg_X Q < n$ , and all coefficients are in  $[0, p]$ . Not stack-clean. Note that  $t$  need not be the same variable as  $Y$ !  
`GEN FpXQX_FpXQ_mul(GEN x, GEN y, GEN T, GEN p)`  
`GEN FpXQX_sqr(GEN x, GEN T, GEN p)`  
`GEN FpXQX_divrem(GEN x, GEN y, GEN T, GEN p, GEN *pr)`  
`GEN FpXQX_div(GEN x, GEN y, GEN T, GEN p)`  
`GEN FpXQX_div_by_X_x(GEN a, GEN x, GEN T, GEN p, GEN *r)`

GEN FpXQX\_rem(GEN x, GEN y, GEN T, GEN p)  
 GEN FpXQX\_powu(GEN x, ulong n, GEN T, GEN p) returns  $x^n$ .  
 GEN FpXQX\_digits(GEN x, GEN B, GEN T, GEN p)  
 GEN FpXQX\_dotproduct(GEN x, GEN y, GEN T, GEN p) returns the scalar product of the coefficients of  $x$  and  $y$ .  
 GEN FpXQXV\_FpXQX\_fromdigits(GEN v, GEN B, GEN T, GEN p)  
 GEN FpXQX\_invBarrett(GEN y, GEN T, GEN p) returns the Barrett inverse of the FpXQX  $y$ , namely a lift of  $1/\text{polrecip}(y) + O(x^{\deg(y)-1})$ .  
 GEN FpXQXV\_prod(GEN V, GEN T, GEN p),  $V$  being a vector of FpXQX, returns their product.  
 GEN FpXQX\_gcd(GEN x, GEN y, GEN T, GEN p)  
 GEN FpXQX\_extgcd(GEN x, GEN y, GEN T, GEN p, GEN \*ptu, GEN \*ptv)  
 GEN FpXQX\_halfgcd(GEN x, GEN y, GEN T, GEN p)  
 GEN FpXQX\_resultant(GEN x, GEN y, GEN T, GEN p) returns the resultant of  $x$  and  $y$ .  
 GEN FpXQX\_disc(GEN x, GEN T, GEN p) returns the discriminant of  $x$ .  
 GEN FpXQX\_FpXQXQ\_eval(GEN f, GEN x, GEN S, GEN T, GEN p) returns  $f(x)$ .

### 7.3.14 FpXQXn, FqXn.

A FpXQXn is a t\_FpXQX which represents an element of the ring  $(Fp[X]/T(X))[Y]/(Y^n)$ , where  $T$  is a FpX.

GEN FpXQXn\_sqr(GEN x, long n, GEN T, GEN p)  
 GEN FqXn\_sqr(GEN x, long n, GEN T, GEN p)  
 GEN FpXQXn\_mul(GEN x, GEN y, long n, GEN T, GEN p)  
 GEN FqXn\_mul(GEN x, GEN y, long n, GEN T, GEN p)  
 GEN FpXQXn\_div(GEN x, GEN y, long n, GEN T, GEN p)  
 GEN FpXQXn\_inv(GEN x, long n, GEN T, GEN p)  
 GEN FqXn\_inv(GEN x, long n, GEN T, GEN p)  
 GEN FpXQXn\_exp(GEN x, long n, GEN T, GEN p) return  $\exp(x)$  as a composition of formal power series. It is required that the valuation of  $x$  is positive and that  $p > n$ .  
 GEN FqXn\_exp(GEN x, long n, GEN T, GEN p)  
 GEN FpXQXn\_expint(GEN f, long n, GEN p) return  $\exp(F)$  where  $F$  is the primitive of  $f$  that vanishes at 0. It is required that  $p > n$ .  
 GEN FqXn\_expint(GEN x, long n, GEN T, GEN p)

### 7.3.15 FpXQXQ, FqXQ.

A FpXQXQ is a t\_FpXQX which represents an element of the ring  $(Fp[X]/T(X))[Y]/S(X, Y)$ , where  $T$  is a FpX and  $S$  a FpXQX modulo  $T$ . A FqXQ is identical except that  $T$  is allowed to be NULL in which case  $S$  must be a FpX.

#### 7.3.15.1 Preconditioned reduction.

For faster reduction, the modulus  $S$  can be replaced by an extended modulus, which is an FpXQXT, in all FpXQXQ- and FqXQ-classes functions, and in FpXQX\_rem and FpXQX\_divrem.

GEN FpXQX\_get\_red(GEN S, GEN T, GEN p) returns the extended modulus eS.

GEN FqX\_get\_red(GEN S, GEN T, GEN p) identical, but allow  $T$  to be NULL, in which case it returns FpX\_get\_red(S, p).

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get\_FpXQX\_mod(GEN eS) returns the underlying modulus  $S$ .

GEN get\_FpXQX\_var(GEN eS) returns the variable number of the modulus.

GEN get\_FpXQX\_degree(GEN eS) returns the degree of the modulus.

Furthermore, ZXXT\_to\_FlxXT allows to convert an extended modulus for a FpXQX to an extended modulus for the corresponding FlxqX.

#### 7.3.15.2 basic operations.

GEN FpXQX\_FpXQXQV\_eval(GEN f, GEN V, GEN S, GEN T, GEN p) returns  $f(x)$ , assuming that  $V$  was computed by FpXQXQ\_powers( $x, n, S, T, p$ ).

GEN FpXQXQ\_div(GEN x, GEN y, GEN S, GEN T, GEN p),  $x, y$  and  $S$  being FpXQXs, returns  $x * y^{-1}$  modulo  $S$ .

GEN FpXQXQ\_inv(GEN x, GEN S, GEN T, GEN p),  $x$  and  $S$  being FpXQXs, returns  $x^{-1}$  modulo  $S$ .

GEN FpXQXQ\_invsafe(GEN x, GEN S, GEN T, GEN p), as FpXQXQ\_inv, returning NULL if  $x$  is not invertible.

GEN FpXQXQ\_mul(GEN x, GEN y, GEN S, GEN T, GEN p),  $x, y$  and  $S$  being FpXQXs, returns  $xy$  modulo  $S$ .

GEN FpXQXQ\_sqr(GEN x, GEN S, GEN T, GEN p),  $x$  and  $S$  being FpXQXs, returns  $x^2$  modulo  $S$ .

GEN FpXQXQ\_pow(GEN x, GEN n, GEN S, GEN T, GEN p),  $x$  and  $S$  being FpXQXs, returns  $x^n$  modulo  $S$ .

GEN FpXQXQ\_powers(GEN x, long n, GEN S, GEN T, GEN p),  $x$  and  $S$  being FpXQXs, returns  $[x^0, \dots, x^n]$  as a t\_VEC of FpXQXs.

GEN FpXQXQ\_halfFrobenius(GEN A, GEN S, GEN T, GEN p) returns  $A(X)^{(q-1)/2} \pmod{S(X)}$  over the finite field  $\mathbf{F}_q$  defined by  $T$  and  $p$ , thus  $q = p^n$  where  $n$  is the degree of  $T$ .

GEN FpXQXQ\_minpoly(GEN x, GEN S, GEN T, GEN p), as FpXQ\_minpoly

GEN FpXQXQ\_matrix\_pow(GEN x, long m, long n, GEN S, GEN T, GEN p) returns the same powers of  $x$  as FpXQXQ\_powers( $x, n-1, S, T, p$ ), but as an  $m \times n$  matrix.

GEN FpXQXQ\_autpow(GEN a, long n, GEN S, GEN T, GEN p)  $\sigma$  being the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$ ,  $\sigma(Y) = a[2] \pmod{S(X,Y), T(X)}$ , returns  $[\sigma^n(X), \sigma^n(Y)]$ .

GEN FpXQXQ\_autsum(GEN a, long n, GEN S, GEN T, GEN p)  $\sigma$  being the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$ ,  $\sigma(Y) = a[2] \pmod{S(X,Y), T(X)}$ , returns the vector  $[\sigma^n(X), \sigma^n(Y), b\sigma(b) \dots \sigma^{n-1}(b)]$  where  $b = a[3]$ .

GEN FpXQXQ\_auttrace(GEN a, long n, GEN S, GEN T, GEN p)  $\sigma$  being the automorphism defined by  $\sigma(X) = X \pmod{T(X)}$ ,  $\sigma(Y) = a[1] \pmod{S(X,Y), T(X)}$ , returns the vector  $[\sigma^n(X), \sigma^n(Y), b + \sigma(b) + \dots + \sigma^{n-1}(b)]$  where  $b = a[2]$ .

GEN FqXQ\_add(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns  $x + y$  modulo S.

GEN FqXQ\_sub(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns  $x - y$  modulo S.

GEN FqXQ\_mul(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns  $xy$  modulo S.

GEN FqXQ\_div(GEN x, GEN y, GEN S, GEN T, GEN p), x and S being FqXs, returns  $x/y$  modulo S.

GEN FqXQ\_inv(GEN x, GEN S, GEN T, GEN p), x and S being FqXs, returns  $x^{-1}$  modulo S.

GEN FqXQ\_invsafe(GEN x, GEN S, GEN T, GEN p), as FqXQ\_inv, returning NULL if x is not invertible.

GEN FqXQ\_sqr(GEN x, GEN S, GEN T, GEN p), x and S being FqXs, returns  $x^2$  modulo S.

GEN FqXQ\_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x and S being FqXs, returns  $x^n$  modulo S.

GEN FqXQ\_powers(GEN x, long n, GEN S, GEN T, GEN p), x and S being FqXs, returns  $[x^0, \dots, x^n]$  as a t\_VEC of FqXs.

GEN FqXQ\_matrix\_pow(GEN x, long m, long n, GEN S, GEN T, GEN p) returns the same powers of x as FqXQ\_powers(x, n-1, S, T, p), but as an  $m \times n$  matrix.

GEN FqV\_roots\_to\_pol(GEN V, GEN T, GEN p, long v), V being a vector of Fqs, returns the monic FqX  $\prod_i (\text{pol\_x}[v] - V[i])$ .

### 7.3.15.3 Miscellaneous operations.

GEN init\_Fq(GEN p, long n, long v) returns an irreducible polynomial of degree  $n > 0$  over  $\mathbf{F}_p$ , in variable v.

int FqX\_is\_squarefree(GEN P, GEN T, GEN p)

GEN FpXQX\_roots(GEN f, GEN T, GEN p) return the roots of  $f$  in  $\mathbf{F}_p[X]/(T)$ . Assumes p is prime and T irreducible in  $\mathbf{F}_p[X]$ .

GEN FqX\_roots(GEN f, GEN T, GEN p) same but allow T = NULL.

GEN FpXQX\_factor(GEN f, GEN T, GEN p) same output convention as FpX\_factor. Assumes p is prime and T irreducible in  $\mathbf{F}_p[X]$ .

GEN FqX\_factor(GEN f, GEN T, GEN p) same but allow T = NULL.



GEN FpXQX\_factor\_squarefree(GEN f, GEN T, GEN p) squarefree factorization of  $f$  modulo  $(T, p)$ ; same output convention as FpX\_factor\_squarefree. Assumes  $p$  is prime and  $T$  irreducible in  $\mathbf{F}_p[X]$ .

GEN FqX\_factor\_squarefree(GEN f, GEN T, GEN p) same but allow  $T = \text{NULL}$ .

GEN FpXQX\_ddf(GEN f, GEN T, GEN p) as FpX\_ddf.

GEN FqX\_ddf(GEN f, GEN T, GEN p) same but allow  $T = \text{NULL}$ .

long FpXQX\_ddf\_degree(GEN f, GEN XP, GEN T, GEN p), as FpX\_ddf\_degree.

GEN FpXQX\_degfact(GEN f, GEN T, GEN p), as FpX\_degfact.

GEN FqX\_degfact(GEN f, GEN T, GEN p) same but allow  $T = \text{NULL}$ .

GEN FpXQX\_split\_part(GEN f, GEN T, GEN p) returns the largest totally split squarefree factor of  $f$ .

long FpXQX\_ispower(GEN f, ulong k, GEN T, GEN p, GEN \*pt) return 1 if the FpXQX  $f$  is a  $k$ -th power, 0 otherwise. If  $pt$  is not NULL, set it to  $g$  such that  $g^k = f$ .

long FqX\_ispower(GEN f, ulong k, GEN T, GEN p, GEN \*pt) same but allow  $T = \text{NULL}$ .

GEN FpX\_factorff(GEN P, GEN T, GEN p). Assumes  $p$  prime and  $T$  irreducible in  $\mathbf{F}_p[X]$ . Factor the FpX  $P$  over the finite field  $\mathbf{F}_p[Y]/(T(Y))$ . See FpX\_factorff\_irred if  $P$  is known to be irreducible of  $\mathbf{F}_p$ .

GEN FpX\_rootsff(GEN P, GEN T, GEN p). Assumes  $p$  prime and  $T$  irreducible in  $\mathbf{F}_p[X]$ . Returns the roots of the FpX  $P$  belonging to the finite field  $\mathbf{F}_p[Y]/(T(Y))$ .

GEN FpX\_factorff\_irred(GEN P, GEN T, GEN p). Assumes  $p$  prime and  $T$  irreducible in  $\mathbf{F}_p[X]$ . Factors the *irreducible* FpX  $P$  over the finite field  $\mathbf{F}_p[Y]/(T(Y))$  and returns the vector of irreducible FqXs factors (the exponents, being all equal to 1, are not included).

GEN FpX\_ffisom(GEN P, GEN Q, GEN p). Assumes  $p$  prime,  $P, Q$  are ZXs, both irreducible mod  $p$ , and  $\deg(P) \mid \deg(Q)$ . Outputs a monomorphism between  $\mathbf{F}_p[X]/(P)$  and  $\mathbf{F}_p[X]/(Q)$ , as a polynomial  $R$  such that  $Q \mid P(R)$  in  $\mathbf{F}_p[X]$ . If  $P$  and  $Q$  have the same degree, it is of course an isomorphism.

void FpX\_ffintersect(GEN P, GEN Q, long n, GEN p, GEN \*SP, GEN \*SQ, GEN MA, GEN MB) Assumes  $p$  is prime,  $P, Q$  are ZXs, both irreducible mod  $p$ , and  $n$  divides both the degree of  $P$  and  $Q$ . Compute  $SP$  and  $SQ$  such that the subfield of  $\mathbf{F}_p[X]/(P)$  generated by  $SP$  and the subfield of  $\mathbf{F}_p[X]/(Q)$  generated by  $SQ$  are isomorphic of degree  $n$ . The polynomials  $P$  and  $Q$  do not need to be of the same variable. If  $MA$  (resp.  $MB$ ) is not NULL, it must be the matrix of the Frobenius map in  $\mathbf{F}_p[X]/(P)$  (resp.  $\mathbf{F}_p[X]/(Q)$ ).

GEN FpXQ\_ffisom\_inv(GEN S, GEN T, GEN p). Assumes  $p$  is prime,  $T$  a ZX, which is irreducible modulo  $p$ ,  $S$  a ZX representing an automorphism of  $\mathbf{F}_q := \mathbf{F}_p[X]/(T)$ . ( $S(X)$  is the image of  $X$  by the automorphism.) Returns the inverse automorphism of  $S$ , in the same format, i.e. an FpX  $H$  such that  $H(S) \equiv X$  modulo  $(T, p)$ .

long FpXQX\_nbfact(GEN S, GEN T, GEN p) returns the number of irreducible factors of the polynomial  $S$  over the finite field  $\mathbf{F}_q$  defined by  $T$  and  $p$ .

long FpXQX\_nbfact\_Frobenius(GEN S, GEN Xq, GEN T, GEN p) as FpXQX\_nbfact where  $Xq$  is FpXQX\_Frobenius( $S, T, p$ ).

long FqX\_nbfact(GEN S, GEN T, GEN p) as above but accept  $T=\text{NULL}$ .

`long FpXQX_nbroots(GEN S, GEN T, GEN p)` returns the number of roots of the polynomial  $S$  over the finite field  $\mathbf{F}_q$  defined by  $T$  and  $p$ .

`long FqX_nbroots(GEN S, GEN T, GEN p)` as above but accept  $T=\text{NULL}$ .

`GEN FpXQX_Frobenius(GEN S, GEN T, GEN p)` returns  $X^q \pmod{S(X)}$  over the finite field  $\mathbf{F}_q$  defined by  $T$  and  $p$ , thus  $q = p^n$  where  $n$  is the degree of  $T$ .

**7.3.16 Flx.** Let  $p$  be an `ulong`, not assumed to be prime unless mentioned otherwise (e.g., all functions involving Euclidean divisions and factorizations), to be given the function arguments; an  $\text{Fl}$  is an `ulong` belonging to  $[0, p - 1]$ , an  $\text{Flx } z$  is a `t_VECSMALL` representing a polynomial with small integer coefficients. Specifically  $z[0]$  is the usual codeword,  $z[1] = \text{evalvarn}(v)$  for some variable  $v$ , then the coefficients by increasing degree. An  $\text{FlxX}$  is a `t_POL` whose coefficients are  $\text{Flxs}$ .

In the following, an argument called  $sv$  is of the form  $\text{evalvarn}(v)$  for some variable number  $v$ .

#### 7.3.16.1 Preconditioned reduction.

For faster reduction, the modulus  $T$  can be replaced by an extended modulus ( $\text{FlxT}$ ) in all  $\text{Flxq}$ -classes functions, and in  $\text{Flx\_divrem}$ .

`GEN Flx_get_red(GEN T, ulong p)` returns the extended modulus  $eT$ .

`GEN Flx_get_red_pre(GEN T, ulong p, ulong pi)` as  $\text{Flx\_get\_red}$ . We assume  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume  $\text{SMALL\_ULONG}(p)$ .

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_Flx_mod(GEN eT)` returns the underlying modulus  $T$ .

`GEN get_Flx_var(GEN eT)` returns the variable number of the modulus.

`GEN get_Flx_degree(GEN eT)` returns the degree of the modulus.

Furthermore,  $\text{ZXT\_to\_FlxT}$  allows to convert an extended modulus for a  $\text{FpX}$  to an extended modulus for the corresponding  $\text{Flx}$ .

#### 7.3.16.2 Basic operations.

In this section,  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume  $\text{SMALL\_ULONG}(p)$ .

`ulong Flx_lead(GEN x)` returns the leading coefficient of  $x$  as a `ulong` (return 0 for the zero polynomial).

`ulong Flx_constant(GEN x)` returns the constant coefficient of  $x$  as a `ulong` (return 0 for the zero polynomial).

`GEN Flx_red(GEN z, ulong p)` converts from  $zx$  with nonnegative coefficients to  $\text{Flx}$  (by reducing them mod  $p$ ).

`int Flx_equal1(GEN x)` returns 1 (true) if the  $\text{Flx } x$  is equal to 1, 0 (false) otherwise.

`int Flx_equal(GEN x, GEN y)` returns 1 (true) if the  $\text{Flx } x$  and  $y$  are equal, and 0 (false) otherwise.

`GEN Flx_copy(GEN x)` returns a copy of  $x$ .

`GEN Flx_add(GEN x, GEN y, ulong p)`

GEN Flx\_Fl\_add(GEN y, ulong x, ulong p)  
 GEN Flx\_neg(GEN x, ulong p)  
 GEN Flx\_neg\_inplace(GEN x, ulong p), same as Flx\_neg, in place (x is destroyed).  
 GEN Flx\_sub(GEN x, GEN y, ulong p)  
 GEN Flx\_Fl\_sub(GEN y, ulong x, ulong p)  
 GEN Flx\_half(GEN x, ulong p) returns  $z$  such that  $2z = x$  modulo  $p$  assuming such  $z$  exists.  
 GEN Flx\_mul(GEN x, GEN y, ulong p)  
 GEN Flx\_mul\_pre(GEN x, GEN y, ulong p, ulong pi)  
 GEN Flx\_Fl\_mul(GEN y, ulong x, ulong p)  
 GEN Flx\_double(GEN y, ulong p) returns  $2y$ .  
 GEN Flx\_triple(GEN y, ulong p) returns  $3y$ .  
 GEN Flx\_mulu(GEN y, ulong x, ulong p) as Flx\_Fl\_mul but do not assume that  $x < p$ .  
 GEN Flx\_Fl\_mul\_to\_monic(GEN y, ulong x, ulong p) returns  $yx$  assuming the result is monic of the same degree as  $y$  (in particular  $x \neq 0$ ).  
 GEN Flx\_sqr(GEN x, ulong p)  
 GEN Flx\_sqr\_pre(GEN x, ulong p, ulong pi)  
 GEN Flx\_powu(GEN x, ulong n, ulong p) return  $x^n$ .  
 GEN Flx\_powu\_pre(GEN x, ulong n, ulong p, ulong pi)  
 GEN Flx\_divrem(GEN x, GEN y, ulong p, GEN \*pr), here  $p$  must be prime.  
 GEN Flx\_divrem\_pre(GEN x, GEN y, ulong p, ulong pi, GEN \*pr)  
 GEN Flx\_div(GEN x, GEN y, ulong p), here  $p$  must be prime.  
 GEN Flx\_div\_pre(GEN x, GEN y, ulong p, ulong pi)  
 GEN Flx\_rem(GEN x, GEN y, ulong p), here  $p$  must be prime.  
 GEN Flx\_rem\_pre(GEN x, GEN y, ulong p)  
 GEN Flx\_deriv(GEN z, ulong p)  
 GEN Flx\_integ(GEN z, ulong p), here  $p$  must be prime.  
 GEN Flx\_translate1(GEN P, ulong p) return  $P(x+1)$ ,  $p$  must be prime. Asymptotically fast (quasi-linear in the degree of  $P$ ).  
 GEN Flx\_translate1\_basecase(GEN P, ulong p) return  $P(x+1)$ ,  $p$  need not be prime. Not asymptotically fast (quadratic in the degree of  $P$ ).  
 GEN zlx\_translate1(GEN P, ulong p, long e) return  $P(x+1)$  modulo  $p^e$  for prime  $p$ . Asymptotically fast (quasi-linear in the degree of  $P$ ).  
 GEN Flx\_diff1(GEN P, ulong p) return  $P(x+1) - P(x)$ ;  $p$  must be prime.  
 GEN Flx\_digits(GEN x, GEN B, ulong p) returns a vector of Flx  $[c_0, \dots, c_n]$  of degree less than the degree of  $B$  and such that  $x = \sum_{i=0}^n c_i B^i$ .

`GEN FlxV_Flx_fromdigits(GEN v, GEN B, ulong p)` where  $v = [c_0, \dots, c_n]$  is a vector of `Flx`, returns  $\sum_{i=0}^n c_i B^i$ .

`GEN Flx_Frobenius(GEN T, ulong p)` here  $p$  must be prime.

`GEN Flx_Frobenius_pre(GEN T, ulong p, ulong pi)`

`GEN Flx_matFrobenius(GEN T, ulong p)` here  $p$  must be prime.

`GEN Flx_matFrobenius_pre(GEN T, ulong p, ulong pi)`

`GEN Flx_gcd(GEN a, GEN b, ulong p)` returns a (not necessarily monic) greatest common divisor of  $x$  and  $y$ . Here  $p$  must be prime.

`GEN Flx_gcd_pre(GEN a, GEN b, ulong p)`

`GEN Flx_halfgcd(GEN x, GEN y, ulong p)` returns a two-by-two `FlxM`  $M$  with determinant  $\pm 1$  such that the image  $(a, b)$  of  $(x, y)$  by  $M$  has the property that  $\deg a \geq \frac{\deg x}{2} > \deg b$ . Assumes that  $p$  is prime.

`GEN Flx_halfgcd_pre(GEN a, GEN b, ulong p)`

`GEN Flx_extgcd(GEN a, GEN b, ulong p, GEN *ptu, GEN *ptv)`, here  $p$  must be prime.

`GEN Flx_extgcd_pre(GEN a, GEN b, ulong p, ulong pi, GEN *ptu, GEN *ptv)`

`GEN Flx_roots(GEN f, ulong p)` returns the vector of roots of  $f$  (without multiplicity, as a `t_VECSMALL`). Assumes that  $p$  is prime.

`GEN Flx_roots_pre(GEN f, ulong p, ulong pi)`

`ulong Flx_oneroot(GEN f, ulong p)` returns one root  $0 \leq r < p$  of the `Flx`  $f$  in  $\mathbf{Z}/p\mathbf{Z}$ . Return  $p$  if no root exists. Assumes that  $p$  is prime.

`GEN Flx_oneroot_pre(GEN f, ulong p)`, as `Flx_oneroot`

`ulong Flx_oneroot_split(GEN f, ulong p)` as `Flx_oneroot` but assume  $f$  is totally split. Assumes that  $p$  is prime.

`ulong Flx_oneroot_split_pre(GEN f, ulong p, ulong pi)`

`long Flx_ispower(GEN f, ulong k, ulong p, GEN *pt)` return 1 if the `Flx`  $f$  is a  $k$ -th power, 0 otherwise. If `pt` is not `NULL`, set it to  $g$  such that  $g^k = f$ .

`GEN Flx_factor(GEN f, ulong p)` Assumes that  $p$  is prime.

`GEN Flx_ddf(GEN f, ulong p)` Assumes that  $p$  is prime.

`GEN Flx_ddf_pre(GEN f, ulong p, ulong pi)`

`GEN Flx_factor_squarefree(GEN f, ulong p)` returns the squarefree factorization of  $f$  modulo  $p$ . This is a vector  $[u_1, \dots, u_k]$  of pairwise coprime `Flx` such that  $u_k \neq 1$  and  $f = \prod u_i$ . Shallow function. Assumes that  $p$  is prime.

`GEN Flx_factor_squarefree_pre(GEN f, ulong p, ulong pi)`

`GEN Flx_mod_Xn1(GEN T, ulong n, ulong p)` return  $T$  modulo  $(X^n + 1, p)$ . Shallow function.

`GEN Flx_mod_Xnm1(GEN T, ulong n, ulong p)` return  $T$  modulo  $(X^n - 1, p)$ . Shallow function.

`GEN Flx_degfact(GEN f, ulong p)` as `FpX_degfact`. Assumes that  $p$  is prime.

GEN Flx\_factorff\_irred(GEN P, GEN Q, ulong p) as FpX\_factorff\_irred. Assumes that  $p$  is prime.

GEN Flx\_rootsff(GEN P, GEN T, ulong p) as FpX\_rootsff. Assumes that  $p$  is prime.

GEN Flx\_factcyclo(ulong n, ulong p, ulong m) returns the factors of the  $n$ -th cyclotomic polynomial over  $\mathbf{F}_p$ . if  $m = 1$  returns a single factor.

GEN Flx\_ffisom(GEN P, GEN Q, ulong l) as FpX\_ffisom. Assumes that  $p$  is prime.

### 7.3.16.3 Miscellaneous operations.

GEN pol0\_Flx(long sv) returns a zero Flx in variable  $v$ .

GEN zero\_Flx(long sv) alias for pol0\_Flx

GEN pol1\_Flx(long sv) returns the unit Flx in variable  $v$ .

GEN polx\_Flx(long sv) returns the variable  $v$  as degree 1 Flx.

GEN polxn\_Flx(long n, long sv) Returns the monomial of degree  $n$  as a Flx in variable  $v$ ; assume that  $n \geq 0$ .

GEN monomial\_Flx(ulong a, long d, long sv) returns the Flx  $aX^d$  in variable  $v$ .

GEN init\_Flxq(ulong p, long n, long sv) returns an irreducible polynomial of degree  $n > 0$  over  $\mathbf{F}_p$ , in variable  $v$ .

GEN Flx\_normalize(GEN z, ulong p), as FpX\_normalize.

GEN Flx\_rescale(GEN P, ulong h, ulong p) returns  $h^{\deg(P)}P(x/h)$ ,  $P$  is a Flx and  $h$  is a nonzero integer.

GEN random\_Flx(long d, long sv, ulong p) returns a random Flx in variable  $v$ , of degree less than  $d$ .

GEN Flx\_recip(GEN x), returns the reciprocal polynomial

ulong Flx\_resultant(GEN a, GEN b, ulong p), returns the resultant of  $a$  and  $b$ . Assumes that  $p$  is prime.

ulong Flx\_resultant\_pre(GEN a, GEN b, ulong p, ulong pi)

ulong Flx\_extresultant(GEN a, GEN b, ulong p, GEN \*ptU, GEN \*ptV) given two Flx  $a$  and  $b$ , returns their resultant and sets Bezout coefficients (if the resultant is 0, the latter are not set). Assumes that  $p$  is prime.

GEN Flx\_invBarrett(GEN T, ulong p), returns the Barrett inverse  $M$  of  $T$  defined by  $M(x) \times x^n T(1/x) \equiv 1 \pmod{x^{n-1}}$  where  $n$  is the degree of  $T$ . Assumes that  $p$  is prime.

GEN Flx\_renormalize(GEN x, long l), as FpX\_renormalize, where  $l = \lg(x)$ , in place.

GEN Flx\_shift(GEN T, long n) returns  $T * x^n$  if  $n \geq 0$ , and  $T \setminus x^{-n}$  otherwise.

long Flx\_val(GEN x) returns the valuation of  $x$ , i.e. the multiplicity of the 0 root.

long Flx\_valrem(GEN x, GEN \*Z) as RgX\_valrem, returns the valuation of  $x$ . In particular, if the valuation is 0, set  $*Z$  to  $x$ , not a copy.

GEN Flx\_div\_by\_X\_x(GEN A, ulong a, ulong p, ulong \*rem), returns the Euclidean quotient of the Flx  $A$  by  $X - a$ , and sets  $rem$  to the remainder  $A(a)$ .

`ulong Flx_eval(GEN x, ulong y, ulong p), as FpX_eval.`  
`ulong Flx_eval_pre(GEN x, ulong y, ulong p, ulong pi)`  
`ulong Flx_eval_powers_pre(GEN P, GEN y, ulong p, ulong pi).` Let  $y$  be the `t_VECSMALL`  $(1, a, \dots, a^n)$ , where  $n$  is the degree of the `Flx`  $P$ , return  $P(a)$ .  
`GEN Flx_Flv_multieval(GEN P, GEN v, ulong p)` returns the vector  $[P(v[1]), \dots, P(v[n])]$  as a `Flv`.  
`ulong Flx_dotproduct(GEN x, GEN y, ulong p)` returns the scalar product of the coefficients of  $x$  and  $y$ .  
`ulong Flx_dotproduct_pre(GEN x, GEN y, ulong p, ulong pi).`  
`GEN Flx_deflate(GEN P, long d)` assuming  $P$  is a polynomial of the form  $Q(X^d)$ , return  $Q$ .  
`GEN Flx_inflate(GEN P, long d)` returns  $P(X^d)$ .  
`GEN Flx_splitting(GEN P, long k), as RgX_splitting.`  
`GEN Flx_blocks(GEN P, long n, long m), as RgX_blocks.`  
`int Flx_is_squarefree(GEN z, ulong p).` Assumes that  $p$  is prime.  
`int Flx_is_irred(GEN f, ulong p), as FpX_is_irred.` Assumes that  $p$  is prime.  
`int Flx_is_totally_split(GEN f, ulong p)` returns 1 if the `Flx`  $f$  splits into a product of distinct linear factors, 0 otherwise. Assumes that  $p$  is prime.  
`int Flx_is_smooth(GEN f, long r, ulong p)` return 1 if all irreducible factors of  $f$  are of degree at most  $r$ , 0 otherwise. Assumes that  $p$  is prime.  
`int Flx_is_smooth_pre(GEN f, long r, ulong p, ulong pi)`  
`long Flx_nbroots(GEN f, ulong p), as FpX_nbroots.` Assumes that  $p$  is prime.  
`long Flx_nbfact(GEN z, ulong p), as FpX_nbfact.` Assumes that  $p$  is prime.  
`long Flx_nbfact_pre(GEN z, ulong p, ulong pi)`  
`long Flx_nbfact_Frobenius(GEN f, GEN XP, ulong p), as FpX_nbfact_Frobenius.` Assumes that  $p$  is prime.  
`long Flx_nbfact_Frobenius_pre(GEN f, GEN XP, ulong p, ulong pi)`  
`GEN Flx_degfact(GEN f, ulong p), as FpX_degfact.` Assumes that  $p$  is prime.  
`GEN Flx_nbfact_by_degree(GEN z, long *nb, ulong p)` Assume that the `Flx`  $z$  is squarefree mod the prime  $p$ . Returns a `t_VECSMALL`  $D$  with  $\deg z$  entries, such that  $D[i]$  is the number of irreducible factors of degree  $i$ . Set `nb` to the total number of irreducible factors (the sum of the  $D[i]$ ). Assumes that  $p$  is prime.  
`void Flx_ffintersect(GEN P, GEN Q, long n, ulong p, GEN*SP, GEN*SQ, GEN MA, GEN MB)`  
`,`  
`as FpX_ffintersect.` Assumes that  $p$  is prime.  
`GEN Flx_Laplace(GEN x, ulong p)`  
`GEN Flx_invLaplace(GEN x, ulong p)`  
`GEN Flx_Newton(GEN x, long n, ulong p)`

GEN Flx\_fromNewton(GEN x, ulong p)

GEN Flx\_Teichmuller(GEN P, ulong p, long n) Return a ZX  $Q$  such that  $P \equiv Q \pmod{p}$  and  $Q(X^p) = 0 \pmod{Q, p^n}$ . Assumes that  $p$  is prime.

GEN Flv\_polint(GEN x, GEN y, ulong p, long sv) as FpV\_polint, returning an Flx in variable  $v$ . Assumes that  $p$  is prime.

GEN Flv\_Flm\_polint(GEN x, GEN V, ulong p, long sv) equivalent (but faster) to applying Flv\_polint(x,...) to all the elements of the vector  $V$  (thus, returns a FlxV). Assumes that  $p$  is prime.

GEN Flv\_invVandermonde(GEN L, ulong d, ulong p)  $L$  being a Flv of length  $n$ , return the inverse  $M$  of the Vandermonde matrix attached to the elements of  $L$ , multiplied by  $d$ . If  $A$  is a Flv and  $B = MA$ , then the polynomial  $P = \sum_{i=1}^n B[i]X^{i-1}$  verifies  $P(L[i]) = dA[i]$  for  $1 \leq i \leq n$ . Assumes that  $p$  is prime.

GEN Flv\_roots\_to\_pol(GEN a, ulong p, long sv) as FpV\_roots.to.pol returning an Flx in variable  $v$ .

**7.3.17** FlxV. See FpXV operations.

GEN FlxV\_Flc\_mul(GEN V, GEN W, ulong p), as FpXV\_FpC\_mul.

GEN FlxV\_red(GEN V, ulong p) reduces each components with Flx\_red.

GEN FlxV\_prod(GEN V, ulong p),  $V$  being a vector of Flx, returns their product.

ulong FlxC\_eval\_powers\_pre(GEN x, GEN y, ulong p, ulong pi) apply Flx\_eval\_powers\_pre to all elements of  $x$ .

GEN FlxV\_Flv\_multieval(GEN F, GEN v, ulong p) assuming  $F$  is a vector of Flx with  $m$  entries and  $v$  is a Flv with  $m$  entries, returns the  $n$ -components vector (FlvV) whose  $j$ -th entry is  $[F_j(v[1]), \dots, F_j(v[n])]$ , with  $F_j = F[j]$ .

GEN FlxC\_neg(GEN x, ulong p)

GEN FlxC\_sub(GEN x, GEN y, ulong p)

GEN zero\_FlxC(long n, long sv)

**7.3.18** FlxM. See FpXM operations.

ulong FlxM\_eval\_powers\_pre(GEN M, GEN y, ulong p, ulong pi) this function applies FlxC\_eval\_powers\_pre to all entries of  $M$ .

GEN FlxM\_neg(GEN x, ulong p)

GEN FlxM\_sub(GEN x, GEN y, ulong p)

GEN zero\_FlxM(long r, long c, long sv)

**7.3.19** FlxT. See FpXT operations.

GEN FlxT\_red(GEN V, ulong p) reduces each leaf with Flx\_red.

**7.3.20 Flxn.** See FpXn operations. In this section,  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume  $\text{SMALL\_ULONG}(p)$ .

`GEN Flxn_mul(GEN a, GEN b, long n, ulong p)` returns  $ab$  modulo  $X^n$ .

`GEN Flxn_mul_pre(GEN a, GEN b, long n, ulong p, ulong pi)`

`GEN Flxn_sqr(GEN a, long n, ulong p)` returns  $a^2$  modulo  $X^n$ .

`GEN Flxn_sqr_pre(GEN a, long n, ulong p, ulong pi)`

`GEN Flxn_inv(GEN a, long n, ulong p)` returns  $1/a$  modulo  $X^n$ .

`GEN Flxn_div(GEN a, GEN b, long n, ulong p)` returns  $a/b$  modulo  $X^n$ .

`GEN Flxn_div_pre(GEN a, GEN b, long n, ulong p, ulong pi)`

`GEN Flxn_red(GEN a, long n)` returns  $a$  modulo  $X^n$ .

`GEN Flxn_exp(GEN x, long n, ulong p)` return  $\exp(x)$  as a composition of formal power series. It is required that the valuation of  $x$  is positive and that  $p > n$ .

`GEN Flxn_expint(GEN f, long n, ulong p)` return  $\exp(F)$  where  $F$  is the primitive of  $f$  that vanishes at 0. It is required that  $p > n$ .

**7.3.21 Flxq.** See FpXQ operations. In this section,  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume  $\text{SMALL\_ULONG}(p)$ .

`GEN Flxq_add(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_sub(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_mul(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_mul_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)`

`GEN Flxq_sqr(GEN y, GEN T, ulong p)`

`GEN Flxq_sqr_pre(GEN y, GEN T, ulong p)`

`GEN Flxq_inv(GEN x, GEN T, ulong p)`

`GEN Flxq_inv_pre(GEN x, GEN T, ulong p, ulong pi)`

`GEN Flxq_invsafe(GEN x, GEN T, ulong p)`

`GEN Flxq_invsafe_pre(GEN x, GEN T, ulong p, ulong pi)`

`GEN Flxq_div(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_div_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)`

`GEN Flxq_pow(GEN x, GEN n, GEN T, ulong p)`

`GEN Flxq_pow_pre(GEN x, GEN n, GEN T, ulong p, ulong pi)`

`GEN Flxq_powu(GEN x, ulong n, GEN T, ulong p)`

`GEN Flxq_powu_pre(GEN x, ulong n, GEN T, ulong p)`

`GEN FlxqV_factorback(GEN L, GEN e, GEN Tp, ulong p)`

`GEN Flxq_pow_init(GEN x, GEN n, long k, GEN T, ulong p)`



GEN Flxq\_pow\_init\_pre(GEN x, GEN n, long k, GEN T, ulong p, ulong pi)  
 GEN Flxq\_pow\_table(GEN R, GEN n, GEN T, ulong p)  
 GEN Flxq\_pow\_table\_pre(GEN R, GEN n, GEN T, ulong p, ulong pi)  
 GEN Flxq\_powers(GEN x, long n, GEN T, ulong p)  
 GEN Flxq\_powers\_pre(GEN x, long n, GEN T, ulong p, ulong pi)  
 GEN Flxq\_matrix\_pow(GEN x, long m, long n, GEN T, ulong p), see FpXQ\_matrix\_pow.  
 GEN Flxq\_matrix\_pow\_pre(GEN x, long m, long n, GEN T, ulong p, ulong pi)  
 GEN Flxq\_autpow(GEN a, long n, GEN T, ulong p) see FpXQ\_autpow.  
 GEN Flxq\_autpow\_pre(GEN a, long n, GEN T, ulong p, ulong pi)  
 GEN Flxq\_autpowers(GEN a, long n, GEN T, ulong p) return  $[X, \sigma(X), \dots, \sigma^n(X)]$ , assuming  $a = \sigma(X)$  where  $\sigma$  is an automorphism of the algebra  $\mathbb{F}_p[X]/T(X)$ .  
 GEN Flxq\_autsum(GEN a, long n, GEN T, ulong p) see FpXQ\_autsum.  
 GEN Flxq\_auttrace(GEN a, ulong n, GEN T, ulong p) see FpXQ\_auttrace.  
 GEN Flxq\_auttrace\_pre(GEN a, ulong n, GEN T, ulong p, ulong pi)  
 GEN Flxq\_ffisom\_inv(GEN S, GEN T, ulong p), as FpXQ\_ffisom\_inv.  
 GEN Flx\_Flxq\_eval(GEN f, GEN x, GEN T, ulong p) returns  $f(x)$ .  
 GEN Flx\_Flxq\_eval\_pre(GEN f, GEN x, GEN T, ulong p, ulong pi)  
 GEN Flx\_FlxqV\_eval(GEN f, GEN x, GEN T, ulong p), see FpX\_FpXQV\_eval.  
 GEN Flx\_FlxqV\_eval\_pre(GEN f, GEN x, GEN T, ulong p, ulong pi)  
 GEN FlxC\_Flxq\_eval(GEN C, GEN x, GEN T, ulong p), see FpXC\_FpXQ\_eval.  
 GEN FlxC\_Flxq\_eval\_pre(GEN C, GEN x, GEN T, ulong p, ulong pi)  
 GEN FlxC\_FlxqV\_eval(GEN C, GEN V, GEN T, ulong p) see FpXC\_FpXQV\_eval.  
 GEN FlxC\_FlxqV\_eval\_pre(GEN C, GEN V, GEN T, ulong p, ulong pi)  
 GEN FlxqV\_roots\_to\_pol(GEN V, GEN T, ulong p, long v) as FqV\_roots\_to\_pol returning an FlxqX in variable  $v$ .  
 int Flxq\_issquare(GEN x, GEN T, ulong p) returns 1 if  $x$  is a square and 0 otherwise. Assume that  $T$  is irreducible mod  $p$ .  
 int Flxq\_is2npower(GEN x, long n, GEN T, ulong p) returns 1 if  $x$  is a  $2^n$ -th power and 0 otherwise. Assume that  $T$  is irreducible mod  $p$ .  
 GEN Flxq\_order(GEN a, GEN ord, GEN T, ulong p) as FpXQ\_order.  
 GEN Flxq\_log(GEN a, GEN g, GEN ord, GEN T, ulong p) as FpXQ\_log  
 GEN Flxq\_sqrtn(GEN x, GEN n, GEN T, ulong p, GEN \*zn) as FpXQ\_sqrtn.  
 GEN Flxq\_sqrt(GEN x, GEN T, ulong p) returns a square root of  $x$ . Return NULL if  $x$  is not a square.  
 GEN Flxq\_lroot(GEN a, GEN T, ulong p) returns  $x$  such that  $x^p = a$ .

`GEN Flxq_lroot_pre(GEN a, GEN T, ulong p, ulong pi)`  
`GEN Flxq_lroot_fast(GEN a, GEN V, GEN T, ulong p)` assuming that  $V = \text{Flxq.powers}(s, p-1, T, p)$  where  $s(x)^p \equiv x \pmod{T(x), p}$ , returns  $b$  such that  $b^p = a$ . Only useful if  $p$  is less than the degree of  $T$ .  
`GEN Flxq_lroot_fast_pre(GEN a, GEN V, GEN T, ulong p, ulong pi)`  
`GEN Flxq_charpoly(GEN x, GEN T, ulong p)` returns the characteristic polynomial of  $x$   
`GEN Flxq_minpoly(GEN x, GEN T, ulong p)` returns the minimal polynomial of  $x$   
`GEN Flxq_minpoly_pre(GEN x, GEN T, ulong p, ulong pi)`  
`ulong Flxq_norm(GEN x, GEN T, ulong p)` returns the norm of  $x$   
`ulong Flxq_trace(GEN x, GEN T, ulong p)` returns the trace of  $x$   
`GEN Flxq_conjvec(GEN x, GEN T, ulong p)` returns the conjugates  $[x, x^p, x^{p^2}, \dots, x^{p^{n-1}}]$  where  $n$  is the degree of  $T$ .  
`GEN gener_Flxq(GEN T, ulong p, GEN *po)` returns a primitive root modulo  $(T, p)$ .  $T$  is an `Flx` assumed to be irreducible modulo the prime  $p$ . If `po` is not `NULL` it is set to  $[o, fa]$ , where  $o$  is the order of the multiplicative group of the finite field, and  $fa$  is its factorization.

**7.3.22 FlxX.** See `FpXX` operations. In this section, we assume  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume `SMALL_ULONG(p)`.

`GEN pol1_FlxX(long vX, long sx)` returns the unit `FlxX` as a `t_POL` in variable `vX` which only coefficient is `pol1_Flx(sx)`.  
`GEN polx_FlxX(long vX, long sx)` returns the variable  $X$  as a degree 1 `t_POL` with `Flx` coefficients in the variable  $x$ .  
`long FlxY_degreeex(GEN P)` return the degree of  $P$  with respect to the secondary variable.  
`GEN FlxX_add(GEN P, GEN Q, ulong p)`  
`GEN FlxX_sub(GEN P, GEN Q, ulong p)`  
`GEN FlxX_Fl_mul(GEN x, ulong y, ulong p)`  
`GEN FlxX_double(GEN x, ulong p)`  
`GEN FlxX_triple(GEN x, ulong p)`  
`GEN FlxX_neg(GEN x, ulong p)`  
`GEN FlxX_Flx_add(GEN x, GEN y, ulong p)`  
`GEN FlxX_Flx_sub(GEN x, GEN y, ulong p)`  
`GEN FlxX_Flx_mul(GEN x, GEN y, ulong p)`  
`GEN FlxY_Flx_div(GEN x, GEN y, ulong p)` divides the coefficients of  $x$  by  $y$  using `Flx_div`.  
`GEN FlxX_deriv(GEN P, ulong p)` returns the derivative of  $P$  with respect to the main variable.  
`GEN FlxX_Laplace(GEN x, ulong p)`  
`GEN FlxX_invLaplace(GEN x, ulong p)`

GEN FlxY\_evalx(GEN P, ulong z, ulong p)  $P$  being an FlxY, returns the Flx  $P(z, Y)$ , where  $Y$  is the main variable of  $P$ .

GEN FlxY\_evalx\_pre(GEN P, ulong z, ulong p, ulong pi)

GEN FlxX\_translate1(GEN P, ulong p, long n)  $P$  being an FlxX with all coefficients of degree at most  $n$ , return  $(P(x, Y + 1))$ , where  $Y$  is the main variable of  $P$ .

GEN zlxX\_translate1(GEN P, ulong p, long e, long n)  $P$  being an zlxX with all coefficients of degree at most  $n$ , return  $(P(x, Y + 1))$  modulo  $p^e$  for prime  $p$ , where  $Y$  is the main variable of  $P$ .

GEN FlxY\_Flx\_translate(GEN P, GEN f, ulong p)  $P$  being an FlxY and  $f$  being an Flx, return  $(P(x, Y + f(x)))$ , where  $Y$  is the main variable of  $P$ .

ulong FlxY\_evalx\_powers\_pre(GEN P, GEN xp, ulong p, ulong pi),  $xp$  being the vector  $[1, x, \dots, x^n]$ , where  $n$  is larger or equal to the degree of  $P$  in  $X$ , return  $P(x, Y)$ , where  $Y$  is the main variable of  $Q$ .

ulong FlxY\_eval\_powers\_pre(GEN P, GEN xp, GEN yp, ulong p, ulong pi),  $xp$  being the vector  $[1, x, \dots, x^n]$ , where  $n$  is larger or equal to the degree of  $P$  in  $X$  and  $yp$  being the vector  $[1, y, \dots, y^m]$ , where  $m$  is larger or equal to the degree of  $P$  in  $Y$  return  $P(x, y)$ .

GEN FlxY\_Flxq\_evalx(GEN x, GEN y, GEN T, ulong p) as FpXY\_FpXQ\_evalx.

GEN FlxY\_Flxq\_evalx\_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)

GEN FlxY\_FlxqV\_evalx(GEN x, GEN V, GEN T, ulong p) as FpXY\_FpXQV\_evalx.

GEN FlxY\_FlxqV\_evalx\_pre(GEN x, GEN V, GEN T, ulong p, ulong pi)

GEN FlxX\_renormalize(GEN x, long l), as normalizopol, where  $l = \lg(x)$ , in place.

GEN FlxX\_resultant(GEN u, GEN v, ulong p, long sv) Returns  $\text{Res}_X(u, v)$ , which is an Flx. The coefficients of  $u$  and  $v$  are assumed to be in the variable  $v$ .

GEN Flx\_FlxY\_resultant(GEN a, GEN b, ulong p) Returns  $\text{Res}_x(a, b)$ , which is an Flx in the main variable of  $b$ .

GEN FlxX\_blocks(GEN P, long n, long m, long sv), as RgX\_blocks, where  $v$  is the secondary variable.

GEN FlxX\_shift(GEN a, long n, long sv), as RgX\_shift\_shallow, where  $v$  is the secondary variable.

GEN FlxX\_swap(GEN x, long n, long ws), as RgXY\_swap.

GEN FlxYqq\_pow(GEN x, GEN n, GEN S, GEN T, ulong p), as FpXYQQ\_pow.

### 7.3.23 FlxXV, FlxXC, FlxXM. See FpXX operations.

GEN FlxXC\_sub(GEN x, GEN y, ulong p)

**7.3.24 FlxqX.** See FpXQX operations.

#### **7.3.24.1 Preconditioned reduction.**

For faster reduction, the modulus  $S$  can be replaced by an extended modulus, which is an  $\text{FlxqXT}$ , in all  $\text{FlxqXQ}$ -classes functions, and in  $\text{FlxqX\_rem}$  and  $\text{FlxqX\_divrem}$ .

$\text{GEN FlxqX\_get\_red}(\text{GEN } S, \text{GEN } T, \text{ulong } p)$  returns the extended modulus  $eS$ .

$\text{GEN FlxqX\_get\_red\_pre}(\text{GEN } S, \text{GEN } T, \text{ulong } p, \text{ulong } pi)$ , where  $pi$  is a pseudoinverse of  $p$ , or 0 in which case we assume  $\text{SMALL\_ULONG}(p)$ .

To write code that works both with plain and extended moduli, the following accessors are defined:

$\text{GEN get\_FlxqX\_mod}(\text{GEN } eS)$  returns the underlying modulus  $S$ .

$\text{GEN get\_FlxqX\_var}(\text{GEN } eS)$  returns the variable number of the modulus.

$\text{GEN get\_FlxqX\_degree}(\text{GEN } eS)$  returns the degree of the modulus.

#### **7.3.24.2 basic functions.**

In this section,  $pi$  is a pseudoinverse of  $p$ , or 0 in which case we assume  $\text{SMALL\_ULONG}(p)$ .

$\text{GEN random\_FlxqX}(\text{long } d, \text{long } v, \text{GEN } T, \text{ulong } p)$  returns a random  $\text{FlxqX}$  in variable  $v$ , of degree less than  $d$ .

$\text{GEN zxX\_to\_Kronecker}(\text{GEN } P, \text{GEN } Q)$  assuming  $P(X, Y)$  is a polynomial of degree in  $X$  strictly less than  $n$ , returns  $P(X, X^{2*n-1})$ , the Kronecker form of  $P$ .

$\text{GEN Kronecker\_to\_FlxqX}(\text{GEN } z, \text{GEN } T, \text{ulong } p)$ . Let  $n = \deg T$  and let  $P(X, Y) \in \mathbf{Z}[X, Y]$  lift a polynomial in  $K[Y]$ , where  $K := \mathbf{F}_p[X]/(T)$  and  $\deg_X P < 2n-1$  — such as would result from multiplying minimal degree lifts of two polynomials in  $K[Y]$ . Let  $z = P(t, t^{2*n-1})$  be a Kronecker form of  $P$ , this function returns  $Q \in \mathbf{Z}[X, t]$  such that  $Q$  is congruent to  $P(X, t) \pmod{(p, T(X))}$ ,  $\deg_X Q < n$ , and all coefficients are in  $[0, p[$ . Not stack-clean. Note that  $t$  need not be the same variable as  $Y$ !

$\text{GEN Kronecker\_to\_FlxqX\_pre}(\text{GEN } z, \text{GEN } T, \text{ulong } p, \text{ulong } pi)$

$\text{GEN FlxqX\_red}(\text{GEN } z, \text{GEN } T, \text{ulong } p)$

$\text{GEN FlxqX\_red\_pre}(\text{GEN } z, \text{GEN } T, \text{ulong } p, \text{ulong } pi)$

$\text{GEN FlxqX\_normalize}(\text{GEN } z, \text{GEN } T, \text{ulong } p)$

$\text{GEN FlxqX\_normalize\_pre}(\text{GEN } z, \text{GEN } T, \text{ulong } p, \text{ulong } pi)$

$\text{GEN FlxqX\_mul}(\text{GEN } x, \text{GEN } y, \text{GEN } T, \text{ulong } p)$

$\text{GEN FlxqX\_mul\_pre}(\text{GEN } x, \text{GEN } y, \text{GEN } T, \text{ulong } p, \text{ulong } pi)$

$\text{GEN FlxqX\_Flxq\_mul}(\text{GEN } P, \text{GEN } U, \text{GEN } T, \text{ulong } p)$

$\text{GEN FlxqX\_Flxq\_mul\_pre}(\text{GEN } P, \text{GEN } U, \text{GEN } T, \text{ulong } p, \text{ulong } pi)$

$\text{GEN FlxqX\_Flxq\_mul\_to\_monic}(\text{GEN } P, \text{GEN } U, \text{GEN } T, \text{ulong } p)$  returns  $P * U$  assuming the result is monic of the same degree as  $P$  (in particular  $U \neq 0$ ).

$\text{GEN FlxqX\_Flxq\_mul\_to\_monic\_pre}(\text{GEN } P, \text{GEN } U, \text{GEN } T, \text{ulong } p, \text{ulong } pi)$

$\text{GEN FlxqX\_sqr}(\text{GEN } x, \text{GEN } T, \text{ulong } p)$

GEN FlxqX\_sqr\_pre(GEN x, GEN T, ulong p, ulong pi)  
 GEN FlxqX\_powu(GEN x, ulong n, GEN T, ulong p)  
 GEN FlxqX\_powu\_pre(GEN x, ulong n, GEN T, ulong p, ulong pi)  
 GEN FlxqX\_divrem(GEN x, GEN y, GEN T, ulong p, GEN \*pr)  
 GEN FlxqX\_divrem\_pre(GEN x, GEN y, GEN T, ulong p, ulong pi, GEN \*pr)  
 GEN FlxqX\_div(GEN x, GEN y, GEN T, ulong p)  
 GEN FlxqX\_div\_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)  
 GEN FlxqX\_rem(GEN x, GEN y, GEN T, ulong p)  
 GEN FlxqX\_rem\_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)  
 GEN FlxqX\_invBarrett(GEN T, GEN Q, ulong p)  
 GEN FlxqX\_invBarrett\_pre(GEN T, GEN Q, ulong p, ulong pi)  
 GEN FlxqX\_gcd(GEN x, GEN y, ulong p) returns a (not necessarily monic) greatest common divisor of  $x$  and  $y$ .  
 GEN FlxqX\_gcd\_pre(GEN x, GEN y, ulong p, ulong pi)  
 GEN FlxqX\_extgcd(GEN x, GEN y, GEN T, ulong p, GEN \*ptu, GEN \*ptv)  
 GEN FlxqX\_extgcd\_pre(GEN x, GEN y, GEN T, ulong p, ulong pi, GEN \*ptu, GEN \*ptv)  
 GEN FlxqX\_halfgcd(GEN x, GEN y, GEN T, ulong p), see FpX\_halfgcd.  
 GEN FlxqX\_halfgcd\_pre(GEN x, GEN y, GEN T, ulong p, ulong pi)  
 GEN FlxqX\_resultant(GEN x, GEN y, GEN T, ulong p)  
 GEN FlxqX\_saferes resultant(GEN P, GEN Q, GEN T, ulong p) Returns the resultant of  $P$  and  $Q$  if Euclid's algorithm succeeds and NULL otherwise. In particular, if  $p$  is not prime or  $T$  is not irreducible over  $\mathbf{F}_p[X]$ , the routine may still be used (but will fail if noninvertible leading terms occur).  
 GEN FlxqX\_disc(GEN x, GEN T, ulong p)  
 GEN FlxqXV\_prod(GEN V, GEN T, ulong p)  
 GEN FlxqX\_safegcd(GEN P, GEN Q, GEN T, ulong p) Returns the *monic* GCD of  $P$  and  $Q$  if Euclid's algorithm succeeds and NULL otherwise. In particular, if  $p$  is not prime or  $T$  is not irreducible over  $\mathbf{F}_p[X]$ , the routine may still be used (but will fail if noninvertible leading terms occur).  
 GEN FlxqX\_dotproduct(GEN x, GEN y, GEN T, ulong p) returns the scalar product of the coefficients of  $x$  and  $y$ .  
 GEN FlxqX\_Newton(GEN x, long n, GEN T, ulong p)  
 GEN FlxqX\_Newton\_pre(GEN x, long n, GEN T, ulong p, ulong pi)  
 GEN FlxqX\_fromNewton(GEN x, GEN T, ulong p)  
 GEN FlxqX\_fromNewton\_pre(GEN x, GEN T, ulong p, ulong pi) We assume  $pi$  is a pseudoinverse of  $p$ , or 0 in which case we assume SMALL\_ULONG( $p$ ).

`long FlxqX_is_squarefree(GEN S, GEN T, ulong p), as FpX_is_squarefree.`  
`long FlxqX_ispower(GEN f, ulong k, GEN T, ulong p, GEN *pt)` return 1 if the FlxqX  $f$  is a  $k$ -th power, 0 otherwise. If  $pt$  is not NULL, set it to  $g$  such that  $g^k = f$ .  
`GEN FlxqX_Frobenius(GEN S, GEN T, ulong p), as FpXQX_Frobenius`  
`GEN FlxqX_Frobenius_pre(GEN S, GEN T, ulong p, ulong pi)`  
`GEN FlxqX_roots(GEN f, GEN T, ulong p)` return the roots of  $f$  in  $\mathbf{F}_p[X]/(T)$ . Assumes  $p$  is prime and  $T$  irreducible in  $\mathbf{F}_p[X]$ .  
`GEN FlxqX_factor(GEN f, GEN T, ulong p)` return the factorization of  $f$  over  $\mathbf{F}_p[X]/(T)$ . Assumes  $p$  is prime and  $T$  irreducible in  $\mathbf{F}_p[X]$ .  
`GEN FlxqX_factor_squarefree(GEN f, GEN T, ulong p)` returns the squarefree factorization of  $f$ , see `FpX_factor_squarefree`.  
`GEN FlxqX_factor_squarefree_pre(GEN f, GEN T, ulong p, ulong pi)`  
`GEN FlxqX_ddf(GEN f, GEN T, ulong p)` as `FpX_ddf`.  
`long FlxqX_ddf_degree(GEN f, GEN XP, GEN T, GEN p), as FpX_ddf_degree.`  
`GEN FlxqX_degfact(GEN f, GEN T, ulong p), as FpX_degfact.`  
`long FlxqX_nbroots(GEN S, GEN T, ulong p), as FpX_nbroots.`  
`long FlxqX_nbfact(GEN S, GEN T, ulong p), as FpX_nbfact.`  
`long FlxqX_nbfact_Frobenius(GEN S, GEN Xq, GEN T, ulong p), as FpX_nbfact_Frobenius.`  
`GEN FlxqX_nbfact_by_degree(GEN z, long *nb, GEN T, ulong p)` Assume that the FlxqX  $z$  is squarefree mod the prime  $p$ . Returns a `t_VECSMALL`  $D$  with  $\deg z$  entries, such that  $D[i]$  is the number of irreducible factors of degree  $i$ . Set  $nb$  to the total number of irreducible factors (the sum of the  $D[i]$ ).  
`GEN FlxqX_FlxqXQ_eval(GEN Q, GEN x, GEN S, GEN T, ulong p)` as `FpX_FpXQ_eval`.  
`GEN FlxqX_FlxqXQ_eval_pre(GEN Q, GEN x, GEN S, GEN T, ulong p, ulong pi)`  
`GEN FlxqX_FlxqXQV_eval(GEN P, GEN V, GEN S, GEN T, ulong p)` as `FpX_FpXQV_eval`.  
`GEN FlxqX_FlxqXQV_eval_pre(GEN P, GEN V, GEN S, GEN T, ulong p, ulong pi)`  
`GEN FlxqXC_FlxqXQ_eval(GEN Q, GEN x, GEN S, GEN T, ulong p)` as `FpXC_FpXQ_eval`.  
`GEN FlxqXC_FlxqXQ_eval_pre(GEN Q, GEN x, GEN S, GEN T, ulong p, ulong pi)`  
`GEN FlxqXC_FlxqXQV_eval(GEN P, GEN V, GEN S, GEN T, ulong p)` as `FpXC_FpXQV_eval`.  
`GEN FlxqXC_FlxqXQV_eval_pre(GEN P, GEN V, GEN S, GEN T, ulong p, ulong pi)`

**7.3.25** FlxqXQ. See FpXQXQ operations. In this section,  $pi$  is a pseudoinverse of  $p$ , or 0 in which case we assume `SMALL_ULONG( $p$ )`.

```

GEN FlxqXQ_mul(GEN x, GEN y, GEN S, GEN T, ulong p)
GEN FlxqXQ_mul_pre(GEN x, GEN y, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_sqr(GEN x, GEN S, GEN T, ulong p)
GEN FlxqXQ_sqr_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_inv(GEN x, GEN S, GEN T, ulong p)
GEN FlxqXQ_inv_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_invsafe(GEN x, GEN S, GEN T, ulong p)
GEN FlxqXQ_invsafe_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_div(GEN x, GEN y, GEN S, GEN T, ulong p)
GEN FlxqXQ_div_pre(GEN x, GEN y, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_pow(GEN x, GEN n, GEN S, GEN T, ulong p)
GEN FlxqXQ_pow_pre(GEN x, GEN n, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_powu(GEN x, ulong n, GEN S, GEN T, ulong p)
GEN FlxqXQ_powu_pre(GEN x, ulong n, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_powers(GEN x, long n, GEN S, GEN T, ulong p)
GEN FlxqXQ_powers_pre(GEN x, long n, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_matrix_pow(GEN x, long n, long m, GEN S, GEN T, ulong p)
GEN FlxqXQ_autpow(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ_autpow
GEN FlxqXQ_autpow_pre(GEN a, long n, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_autsum(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ_autsum
GEN FlxqXQ_autsum_pre(GEN a, long n, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_auttrace(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ_auttrace
GEN FlxqXQ_auttrace_pre(GEN a, long n, GEN S, GEN T, ulong p, ulong pi)
GEN FlxqXQ_halfFrobenius(GEN A, GEN S, GEN T, ulong p), as FpXQXQ_halfFrobenius
GEN FlxqXQ_minpoly(GEN x, GEN S, GEN T, ulong p), as FpXQ_minpoly
GEN FlxqXQ_minpoly_pre(GEN x, GEN S, GEN T, ulong p, ulong pi)

```

**7.3.26 FlxqXn.** See FpXn operations. In this section, we assume  $pi$  is the pseudoinverse of  $p$ , or 0 in which case we assume `SMALL_ULONG( $p$ )`.

GEN FlxXn\_red(GEN a, long n) returns  $a$  modulo  $X^n$ .  
 GEN FlxqXn\_mul(GEN a, GEN b, long n, GEN T, ulong p)  
 GEN FlxqXn\_mul\_pre(GEN a, GEN b, long n, GEN T, ulong p, ulong pi)  
 GEN FlxqXn\_sqr(GEN a, long n, GEN T, ulong p)  
 GEN FlxqXn\_sqr\_pre(GEN a, long n, GEN T, ulong p, ulong pi)  
 GEN FlxqXn\_inv(GEN a, long n, GEN T, ulong p)  
 GEN FlxqXn\_inv\_pre(GEN a, long n, GEN T, ulong p, ulong pi)  
 GEN FlxqXn\_expint(GEN a, long n, GEN T, ulong p)  
 GEN FlxqXn\_expint\_pre(GEN a, long n, GEN T, ulong p, ulong pi)

**7.3.27 F2x.** An F2x  $z$  is a `t_VECSMALL` representing a polynomial over  $\mathbf{F}_2[X]$ . Specifically  $z[0]$  is the usual codeword,  $z[1] = \text{evalvarn}(v)$  for some variable  $v$  and the coefficients are given by the bits of remaining words by increasing degree.

#### 7.3.27.1 Preconditioned reduction.

For faster reduction, the modulus  $T$  can be replaced by an extended modulus (`FlxT`) in all `Flxq`-classes functions, and in `Flx_divrem`.

GEN F2x\_get\_red(GEN T) returns the extended modulus `eT`.

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get\_F2x\_mod(GEN eT) returns the underlying modulus  $T$ .  
 GEN get\_F2x\_var(GEN eT) returns the variable number of the modulus.  
 GEN get\_F2x\_degree(GEN eT) returns the degree of the modulus.

#### 7.3.27.2 Basic operations.

ulong F2x\_coeff(GEN x, long i) returns the coefficient  $i \geq 0$  of  $x$ .  
 void F2x\_clear(GEN x, long i) sets the coefficient  $i \geq 0$  of  $x$  to 0.  
 void F2x\_flip(GEN x, long i) adds 1 to the coefficient  $i \geq 0$  of  $x$ .  
 void F2x\_set(GEN x, long i) sets the coefficient  $i \geq 0$  of  $x$  to 1.  
 GEN F2x\_copy(GEN x)  
 GEN Flx\_to\_F2x(GEN x)  
 GEN Z\_to\_F2x(GEN x, long sv)  
 GEN ZX\_to\_F2x(GEN x)  
 GEN F2v\_to\_F2x(GEN x, long sv)  
 GEN F2x\_to\_Flx(GEN x)



GEN F2x\_to\_F2xX(GEN x, long sv)  
 GEN F2x\_to\_ZX(GEN x)  
 GEN pol0\_F2x(long sv) returns a zero F2x in variable  $v$ .  
 GEN zero\_F2x(long sv) alias for pol0\_F2x.  
 GEN pol1\_F2x(long sv) returns the F2x in variable  $v$  constant to 1.  
 GEN polx\_F2x(long sv) returns the variable  $v$  as degree 1 F2x.  
 GEN monomial\_F2x(long d, long sv) returns the F2x  $X^d$  in variable  $v$ .  
 GEN random\_F2x(long d, long sv) returns a random F2x in variable  $v$ , of degree less than  $d$ .  
 long F2x\_degree(GEN x) returns the degree of the F2x  $x$ . The degree of 0 is defined as  $-1$ .  
 GEN F2x\_recip(GEN x)  
 int F2x\_equal1(GEN x)  
 int F2x\_equal(GEN x, GEN y)  
 GEN F2x\_1\_add(GEN y) returns  $y+1$  where  $y$  is a F1x.  
 GEN F2x\_add(GEN x, GEN y)  
 GEN F2x\_mul(GEN x, GEN y)  
 GEN F2x\_sqr(GEN x)  
 GEN F2x\_divrem(GEN x, GEN y, GEN \*pr)  
 GEN F2x\_rem(GEN x, GEN y)  
 GEN F2x\_div(GEN x, GEN y)  
 GEN F2x\_renormalize(GEN x, long lx)  
 GEN F2x\_deriv(GEN x)  
 GEN F2x\_deflate(GEN x, long d)  
 ulong F2x\_eval(GEN P, ulong u) returns  $P(u)$ .  
 void F2x\_shift(GEN x, long d) as RgX\_shift  
 void F2x\_even\_odd(GEN P, GEN \*pe, GEN \*po) as RgX\_even\_odd  
 long F2x\_valrem(GEN x, GEN \*Z)  
 GEN F2x\_extgcd(GEN a, GEN b, GEN \*ptu, GEN \*ptv)  
 GEN F2x\_gcd(GEN a, GEN b)  
 GEN F2x\_halfgcd(GEN a, GEN b)  
 int F2x\_issquare(GEN x) returns 1 if  $x$  is a square of a F2x and 0 otherwise.  
 int F2x\_is\_irred(GEN f), as FpX\_is\_irred.  
 GEN F2x\_degfact(GEN f) as FpX\_degfact.  
 GEN F2x\_sqrt(GEN x) returns the squareroot of  $x$ , assuming  $x$  is a square of a F2x.

GEN F2x\_Frobenius(GEN T)  
 GEN F2x\_matFrobenius(GEN T)  
 GEN F2x\_factor(GEN f)  
 GEN F2x\_factor\_squarefree(GEN f)  
 GEN F2x\_ddf(GEN f)  
 GEN F2x\_Teichmuller(GEN P, long n) Return a ZX  $Q$  such that  $P \equiv Q \pmod{2}$  and  $Q(X^p) = 0 \pmod{Q, 2^n}$ .

**7.3.28 F2xq.** See FpXQ operations.

GEN F2xq\_mul(GEN x, GEN y, GEN T)  
 GEN F2xq\_sqr(GEN x, GEN T)  
 GEN F2xq\_div(GEN x, GEN y, GEN T)  
 GEN F2xq\_inv(GEN x, GEN T)  
 GEN F2xq\_invsafe(GEN x, GEN T)  
 GEN F2xq\_pow(GEN x, GEN n, GEN T)  
 GEN F2xq\_powu(GEN x, ulong n, GEN T)  
 GEN F2xq\_pow\_init(GEN x, GEN n, long k, GEN T)  
 GEN F2xq\_pow\_table(GEN R, GEN n, GEN T)  
 ulong F2xq\_trace(GEN x, GEN T)  
 GEN F2xq\_conjvec(GEN x, GEN T) returns the vector of conjugates  $[x, x^2, x^{2^2}, \dots, x^{2^{n-1}}]$  where  $n$  is the degree of  $T$ .  
 GEN F2xq\_log(GEN a, GEN g, GEN ord, GEN T)  
 GEN F2xq\_order(GEN a, GEN ord, GEN T)  
 GEN F2xq\_Artin\_Schreier(GEN a, GEN T) returns a solution of  $x^2 + x = a$ , assuming it exists.  
 GEN F2xq\_sqrt(GEN a, GEN T)  
 GEN F2xq\_sqrt\_fast(GEN a, GEN s, GEN T) assuming that  $s^2 \equiv x \pmod{T(x)}$ , computes  $b \equiv a(s) \pmod{T}$  so that  $b^2 = a$ .  
 GEN F2xq\_sqrtn(GEN a, GEN n, GEN T, GEN \*zeta)  
 GEN gener\_F2xq(GEN T, GEN \*po)  
 GEN F2xq\_powers(GEN x, long n, GEN T)  
 GEN F2xq\_matrix\_pow(GEN x, long m, long n, GEN T)  
 GEN F2x\_F2xq\_eval(GEN f, GEN x, GEN T)  
 GEN F2x\_F2xqV\_eval(GEN f, GEN x, GEN T), see FpX\_FpXQV\_eval.  
 GEN F2xq\_outpow(GEN a, long n, GEN T) computes  $\sigma^n(X)$  assuming  $a = \sigma(X)$  where  $\sigma$  is an automorphism of the algebra  $\mathbf{F}_2[X]/T(X)$ .

**7.3.29** F2xn. See FpXn operations.

GEN F2xn\_red(GEN a, long n)

GEN F2xn\_div(GEN x, GEN y, long e)

GEN F2xn\_inv(GEN x, long e)

**7.3.30** F2xqV, F2xqM.. See FqV, FqM operations.

GEN F2xqM\_F2xqC\_gauss(GEN a, GEN b, GEN T)

GEN F2xqM\_F2xqC\_invimage(GEN a, GEN b, GEN T)

GEN F2xqM\_F2xqC\_mul(GEN a, GEN b, GEN T)

GEN F2xqM\_deplin(GEN x, GEN T)

GEN F2xqM\_det(GEN a, GEN T)

GEN F2xqM\_gauss(GEN a, GEN b, GEN T)

GEN F2xqM\_image(GEN x, GEN T)

GEN F2xqM\_indexrank(GEN x, GEN T)

GEN F2xqM\_inv(GEN a, GEN T)

GEN F2xqM\_invimage(GEN a, GEN b, GEN T)

GEN F2xqM\_ker(GEN x, GEN T)

GEN F2xqM\_mul(GEN a, GEN b, GEN T)

long F2xqM\_rank(GEN x, GEN T)

GEN F2xqM\_suppl(GEN x, GEN T)

GEN matid\_F2xqM(long n, GEN T)

**7.3.31** F2xX.. See FpXX operations.

GEN ZXX\_to\_F2xX(GEN x, long v)

GEN FlxX\_to\_F2xX(GEN x)

GEN F2xX\_to\_FlxX(GEN B)

GEN F2xX\_to\_F2xC(GEN B, long N, long sv)

GEN F2xXV\_to\_F2xM(GEN B, long N, long sv)

GEN F2xX\_to\_ZXX(GEN B)

GEN F2xX\_renormalize(GEN x, long lx)

long F2xY\_degreeex(GEN P) return the degree of  $P$  with respect to the secondary variable.

GEN pol1\_F2xX(long v, long sv)

GEN polx\_F2xX(long v, long sv)

GEN F2xX\_add(GEN x, GEN y)

GEN F2xX\_F2x\_add(GEN x, GEN y)  
 GEN F2xX\_F2x\_mul(GEN x, GEN y)  
 GEN F2xX\_deriv(GEN P) returns the derivative of P with respect to the main variable.  
 GEN Kronecker\_to\_F2xqX(GEN z, GEN T)  
 GEN F2xX\_to\_Kronecker(GEN z, GEN T)  
 GEN F2xY\_F2xq\_evalx(GEN x, GEN y, GEN T) as FpXY\_FpXQ\_evalx.  
 GEN F2xY\_F2xqV\_evalx(GEN x, GEN V, GEN T) as FpXY\_FpXQV\_evalx.

**7.3.32** F2xXV/F2xXC.. See FpXXV operations.

GEN FlxXC\_to\_F2xXC(GEN B)  
 GEN F2xXC\_to\_ZXXC(GEN B)

**7.3.33** F2xqX.. See FlxqX operations.

#### **7.3.33.1 Preconditioned reduction.**

For faster reduction, the modulus  $S$  can be replaced by an extended modulus, which is an  $F2xqXT$ , in all  $F2xqXQ$ -classes functions, and in  $F2xqX_{\text{rem}}$  and  $F2xqX_{\text{divrem}}$ .

GEN  $F2xqX_{\text{get\_red}}$ (GEN  $S$ , GEN  $T$ ) returns the extended modulus  $eS$ .

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN  $get\_F2xqX_{\text{mod}}$ (GEN  $eS$ ) returns the underlying modulus  $S$ .  
 GEN  $get\_F2xqX_{\text{var}}$ (GEN  $eS$ ) returns the variable number of the modulus.  
 GEN  $get\_F2xqX_{\text{degree}}$ (GEN  $eS$ ) returns the degree of the modulus.

#### **7.3.33.2 basic functions.**

GEN  $random\_F2xqX$ (long  $d$ , long  $v$ , GEN  $T$ , ulong  $p$ ) returns a random  $F2xqX$  in variable  $v$ , of degree less than  $d$ .

GEN  $F2xqX_{\text{red}}$ (GEN  $z$ , GEN  $T$ )  
 GEN  $F2xqX_{\text{normalize}}$ (GEN  $z$ , GEN  $T$ )  
 GEN  $F2xqX_{\text{F2xq\_mul}}$ (GEN  $P$ , GEN  $U$ , GEN  $T$ )  
 GEN  $F2xqX_{\text{F2xq\_mul\_to\_monic}}$ (GEN  $P$ , GEN  $U$ , GEN  $T$ )  
 GEN  $F2xqX_{\text{mul}}$ (GEN  $x$ , GEN  $y$ , GEN  $T$ )  
 GEN  $F2xqX_{\text{sqr}}$ (GEN  $x$ , GEN  $T$ )  
 GEN  $F2xqX_{\text{powu}}$ (GEN  $x$ , ulong  $n$ , GEN  $T$ )  
 GEN  $F2xqX_{\text{rem}}$ (GEN  $x$ , GEN  $y$ , GEN  $T$ )  
 GEN  $F2xqX_{\text{div}}$ (GEN  $x$ , GEN  $y$ , GEN  $T$ )  
 GEN  $F2xqX_{\text{divrem}}$ (GEN  $x$ , GEN  $y$ , GEN  $T$ , GEN  $*pr$ )

GEN F2xqXQ\_inv(GEN x, GEN S, GEN T)  
 GEN F2xqXQ\_invsafe(GEN x, GEN S, GEN T)  
 GEN F2xqX\_invBarrett(GEN T, GEN Q)  
 GEN F2xqX\_extgcd(GEN x, GEN y, GEN T, GEN \*ptu, GEN \*ptv)  
 GEN F2xqX\_gcd(GEN x, GEN y, GEN T)  
 GEN F2xqX\_halfgcd(GEN x, GEN y, GEN T)  
 GEN F2xqX\_resultant(GEN x, GEN y, GEN T)  
 GEN F2xqX\_disc(GEN x, GEN T)  
 long F2xqX\_ispower(GEN f, ulong k, GEN T, GEN \*pt)  
 GEN F2xqX\_F2xqXQ\_eval(GEN Q, GEN x, GEN S, GEN T) as FpX\_FpXQ\_eval.  
 GEN F2xqX\_F2xqXQV\_eval(GEN P, GEN V, GEN S, GEN T) as FpX\_FpXQV\_eval.  
 GEN F2xqX\_roots(GEN f, GEN T) return the roots of  $f$  in  $\mathbf{F}_2[X]/(T)$ . Assumes  $T$  irreducible in  $\mathbf{F}_2[X]$ .  
 GEN F2xqX\_factor(GEN f, GEN T) return the factorization of  $f$  over  $\mathbf{F}_2[X]/(T)$ . Assumes  $T$  irreducible in  $\mathbf{F}_2[X]$ .  
 GEN F2xqX\_factor\_squarefree(GEN f, GEN T) as FlxqX\_factor\_squarefree.  
 GEN F2xqX\_ddf(GEN f, GEN T) as FpX\_ddf.  
 GEN F2xqX\_degfact(GEN f, GEN T) as FpX\_degfact.

**7.3.34** F2xqXQ.. See FlxqXQ operations.

GEN FlxqXQ\_inv(GEN x, GEN S, GEN T)  
 GEN FlxqXQ\_invsafe(GEN x, GEN S, GEN T)  
 GEN F2xqXQ\_mul(GEN x, GEN y, GEN S, GEN T)  
 GEN F2xqXQ\_sqr(GEN x, GEN S, GEN T)  
 GEN F2xqXQ\_pow(GEN x, GEN n, GEN S, GEN T)  
 GEN F2xqXQ\_powers(GEN x, long n, GEN S, GEN T)  
 GEN F2xqXQ\_autpow(GEN a, long n, GEN S, GEN T) as FpXQXQ\_autpow  
 GEN F2xqXQ\_auttrace(GEN a, long n, GEN S, GEN T). Let  $\sigma$  be the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$  and  $\sigma(Y) = a[2] \pmod{S(X,Y),T(X)}$ ; returns the vector  $[\sigma^n(X), \sigma^n(Y), b + \sigma(b) + \dots + \sigma^{n-1}(b)]$  where  $b = a[3]$ .  
 GEN F2xqXQV\_red(GEN x, GEN S, GEN T)

### 7.3.35 Functions returning objects with `t_INTMOD` coefficients.

Those functions are mostly needed for interface reasons: `t_INTMOD`s should not be used in library mode since the modular kernel is more flexible and more efficient, but GP users do not have access to the modular kernel. We document them for completeness:

`GEN Fp_to_mod(GEN z, GEN p)`,  $z$  a `t_INT`. Returns  $z * \text{Mod}(1, p)$ , normalized. Hence the returned value is a `t_INTMOD`.

`GEN FpX_to_mod(GEN z, GEN p)`,  $z$  a `ZX`. Returns  $z * \text{Mod}(1, p)$ , normalized. Hence the returned value has `t_INTMOD` coefficients.

`GEN FpC_to_mod(GEN z, GEN p)`,  $z$  a `ZC`. Returns  $\text{Col}(z) * \text{Mod}(1, p)$ , a `t_COL` with `t_INTMOD` coefficients.

`GEN FpV_to_mod(GEN z, GEN p)`,  $z$  a `ZV`. Returns  $\text{Vec}(z) * \text{Mod}(1, p)$ , a `t_VEC` with `t_INTMOD` coefficients.

`GEN FpVV_to_mod(GEN z, GEN p)`,  $z$  a `ZVV`. Returns  $\text{Vec}(z) * \text{Mod}(1, p)$ , a `t_VEC` of `t_VEC` with `t_INTMOD` coefficients.

`GEN FpM_to_mod(GEN z, GEN p)`,  $z$  a `ZM`. Returns  $z * \text{Mod}(1, p)$ , with `t_INTMOD` coefficients.

`GEN F2c_to_mod(GEN x)`

`GEN F3c_to_mod(GEN x)`

`GEN F2m_to_mod(GEN x)`

`GEN F3m_to_mod(GEN x)`

`GEN Flc_to_mod(GEN z)`

`GEN Flm_to_mod(GEN z)`

`GEN FqC_to_mod(GEN z, GEN T, GEN p)`

`GEN FqM_to_mod(GEN z, GEN T, GEN p)`

`GEN FpXC_to_mod(GEN V, GEN p)`

`GEN FpXM_to_mod(GEN V, GEN p)`

`GEN FpXQC_to_mod(GEN V, GEN T, GEN p)`  $V$  being a vector of `FpXQ`, converts each entry to a `t_POLMOD` with `t_INTMOD` coefficients, and return a `t_COL`.

`GEN FpQXQ_to_mod(GEN P, GEN T, GEN p)`  $P$  being a `FpQXQ`, converts each coefficient to a `t_POLMOD` with `t_INTMOD` coefficients.

`GEN FqX_to_mod(GEN P, GEN T, GEN p)` same but allow  $T = \text{NULL}$ .

`GEN FqXC_to_mod(GEN P, GEN T, GEN p)`

`GEN FqXM_to_mod(GEN P, GEN T, GEN p)`

`GEN QXQ_to_mod_shallow(GEN x, GEN T)`  $x$  a `QXQ`, which is a lifted representative of elements of  $\mathbf{Q}[X]/(T)$  (number field elements in most applications) and  $T$  is in  $\mathbf{Z}[X]$ . Convert it to a `t_POLMOD` modulo  $T$ ; no reduction mod  $T$  is attempted: the representatives should be already reduced. Shallow function.

GEN QXQV\_to\_mod(GEN V, GEN T)  $V$  a vector of QXQ, which are lifted representatives of elements of  $\mathbf{Q}[X]/(T)$  (number field elements in most applications) and  $T$  is in  $\mathbf{Z}[X]$ . Return a vector where all nonrational entries are converted to `t_POLMOD` modulo  $T$ ; no reduction mod  $T$  is attempted: the representatives should be already reduced. Used to normalize the output of `nfroots`.

GEN QXQX\_to\_mod\_shallow(GEN P, GEN T)  $P$  a polynomial with QXQ coefficients; replace them by `mkpolmod(.,T)`. Shallow function.

GEN QXQC\_to\_mod\_shallow(GEN V, GEN T)  $V$  a vector with QXQ coefficients; replace them by `mkpolmod(.,T)`. Shallow function.

GEN QXQM\_to\_mod\_shallow(GEN M, GEN T)  $M$  a matrix with QXQ coefficients; replace them by `mkpolmod(.,T)`. Shallow function.

GEN QXQXV\_to\_mod(GEN V, GEN T)  $V$  a vector of polynomials whose coefficients are QXQ. Analogous to `QXQV_to_mod`. Used to normalize the output of `nfactor`.

The following functions are obsolete and should not be used: they receive a polynomial with arbitrary coefficients, apply a conversion function to map them to a finite field, a function from the modular kernel, then `*_to_mod`:

GEN rootmod(GEN f, GEN p), applies `FpX_roots`.

GEN rootmod2(GEN f, GEN p), (now) identical to `rootmod`.

GEN rootmod0(GEN f, GEN p, long flag), (now) identical to `rootmod`; ignores *flag*.

GEN factmod(GEN f, GEN p) applies `*_factor`.

GEN simplefactmod(GEN f, GEN p) applies `*_degfact`.

**7.3.36 Slow Chinese remainder theorem over  $\mathbf{Z}$ .** The routines in this section have quadratic time complexity with respect to the input size; see the routines in the next two sections for quasi-linear time variants.

GEN Z\_chinese(GEN a, GEN b, GEN A, GEN B) returns the integer in  $[0, \text{lcm}(A, B)[$  congruent to  $a \bmod A$  and  $b \bmod B$ , assuming it exists; in other words, that  $a$  and  $b$  are congruent mod  $\text{gcd}(A, B)$ .

GEN Z\_chinese\_all(GEN a, GEN b, GEN A, GEN B, GEN \*pC) as `Z_chinese`, setting `*pC` to the lcm of  $A$  and  $B$ .

GEN Z\_chinese\_coprime(GEN a, GEN b, GEN A, GEN B, GEN C), as `Z_chinese`, assuming that  $\text{gcd}(A, B) = 1$  and that  $C = \text{lcm}(A, B) = AB$ .

ulong u\_chinese\_coprime(ulong a, ulong b, ulong A, ulong B, ulong C), as `Z_chinese_coprime` for ulong inputs and output.

void Z\_chinese\_pre(GEN A, GEN B, GEN \*pC, GEN \*pU, GEN \*pd) initializes chinese remainder computations modulo  $A$  and  $B$ . Sets `*pC` to  $\text{lcm}(A, B)$ , `*pd` to  $\text{gcd}(A, B)$ , `*pU` to an integer congruent to 0 mod  $(A/d)$  and 1 mod  $(B/d)$ . It is allowed to set `pd = NULL`, in which case,  $d$  is still computed, but not saved.

GEN Z\_chinese\_post(GEN a, GEN b, GEN C, GEN U, GEN d) returns the solution to the chinese remainder problem  $x$  congruent to  $a \bmod A$  and  $b \bmod B$ , where  $C, U, d$  were set in `Z_chinese_pre`. If  $d$  is NULL, assume the problem has a solution. Otherwise, return NULL if it has no solution.

The following pair of functions is used in homomorphic imaging schemes, when reconstructing an integer from its images modulo pairwise coprime integers. The idea is as follows: we want to discover an integer  $H$  which satisfies  $|H| < B$  for some known bound  $B$ ; we are given pairs  $(H_p, p)$  with  $H$  congruent to  $H_p \bmod p$  and all  $p$  pairwise coprime.

Given  $H$  congruent to  $H_p$  modulo a number of  $p$ , whose product is  $q$ , and a new pair  $(H_p, p)$ ,  $p$  coprime to  $q$ , the following incremental functions use the chinese remainder theorem (CRT) to find a new  $H$ , congruent to the preceding one modulo  $q$ , but also to  $H_p$  modulo  $p$ . It is defined uniquely modulo  $qp$ , and we choose the centered representative. When  $P$  is larger than  $2B$ , we have  $H = H$ , but of course, the value of  $H$  may stabilize sooner. In many applications it is possible to directly check that such a partial result is correct.

`GEN Z_init_CRT(ulong Hp, ulong p)` given a `Fl Hp` in  $[0, p-1]$ , returns the centered representative  $H$  congruent to  $H_p$  modulo  $p$ .

`int Z_incremental_CRT(GEN *H, ulong Hp, GEN *q, ulong p)` given a `t_INT *H`, centered modulo  $*q$ , a new pair  $(H_p, p)$  with  $p$  coprime to  $q$ , this function updates  $*H$  so that it also becomes congruent to  $(H_p, p)$ , and  $*q$  to the product  $qp = p \cdot *q$ . It returns 1 if the new value is equal to the old one, and 0 otherwise.

`GEN chinese1_coprime_Z(GEN v)` an alternative divide-and-conquer implementation:  $v$  is a vector of `t_INTMOD` with pairwise coprime moduli. Return the `t_INTMOD` solving the corresponding chinese remainder problem. This is a streamlined version of

`GEN chinese1(GEN v)`, which solves a general chinese remainder problem (not necessarily over  $\mathbf{Z}$ , moduli not assumed coprime).

As above, for  $H$  a  $\mathbf{ZM}$ : we assume that  $H$  and all  $H_p$  have dimension  $> 0$ . The original  $*H$  is destroyed.

`GEN ZM_init_CRT(GEN Hp, ulong p)`

`int ZM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above for  $H$  a  $\mathbf{ZX}$ : note that the degree may increase or decrease. The original  $*H$  is destroyed.

`GEN ZX_init_CRT(GEN Hp, ulong p, long v)`

`int ZX_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above, for  $H$  a matrix whose coefficient are  $\mathbf{ZX}$ . The original  $*H$  is destroyed. The entries of  $H$  are not normalized, use `ZX.renormalize` for this.

`GEN ZXM_init_CRT(GEN Hp, long deg, ulong p)` where `deg` is the maximal degree of all the  $H_p$

`int ZXM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`



### 7.3.37 Fast remainders.

The routines in these section are asymptotically fast (quasi-linear time in the input size).

**GEN Z\_ZV\_mod**(GEN A, GEN P) given a **t\_INT**  $A$  and a vector  $P$  of positive pairwise coprime integers of length  $n \geq 1$ , return a vector  $B$  of the same length such that  $B[i] = A \pmod{P[i]}$  and  $0 \leq B[i] < P[i]$  for all  $1 \leq i \leq n$ . The vector  $P$  may be a **t\_VEC** or a **t\_VECSMALL** (treated as **ulongs**) and  $B$  has the same type as  $P$ .

**GEN Z\_nv\_mod**(GEN A, GEN P) given a **t\_INT**  $A$  and a **t\_VECSMALL**  $P$  of positive pairwise coprime integers of length  $n \geq 1$ , return a **t\_VECSMALL**  $B$  of the same length such that  $B[i] = A \pmod{P[i]}$  and  $0 \leq B[i] < P[i]$  for all  $1 \leq i \leq n$ . The entries of  $P$  and  $B$  are treated as **ulongs**.

The following low level functions allow precomputations:

**GEN ZV\_producttree**(GEN P) where  $P$  is a vector of integers (or **t\_VECSMALL**) of length  $n \geq 1$ , return the vector of **t\_VECS**  $[f(P), f^2(P), \dots, f^k(P)]$  where  $f$  is the transformation  $[p_1, p_2, \dots, p_m] \mapsto [p_1 p_2, p_3 p_4, \dots, p_{m-1} p_m]$  if  $m$  is even and  $[p_1 p_2, p_3 p_4, \dots, p_{m-2} p_{m-1}, p_m]$  if  $m$  is odd, and  $k = O(\log m)$  is minimal so that  $f^k(P)$  has length 1; in other words,  $f^k(P) = [p_1 p_2 \dots p_m]$ .

**GEN Z\_ZV\_mod\_tree**(GEN A, GEN P, GEN T) as **Z\_ZV\_mod** where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZV\_nv\_mod\_tree**(GEN A, GEN P, GEN T)  $A$  being a **ZV** and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **Flv**  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZM\_nv\_mod\_tree**(GEN A, GEN P, GEN T)  $A$  being a **ZM** and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **Flm**  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZX\_nv\_mod\_tree**(GEN A, GEN P, GEN T)  $A$  being a **ZX** and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **Flx** polynomials  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZXC\_nv\_mod\_tree**(GEN A, GEN P, GEN T)  $A$  being a **ZXC** and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **FlxC**  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZXM\_nv\_mod\_tree**(GEN A, GEN P, GEN T)  $A$  being a **ZXM** and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **FlxM**  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZXX\_nv\_mod\_tree**(GEN A, GEN P, GEN T, long v)  $A$  being a **ZXX**, and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **FlxX**  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is assumed to be the tree created by **ZV\_producttree**(P).

**7.3.38 Fast Chinese remainder theorem over  $\mathbb{Z}$ .** The routines in these section are asymptotically fast (quasi-linear time in the input size) and should be used whenever the moduli are known from the start.

The simplest function is

**GEN ZV\_chinese**(GEN A, GEN P, GEN \*pM) let  $P$  be a vector of positive pairwise coprime integers, let  $A$  be a vector of integers of the same length  $n \geq 1$  such that  $0 \leq A[i] < P[i]$  for all  $i$ , and let  $M$  be the product of the elements of  $P$ . Returns the integer in  $[0, M[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If pM is not NULL, set \*pM to  $M$ . We also allow t\_VECSMALLs for  $A$  and  $P$  (seen as vectors of unsigned integers).

**GEN ZV\_chinese\_center**(GEN A, GEN P, GEN \*pM) As ZV\_chinese but return integers in  $[-M/2, M/2[$  instead.

The following functions allow to solve many Chinese remainder problems simultaneously, for a given set of moduli:

**GEN nxV\_chinese\_center**(GEN A, GEN P, GEN \*pt\_mod) where  $A$  is a vector of nx and  $P$  a t\_VECSMALL of the same length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, and  $M$  being the product of the elements of  $P$ , returns the t\_POL whose entries are integers in  $[-M/2, M/2[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If pt\_mod is not NULL, set \*pt\_mod to  $M$ .

**GEN ncV\_chinese\_center**(GEN A, GEN P, GEN \*pM) where  $A$  is a vector of VECSMALLs (seen as vectors of unsigned integers) and  $P$  a t\_VECSMALL of the same length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, and  $M$  being the product of the elements of  $P$ , returns the t\_COL whose entries are integers in  $[-M/2, M/2[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If pM is not NULL, set \*pt\_mod to  $M$ .

**GEN nmV\_chinese\_center**(GEN A, GEN P, GEN \*pM) where  $A$  is a vector of MATSMALLs (seen as matrices of unsigned integers) and  $P$  a t\_VECSMALL of the same length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, and  $M$  being the product of the elements of  $P$ , returns the matrix whose entries are integers in  $[-M/2, M/2[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If pM is not NULL, set \*pM to  $M$ . N.B.: this function uses the parallel GP interface.

**GEN nxCV\_chinese\_center**(GEN A, GEN P, GEN \*pM) where  $A$  is a vector of nxCs and  $P$  a t\_VECSMALL of the same length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, and  $M$  being the product of the elements of  $P$ , returns the t\_COL whose entries are integers in  $[-M/2, M/2[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If pM is not NULL, set \*pt\_mod to  $M$ .

**GEN nxMV\_chinese\_center**(GEN A, GEN P, GEN \*pM) where  $A$  is a vector of nxMs and  $P$  a t\_VECSMALL of the same length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, and  $M$  being the product of the elements of  $P$ , returns the matrix whose entries are integers in  $[-M/2, M/2[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If pM is not NULL, set \*pM to  $M$ . N.B.: this function uses the parallel GP interface.

The other routines allow for various precomputations :

**GEN ZV\_chinesetree**(GEN P, GEN T) given  $P$  a vector of integers (or t\_VECSMALL) and a product tree  $T$  from ZV\_producttree( $P$ ) for the same  $P$ , return a “chinese remainder tree”  $R$ , preconditionning the solution of Chinese remainder problems modulo the  $P[i]$ .

**GEN ZV\_chinese\_tree**(GEN A, GEN P, GEN T, GEN R) return ZV\_chinese( $A, P, \text{NULL}$ ), where  $T$  is created by ZV\_producttree( $P$ ) and  $R$  by ZV\_chinesetree( $P, T$ ).

GEN `ncV_chinese_center_tree`(GEN A, GEN P, GEN T, GEN R) as `ncV_chinese_center` where  $T$  is assumed to be the tree created by `ZV_producttree(P)` and  $R$  by `ZV_chinesetree(P, T)`.

GEN `nmV_chinese_center_tree`(GEN A, GEN P, GEN T, GEN R) as `nmV_chinese_center` where  $T$  is assumed to be the tree created by `ZV_producttree(P)` and  $R$  by `ZV_chinesetree(P, T)`.

GEN `nxV_chinese_center_tree`(GEN A, GEN P, GEN T, GEN R) as `nxV_chinese_center` where  $T$  is assumed to be the tree created by `ZV_producttree(P)` and  $R$  by `ZV_chinesetree(P, T)`.

GEN `nxCV_chinese_center_tree`(GEN A, GEN P, GEN T, GEN R) as `nxCV_chinese_center` where  $T$  is assumed to be the tree created by `ZV_producttree(P)` and  $R$  by `ZV_chinesetree(P, T)`.

### 7.3.39 Rational reconstruction.

`int Fp_ratlift`(GEN x, GEN m, GEN amax, GEN bmax, GEN \*a, GEN \*b). Assuming that  $0 \leq x < m$ ,  $\text{amax} \geq 0$ , and  $\text{bmax} > 0$  are `t_INTs`, and that  $2\text{amaxbmax} < m$ , attempts to recognize  $x$  as a rational  $a/b$ , i.e. to find `t_INTs`  $a$  and  $b$  such that

- $a \equiv bx \text{ modulo } m$ ,
- $|a| \leq \text{amax}$ ,  $0 < b \leq \text{bmax}$ ,
- $\gcd(m, b) = \gcd(a, b)$ .

If unsuccessful, the routine returns 0 and leaves  $a$ ,  $b$  unchanged; otherwise it returns 1 and sets  $a$  and  $b$ .

In almost all applications, we actually know that a solution exists, as well as a nonzero multiple  $B$  of  $b$ , and  $m = p^\ell$  is a prime power, for a prime  $p$  chosen coprime to  $B$  hence to  $b$ . Under the single assumption  $\gcd(m, b) = 1$ , if a solution  $a, b$  exists satisfying the three conditions above, then it is unique.

GEN `FpM_ratlift`(GEN M, GEN m, GEN amax, GEN bmax, GEN denom) given an `FpM` modulo  $m$  with reduced or `Fp_center`-ed entries, reconstructs a matrix with rational coefficients by applying `Fp_ratlift` to all entries. Assume that all preconditions for `Fp_ratlift` are satisfied, as well  $\gcd(m, b) = 1$  (so that the solution is unique if it exists). Return `NULL` if the reconstruction fails, and the rational matrix otherwise. If `denom` is not `NULL` check further that all denominators divide `denom`.

The function is not stack clean if one of the coefficients of  $M$  is negative (centered residues), but still suitable for `gerepileupto`.

GEN `FpX_ratlift`(GEN P, GEN m, GEN amax, GEN bmax, GEN denom) as `FpM_ratlift`, where  $P$  is an `FpX`.

GEN `FpC_ratlift`(GEN P, GEN m, GEN amax, GEN bmax, GEN denom) as `FpM_ratlift`, where  $P$  is an `FpC`.

### 7.3.40 Zp.

GEN Zp\_invlift(GEN b, GEN a, GEN p, long e) let  $p$  be a prime  $\mathbf{t\_INT}$ ,  $a$  be a  $\mathbf{t\_INT}$  and  $b$  a  $\mathbf{t\_INT}$  such that  $ab \equiv 1 \pmod{p}$ . Returns an  $\mathbf{t\_INT}$   $A$  such that  $A \equiv a^{-1} \pmod{p}$  and  $Ab \equiv 1 \pmod{p^e}$ .

GEN Zp\_inv(GEN b, GEN p, long e) let  $p$  be a prime  $\mathbf{t\_INT}$  and  $b$  be a  $\mathbf{t\_INT}$  Returns an  $\mathbf{t\_INT}$   $A$  such that  $Ab \equiv 1 \pmod{p^e}$ .

GEN Zp\_div(GEN a, GEN b, GEN p, long e) let  $p$  be a prime  $\mathbf{t\_INT}$  and  $a$  and  $b$  be a  $\mathbf{t\_INT}$  Returns an  $\mathbf{t\_INT}$   $c$  such that  $cb \equiv a \pmod{p^e}$ .

GEN Zp\_sqrt(GEN b, GEN p, long e)  $b$  and  $p$  being  $\mathbf{t\_INT}$ s, with  $p$  a prime (possibly 2), returns a  $\mathbf{t\_INT}$   $a$  such that  $a^2 \equiv b \pmod{p^e}$ .

GEN Z2\_sqrt(GEN b, long e)  $b$  being a  $\mathbf{t\_INT}$ s returns a  $\mathbf{t\_INT}$   $a$  such that  $a^2 \equiv b \pmod{2^e}$ .

GEN Zp\_sqrtlift(GEN b, GEN a, GEN p, long e) let  $a, b, p$  be  $\mathbf{t\_INT}$ s, with  $p > 2$ , such that  $a^2 \equiv b \pmod{p}$ . Returns a  $\mathbf{t\_INT}$   $A$  such that  $A^2 \equiv b \pmod{p^e}$ . Special case of Zp\_sqrtnlift.

GEN Zp\_sqrtnlift(GEN b, GEN n, GEN a, GEN p, long e) let  $a, b, n, p$  be  $\mathbf{t\_INT}$ s, with  $n, p > 1$ , and  $p$  coprime to  $n$ , such that  $a^n \equiv b \pmod{p}$ . Returns a  $\mathbf{t\_INT}$   $A$  such that  $A^n \equiv b \pmod{p^e}$ . Special case of ZpX\_liftroot.

GEN Zp\_teichmuller(GEN x, GEN p, long e, GEN pe) for  $p$  an odd prime,  $x$  a  $\mathbf{t\_INT}$  coprime to  $p$ , and  $pe = p^e$ , returns the  $(p-1)$ -th root of 1 congruent to  $x$  modulo  $p$ , modulo  $p^e$ . For convenience,  $p = 2$  is also allowed and we return 1 ( $x$  is 1 mod 4) or  $2^e - 1$  ( $x$  is 3 mod 4).

GEN teichmullerinit(long p, long n) returns the values of Zp\_teichmuller at all  $x = 1, \dots, p-1$ .

GEN Zp\_exp(GEN z, GEN p, ulong e) given a  $\mathbf{t\_INT}$   $z$  (preferably reduced mod  $p^e$ ), return  $\exp_p(a) \pmod{p^e}$  ( $\mathbf{t\_INT}$ ).

GEN Zp\_log(GEN z, GEN p, ulong e) given a  $\mathbf{t\_INT}$   $z$  (preferably reduced mod  $p^e$ ), such that  $a \equiv 1 \pmod{p}$ , return  $\log_p(a) \pmod{p^e}$  ( $\mathbf{t\_INT}$ ).

### 7.3.41 ZpM.

GEN ZpM\_invlift(GEN M, GEN Np, GEN p, long e) let  $p$  be a prime  $\mathbf{t\_INT}$ ,  $Np$  be a  $\mathbf{FpM}$  (modulo  $p$ ) and  $M$  a  $\mathbf{ZpM}$  such that  $MNp \equiv 1 \pmod{p}$ . Returns an  $\mathbf{ZpM}$   $N$  such that  $N \equiv Np^{-1} \pmod{p}$  and  $MN \equiv 1 \pmod{p^e}$ .

### 7.3.42 ZpX.

GEN ZpX\_roots(GEN f, GEN p, long e)  $f$  a  $\mathbf{ZX}$  with leading term prime to  $p$ , and without multiple roots mod  $p$ . Return a vector of  $\mathbf{t\_INT}$ s which are the roots of  $f \pmod{p^e}$ .

GEN ZpX\_liftroot(GEN f, GEN a, GEN p, long e)  $f$  a  $\mathbf{ZX}$  with leading term prime to  $p$ , and  $a$  a root mod  $p$  such that  $v_p(f'(a)) = 0$ . Return a  $\mathbf{t\_INT}$  which is the root of  $f \pmod{p^e}$  congruent to  $a \pmod{p}$ .

GEN ZX\_Zp\_root(GEN f, GEN a, GEN p, long e) same as ZpX\_liftroot without the assumption  $v_p(f'(a)) = 0$ . Return a  $\mathbf{t\_VEC}$  of  $\mathbf{t\_INT}$ s, which are the  $p$ -adic roots of  $f$  congruent to  $a \pmod{p}$  (given modulo  $p^e$ ). Assume that  $0 \leq a < p$ .

GEN ZpX\_liftroots(GEN f, GEN S, GEN p, long e)  $f$  a  $\mathbf{ZX}$  with leading term prime to  $p$ , and  $S$  a vector of simple roots mod  $p$ . Return a vector of  $\mathbf{t\_INT}$ s which are the root of  $f \pmod{p^e}$  congruent to the  $S[i] \pmod{p}$ .

`GEN ZpX_liftfact(GEN A, GEN B, GEN pe, GEN p, long e)` is the routine underlying `pol-hensellift`. Here,  $p$  is prime defines a finite field  $\mathbf{F}_p$ .  $A$  is a polynomial in  $\mathbf{Z}[X]$ , whose leading coefficient is nonzero in  $\mathbf{F}_p$ .  $B$  is a vector of monic  $\mathbf{F}_p[X]$ , pairwise coprime in  $\mathbf{F}_p[X]$ , whose product is congruent to  $A/\text{lc}(A)$  in  $\mathbf{F}_p[X]$ . Lifts the elements of  $B \bmod p = p^e$ .

`GEN ZpX_Frobenius(GEN T, GEN p, ulong e)` returns the  $p$ -adic lift of the Frobenius automorphism of  $\mathbf{F}_p[X]/(T)$  to precision  $e$ .

`long ZpX_disc_val(GEN f, GEN p)` returns the valuation at  $p$  of the discriminant of  $f$ . Assume that  $f$  is a monic *separable* ZX and that  $p$  is a prime number. Proceeds by dynamically increasing the  $p$ -adic accuracy; infinite loop if the discriminant of  $f$  is 0.

`long ZpX_resultant_val(GEN f, GEN g, GEN p, long M)` returns the valuation at  $p$  of  $\text{Res}(f, g)$ . Assume  $f, g$  are both ZX, and that  $p$  is a prime number coprime to the leading coefficient of  $f$ . Proceeds by dynamically increasing the  $p$ -adic accuracy. To avoid an infinite loop when the resultant is 0, we return  $M$  if the Sylvester matrix mod  $p^M$  still does not have maximal rank.

`GEN ZpX_gcd(GEN f, GEN g, GEN p, GEN pm)`  $f$  a monic ZX,  $g$  a ZX,  $pm = p^m$  a prime power. There is a unique integer  $r \geq 0$  and a monic  $h \in \mathbf{Q}_p[X]$  such that

$$p^r h \mathbf{Z}_p[X] + p^m \mathbf{Z}_p[X] = f \mathbf{Z}_p[X] + g \mathbf{Z}_p[X] + p^m \mathbf{Z}_p[X].$$

Return the 0 polynomial if  $r \geq m$  and a monic  $h \in \mathbf{Z}[1/p][X]$  otherwise (whose valuation at  $p$  is  $> -m$ ).

`GEN ZpX_reduced_resultant(GEN f, GEN g, GEN p, GEN pm)`  $f$  a monic ZX,  $g$  a ZX,  $pm = p^m$  a prime power. The  $p$ -adic *reduced resultant* of  $f$  and  $g$  is 0 if  $f, g$  not coprime in  $\mathbf{Z}_p[X]$ , and otherwise the generator of the form  $p^d$  of

$$(f \mathbf{Z}_p[X] + g \mathbf{Z}_p[X]) \cap \mathbf{Z}_p.$$

Return the reduced resultant modulo  $p^m$ .

`GEN ZpX_reduced_resultant_fast(GEN f, GEN g, GEN p, long M)`  $f$  a monic ZX,  $g$  a ZX,  $p$  a prime. Returns the  $p$ -adic reduced resultant of  $f$  and  $g$  modulo  $p^M$ . This function computes resultants for a sequence of increasing  $p$ -adic accuracies (up to  $M$   $p$ -adic digits), returning as soon as it obtains a nonzero result. It is very inefficient when the resultant is 0, but otherwise usually more efficient than computations using a priori bounds.

`GEN ZpX_monice_factor(GEN f, GEN p, long M)`  $f$  a monic ZX,  $p$  a prime, return the  $p$ -adic factorization of  $f$ , modulo  $p^M$ . This is the underlying low-level recursive function behind `factorpadic` (using a combination of Round 4 factorization and Hensel lifting); the factors are not sorted and the function is not `gerepile-clean`.

`GEN ZpX_primedec(GEN T, GEN p)`  $T$  a monic separable ZX,  $p$  a prime, return as a factorization matrix the shape of the prime ideal decomposition of  $(p)$  in  $\mathbf{Q}[X]/(T)$ : the first column contains inertia degrees, the second columns contains ramification degrees.

### 7.3.43 ZpXQ.

**GEN ZpXQ\_invlift**(GEN b, GEN a, GEN T, GEN p, long e) let  $p$  be a prime  $t\_INT$ ,  $a$  be a FpXQ (modulo  $(p, T)$ ) and  $b$  a ZpXQ such that  $ab \equiv 1 \pmod{(p, T)}$ . Returns an ZpXQ  $A$  such that  $A \equiv a \pmod{p}$  and  $Ab \equiv 1 \pmod{(p^e, T)}$ .

**GEN ZpXQ\_inv**(GEN b, GEN T, GEN p, long e) let  $p$  be a prime  $t\_INT$  and  $b$  be a FpXQ (modulo  $T, p^e$ ). Returns an FpXQ  $A$  such that  $Ab \equiv 1 \pmod{(p^e, T)}$ .

**GEN ZpXQ\_div**(GEN a, GEN b, GEN T, GEN q, GEN p, long e) let  $p$  be a prime  $t\_INT$  and  $a$  and  $b$  be a FpXQ (modulo  $T, p^e$ ). Returns an FpXQ  $c$  such that  $cb \equiv a \pmod{(p^e, T)}$ . The parameter  $q$  must be equal to  $p^e$ .

**GEN ZpXQ\_sqrtnlift**(GEN b, GEN n, GEN a, GEN T, GEN p, long e) let  $n, p$  be  $t\_INT$ s, with  $n, p > 1$  and  $p$  coprime to  $n$ , and  $a, b$  be FpXQs (modulo  $T$ ) such that  $a^n \equiv b \pmod{(p, T)}$ . Returns an Fq  $A$  such that  $A^n \equiv b \pmod{(p^e, T)}$ .

**GEN ZpXQ\_sqrt**(GEN b, GEN T, GEN p, long e) let  $p$  being a odd prime and  $b$  be a FpXQ (modulo  $T, p^e$ ), returns  $a$  such that  $a^2 \equiv b \pmod{(p^e, T)}$ .

**GEN ZpX\_ZpXQ\_liftroot**(GEN f, GEN a, GEN T, GEN p, long e) as ZpXQX\_liftroot, but  $f$  is a polynomial in  $\mathbf{Z}[X]$ .

**GEN ZpX\_ZpXQ\_liftroot\_ea**(GEN f, GEN a, GEN T, GEN p, long e, void \*E, GEN early(void \*E, GEN x, GEN q)) as ZpX\_ZpXQ\_liftroot with early abort: the function `early(E,x,q)` will be called with  $x$  is a root of  $f$  modulo  $q = p^n$  for some  $n$ . If `early` returns a non-NULL value  $z$ , the function returns  $z$  immediately.

**GEN ZpXQ\_log**(GEN a, GEN T, GEN p, long e)  $T$  being a ZpX irreducible modulo  $p$ , return the logarithm of  $a$  in  $\mathbf{Z}_p[X]/(T)$  to precision  $e$ , assuming that  $a \equiv 1 \pmod{p\mathbf{Z}_p[X]}$  if  $p$  odd or  $a \equiv 1 \pmod{4\mathbf{Z}_2[X]}$  if  $p = 2$ .

### 7.3.44 Zq.

**GEN Zq\_sqrtnlift**(GEN b, GEN n, GEN a, GEN T, GEN p, long e)

### 7.3.45 ZpXQM.

**GEN ZpXQM\_prodFrobenius**(GEN M, GEN T, GEN p, long e) returns the product of matrices  $M\sigma(M)\sigma^2(M)\dots\sigma^{n-1}(M)$  to precision  $e$  where  $\sigma$  is the lift of the Frobenius automorphism over  $\mathbf{Z}_p[X]/(T)$  and  $n$  is the degree of  $T$ .

### 7.3.46 ZpXQX.

**GEN ZpXQX\_liftfact**(GEN A, GEN B, GEN T, GEN pe, GEN p, long e) is the routine underlying `polhensellift`. Here,  $p$  is prime,  $T(Y)$  defines a finite field  $\mathbf{F}_q$ .  $A$  is a polynomial in  $\mathbf{Z}[X, Y]$ , whose leading coefficient is nonzero in  $\mathbf{F}_q$ .  $B$  is a vector of monic or FqX, pairwise coprime in  $\mathbf{F}_q[X]$ , whose product is congruent to  $A/\text{lc}(A)$  in  $\mathbf{F}_q[X]$ . Lifts the elements of  $B \pmod{pe = p^e}$ , such that the congruence now holds  $\pmod{(T, p^e)}$ .

**GEN ZpXQX\_liftroot**(GEN f, GEN a, GEN T, GEN p, long e) as ZpX\_liftroot, but  $f$  is now a polynomial in  $\mathbf{Z}[X, Y]$  and lift the root  $a$  in the unramified extension of  $\mathbf{Q}_p$  with residue field  $\mathbf{F}_p[Y]/(T)$ , assuming  $v_p(f(a)) > 0$  and  $v_p(f'(a)) = 0$ .

GEN ZpXQX\_liftroot\_vald(GEN f, GEN a, long v, GEN T, GEN p, long e) returns the roots of  $f$  as ZpXQX\_liftroot, where  $v$  is the valuation of the content of  $f'$  and it is required that  $v_p(f(a)) > v$  and  $v_p(f'(a)) = v$ .

GEN ZpXQX\_roots(GEN F, GEN T, GEN p, long e)

GEN ZpXQX\_liftroots(GEN F, GEN S, GEN T, GEN p, long e)

GEN ZpXQX\_divrem(GEN x, GEN Sp, GEN T, GEN q, GEN p, long e, GEN \*pr) as FpXQX\_divrem. The parameter  $q$  must be equal to  $p^e$ .

GEN ZpXQX\_digits(GEN x, GEN B, GEN T, GEN q, GEN p, long e) As FpXQX\_digits. The parameter  $q$  must be equal to  $p^e$ .

GEN ZpXQX\_ZpXQX\_liftroot(GEN f, GEN a, GEN S, GEN T, GEN p, long e) as ZpXQX\_liftroot, except that  $a$  is an element of  $\mathbf{Z}_p[X, Y]/(S(X, Y), T(X))$ .

**7.3.47 ZqX.** ZqX are either ZpX or ZpXQX depending whether T is NULL or not.

GEN ZqX\_roots(GEN F, GEN T, GEN p, long e)

GEN ZqX\_liftfact(GEN A, GEN B, GEN T, GEN pe, GEN p, long e)

GEN ZqX\_liftroot(GEN f, GEN a, GEN T, GEN p, long e)

GEN ZqX\_ZqX\_liftroot(GEN f, GEN a, GEN P, GEN T, GEN p, long e)

### 7.3.48 Other $p$ -adic functions.

GEN ZpM\_echelon(GEN M, long early\_abort, GEN p, GEN pm) given a ZM  $M$ , a prime  $p$  and  $pm = p^m$ , returns an echelon form  $E$  for  $M \bmod p^m$ . I.e. there exist a square integral matrix  $U$  with  $\det U$  coprime to  $p$  such that  $E = MU$  modulo  $p^m$ . If  $\text{early\_abort}$  is nonzero, return NULL as soon as one pivot in the echelon form is divisible by  $p^m$ . The echelon form is an upper triangular HNF, we do not waste time to reduce it to Gauss-Jordan form.

GEN zlm\_echelon(GEN M, long early\_abort, ulong p, ulong pm) variant of ZpM\_echelon, for a Zlm  $M$ .

GEN Zlm\_gauss(GEN a, GEN b, ulong p, long e, GEN C) as gauss with the following peculiarities:  $a$  and  $b$  are ZM, such that  $a$  is invertible modulo  $p$ . Optional  $C$  is an Flm that is an inverse of  $a \bmod p$  or NULL. Return the matrix  $x$  such that  $ax = b \bmod p^e$  and all elements of  $x$  are in  $[0, p^e - 1]$ . For efficiency, it is better to reduce  $a$  and  $b \bmod p^e$  first.

GEN padic\_to\_Q(GEN x) truncate the t\_PADIC to a t\_INT or t\_FRAC.

GEN padic\_to\_Q\_shallow(GEN x) shallow version of padic\_to\_Q

GEN QpV\_to\_QV(GEN v) apply padic\_to\_Q\_shallow

long padicprec(GEN x, GEN p) returns the absolute  $p$ -adic precision of the object  $x$ , by definition the minimum precision of the components of  $x$ . For a nonzero t\_PADIC, this returns  $\text{valp}(x) + \text{precp}(x)$ .

long padicprec\_relative(GEN x) returns the relative  $p$ -adic precision of the t\_INT, t\_FRAC, or t\_PADIC  $x$  (minimum precision of the components of  $x$  for t\_POL or vector/matrices). For a t\_PADIC, this returns  $\text{precp}(x)$  if  $x \neq 0$ , and 0 for  $x = 0$ .

### 7.3.48.1 low-level.

The following technical function returns an optimal sequence of  $p$ -adic accuracies, for a given target accuracy:

`ulong quadratic_prec_mask(long n)` we want to reach accuracy  $n \geq 1$ , starting from accuracy 1, using a quadratically convergent, self-correcting, algorithm; in other words, from inputs correct to accuracy  $l$  one iteration outputs a result correct to accuracy  $2l$ . For instance, to reach  $n = 9$ , we want to use accuracies  $[1, 2, 3, 5, 9]$  instead of  $[1, 2, 4, 8, 9]$ . The idea is to essentially double the accuracy at each step, and not overshoot in the end.

Let  $a_0 = 1, a_1 = 2, \dots, a_k = n$ , be the desired sequence of accuracies. To obtain it, we work backwards and set

$$a_k = n, \quad a_{i-1} = (a_i + 1) \setminus 2.$$

This is in essence what the function returns. But we do not want to store the  $a_i$  explicitly, even as a `t_VECSMALL`, since this would leave an object on the stack. Instead, we store  $a_i$  implicitly in a bitmask `MASK`: let  $a_0 = 1$ , if the  $i$ -th bit of the mask is set, set  $a_{i+1} = 2a_i - 1$ , and  $2a_i$  otherwise; in short the bits indicate the places where we do something special and do not quite double the accuracy (which would be the straightforward thing to do).

In fact, to avoid returning separately the mask and the sequence length  $k + 1$ , the function returns `MASK + 2k+1`, so the highest bit of the mask indicates the length of the sequence, and the following ones give an algorithm to obtain the accuracies. This is much simpler than it sounds, here is what it looks like in practice:

```
ulong mask = quadratic_prec_mask(n);
long l = 1;
while (mask > 1) {
    /* here, the result is known to accuracy l */
    l = 2*l; if (mask & 1) l--; /* new accuracy l for the iteration */
    mask >>= 1; /* pop low order bit */
    /* ... lift to the new accuracy ... */
}
/* we are done. At this point l = n */
```

We just pop the bits in `mask` starting from the low order bits, stop when `mask` is 1 (that last bit corresponds to the  $2^{k+1}$  that we added to the mask proper). Note that there is nothing specific to Hensel lifts in that function: it would work equally well for an Archimedean Newton iteration.

Note that in practice, we rather use an infinite loop, and insert an

```
if (mask == 1) break;
```

in the middle of the loop: the loop body usually includes preparations for the next iterations (e.g. lifting Bezout coefficients in a quadratic Hensel lift), which are costly and useless in the *last* iteration.



### 7.3.49 Conversions involving single precision objects.

#### 7.3.49.1 To single precision.

`ulong Rg_to_Fl(GEN z, ulong p)`,  $z$  which can be mapped to  $\mathbf{Z}/p\mathbf{Z}$ : a `t_INT`, a `t_INTMOD` whose modulus is divisible by  $p$ , a `t_FRAC` whose denominator is coprime to  $p$ , or a `t_PADIC` with underlying prime  $\ell$  satisfying  $p = \ell^n$  for some  $n$  (less than the accuracy of the input). Returns `lift(z * Mod(1,p))`, normalized, as an `Fl`.

`ulong Rg_to_F2(GEN z)`, as `Rg_to_Fl` for  $p = 2$ .

`ulong padic_to_Fl(GEN x, ulong p)` special case of `Rg_to_Fl`, for a  $x$  a `t_PADIC`.

`GEN RgX_to_F2x(GEN x)`,  $x$  a `t_POL`, returns the `F2x` obtained by applying `Rg_to_Fl` coefficientwise.

`GEN RgX_to_Flx(GEN x, ulong p)`,  $x$  a `t_POL`, returns the `Flx` obtained by applying `Rg_to_Fl` coefficientwise.

`GEN RgXV_to_FlxV(GEN x, ulong p)`,  $x$  a vector, returns the `FlxV` obtained by applying `RgX_to_Flx` coefficientwise.

`GEN Rg_to_F2xq(GEN z, GEN T)`,  $z$  a `GEN` which can be mapped to  $\mathbf{F}_2[X]/(T)$ : anything `Rg_to_Fl` can be applied to, a `t_POL` to which `RgX_to_F2x` can be applied to, a `t_POLMOD` whose modulus is divisible by  $T$  (once mapped to a `F2x`), a suitable `t_RFRAC`. Returns  $z$  as an `F2xq`, normalized.

`GEN Rg_to_Flxq(GEN z, GEN T, ulong p)`,  $z$  a `GEN` which can be mapped to  $\mathbf{F}_p[X]/(T)$ : anything `Rg_to_Fl` can be applied to, a `t_POL` to which `RgX_to_Flx` can be applied to, a `t_POLMOD` whose modulus is divisible by  $T$  (once mapped to a `Flx`), a suitable `t_RFRAC`. Returns  $z$  as an `Flxq`, normalized.

`GEN RgX_to_FlxqX(GEN z, GEN T, ulong p)`,  $z$  a `GEN` which can be mapped to  $\mathbf{F}_p[x]/(T)[X]$ : anything `Rg_to_Flxq` can be applied to, a `t_POL` to which `RgX_to_Flx` can be applied to, a `t_POLMOD` whose modulus is divisible by  $T$  (once mapped to a `Flx`), a suitable `t_RFRAC`. Returns  $z$  as an `FlxqX`, normalized.

`GEN ZX_to_Flx(GEN x, ulong p)` reduce  $ZX$  modulo  $p$  (yielding an `Flx`). Faster than `RgX_to_Flx`.

`GEN ZV_to_Flv(GEN x, ulong p)` reduce  $ZV$  modulo  $p$  (yielding an `Flv`).

`GEN ZXV_to_FlxV(GEN v, ulong p)`, as `ZX_to_Flx`, repeatedly called on the vector's coefficients.

`GEN ZXT_to_FlxT(GEN v, ulong p)`, as `ZX_to_Flx`, repeatedly called on the tree leaves.

`GEN ZXX_to_FlxX(GEN B, ulong p, long v)`, as `ZX_to_Flx`, repeatedly called on the polynomial's coefficients.

`GEN zxX_to_FlxX(GEN z, ulong p)` as `zx_to_Flx`, repeatedly called on the polynomial's coefficients.

`GEN ZXXV_to_FlxXV(GEN V, ulong p, long v)`, as `ZXX_to_FlxX`, repeatedly called on the vector's coefficients.

`GEN ZXXT_to_FlxXT(GEN V, ulong p, long v)`, as `ZXX_to_FlxX`, repeatedly called on the tree leaves.

`GEN RgV_to_Flv(GEN x, ulong p)` reduce the `t_VEC/t_COL` modulo  $p$ , yielding a `t_VECSMALL`.

`GEN RgM_to_Flm(GEN x, ulong p)` reduce the `t_MAT` modulo  $p$ .

`GEN ZM_to_Flm(GEN x, ulong p)` reduce  $ZM$  modulo  $p$  (yielding an `Flm`).

GEN ZXC\_to\_FlxC(GEN x, ulong p, long sv) reduce ZXC  $x$  modulo  $p$  (yielding an FlxC). Assume that  $sv = \text{evalvarn}(v)$  where  $v$  is the variable number of the entries of  $x$ . It is allowed for the entries of  $x$  to be t\_INT.

GEN ZXM\_to\_FlxM(GEN x, ulong p, long sv) reduce ZXM  $x$  modulo  $p$  (yielding an FlxM). Assume that  $sv = \text{evalvarn}(v)$  where  $v$  is the variable number of the entries of  $x$ . It is allowed for the entries of  $x$  to be t\_INT.

GEN ZV\_to\_zv(GEN z), converts coefficients using itos

GEN ZV\_to\_nv(GEN z), converts coefficients using itou

GEN ZM\_to\_zm(GEN z), converts coefficients using itos

### 7.3.49.2 From single precision.

GEN Flx\_to\_ZX(GEN z), converts to ZX (t\_POL of nonnegative t\_INTs in this case)

GEN Flx\_to\_FlxX(GEN z), converts to FlxX (t\_POL of constant Flx in this case).

GEN Flx\_to\_ZX\_inplace(GEN z), same as Flx\_to\_ZX, in place (z is destroyed).

GEN FlxX\_to\_ZXX(GEN B), converts an FlxX to a polynomial with ZX or t\_INT coefficients (repeated calls to Flx\_to\_ZX).

GEN FlxXC\_to\_ZXXC(GEN B), converts an FlxXC to a t\_COL with ZXX coefficients (repeated calls to FlxX\_to\_ZXX).

GEN FlxXM\_to\_ZXXM(GEN B), converts an FlxXM to a t\_MAT with ZXX coefficients (repeated calls to FlxX\_to\_ZXX).

GEN FlxC\_to\_ZXC(GEN x), converts a vector of Flx to a column vector of polynomials with t\_INT coefficients (repeated calls to Flx\_to\_ZX).

GEN FlxV\_to\_ZXV(GEN x), as above but return a t\_VEC.

void F2xV\_to\_FlxV\_inplace(GEN v) v is destroyed.

void F2xV\_to\_ZXV\_inplace(GEN v) v is destroyed.

void FlxV\_to\_ZXV\_inplace(GEN v) v is destroyed.

GEN FlxM\_to\_ZXM(GEN z), converts a matrix of Flx to a matrix of polynomials with t\_INT coefficients (repeated calls to Flx\_to\_ZX).

GEN zx\_to\_ZX(GEN z), as Flx\_to\_ZX, without assuming the coefficients to be nonnegative.

GEN zx\_to\_Flx(GEN z, ulong p) as Flx\_red without assuming the coefficients to be nonnegative.

GEN Flc\_to\_ZC(GEN z), converts to ZC (t\_COL of nonnegative t\_INTs in this case)

GEN Flc\_to\_ZC\_inplace(GEN z), same as Flc\_to\_ZC, in place (z is destroyed).

GEN Flv\_to\_ZV(GEN z), converts to ZV (t\_VEC of nonnegative t\_INTs in this case)

GEN Flm\_to\_ZM(GEN z), converts to ZM (t\_MAT with nonnegative t\_INTs coefficients in this case)

GEN Flm\_to\_ZM\_inplace(GEN z), same as Flm\_to\_ZM, in place (z is destroyed).

GEN zc\_to\_ZC(GEN z) as Flc\_to\_ZC, without assuming coefficients are nonnegative.

GEN zv\_to\_ZV(GEN z) as Flv\_to\_ZV, without assuming coefficients are nonnegative.

GEN zm\_to\_ZM(GEN z) as Flm\_to\_ZM, without assuming coefficients are nonnegative.

GEN zv\_to\_Flv(GEN z, ulong p)

GEN zm\_to\_Flm(GEN z, ulong p)

**7.3.49.3 Mixed precision linear algebra.** Assumes dimensions are compatible. Multiply a multiprecision object by a single-precision one.

GEN RgM\_zc\_mul(GEN x, GEN y)

GEN RgMrow\_zc\_mul(GEN x, GEN y, long i)

GEN RgM\_zm\_mul(GEN x, GEN y)

GEN RgV\_zc\_mul(GEN x, GEN y)

GEN RgV\_zm\_mul(GEN x, GEN y)

GEN ZM\_zc\_mul(GEN x, GEN y)

GEN zv\_ZM\_mul(GEN x, GEN y)

GEN ZV\_zc\_mul(GEN x, GEN y)

GEN ZM\_zm\_mul(GEN x, GEN y)

GEN ZC\_z\_mul(GEN x, long y)

GEN ZM\_nm\_mul(GEN x, GEN y) the entries of  $y$  are ulongs.

GEN nm\_Z\_mul(GEN y, GEN c) the entries of  $y$  are ulongs.

**7.3.49.4 Miscellaneous involving Fl.**

GEN Fl\_to\_Flx(ulong x, long evx) converts a unsigned long to a scalar Flx. Assume that  $evx = evalvarn(vx)$  for some variable number  $vx$ .

GEN Z\_to\_Flx(GEN x, ulong p, long sv) converts a  $t\_INT$  to a scalar Flx polynomial. Assume that  $sv = evalvarn(v)$  for some variable number  $v$ .

GEN Flx\_to\_Flv(GEN x, long n) converts from Flx to Flv with  $n$  components (assumed larger than the number of coefficients of  $x$ ).

GEN zx\_to\_zv(GEN x, long n) as Flx\_to\_Flv.

GEN Flv\_to\_Flx(GEN x, long sv) converts from vector (coefficient array) to (normalized) polynomial in variable  $v$ .

GEN zv\_to\_zx(GEN x, long n) as Flv\_to\_Flx.

GEN Flm\_to\_FlxV(GEN x, long sv) converts the columns of Flm  $x$  to an array of Flx in the variable  $v$  (repeated calls to Flv\_to\_Flx).

GEN FlxM\_to\_FlxXV(GEN V, long v) see RgM\_to\_RgXV

GEN zm\_to\_zxV(GEN x, long n) as Flm\_to\_FlxV.

GEN Flm\_to\_FlxX(GEN x, long sw, long sv) same as Flm\_to\_FlxV( $x, sv$ ) but returns the result as a (normalized) polynomial in variable  $w$ .

GEN FlxV\_to\_Flm(GEN v, long n) reverse Flm\_to\_FlxV, to obtain an Flm with  $n$  rows (repeated calls to Flx\_to\_Flv).

GEN FlxX\_to\_Flx(GEN P) Let  $P(x, X)$  be a FlxX, return  $P(0, X)$  as a Flx.

GEN FlxX\_to\_Flm(GEN v, long n) reverse Flm\_to\_FlxX, to obtain an Flm with n rows (repeated calls to Flx\_to\_Flv).

GEN FlxX\_to\_FlxC(GEN B, long n, long sv) see RgX\_to\_RgV. The coefficients of B are assumed to be in the variable v.

GEN FlxV\_to\_FlxX(GEN x, long v) see RgV\_to\_RgX.

GEN FlxXV\_to\_FlxM(GEN V, long n, long sv) see RgXV\_to\_RgM. The coefficients of  $V[i]$  are assumed to be in the variable v.

GEN Fly\_to\_FlxY(GEN a, long sv) convert coefficients of a to constant Flx in variable v.

### 7.3.49.5 Miscellaneous involving F2x.

GEN F2x\_to\_F2v(GEN x, long n) converts from F2x to F2v with n components (assumed larger than the number of coefficients of x).

GEN F2xC\_to\_ZXC(GEN x), converts a vector of F2x to a column vector of polynomials with t\_INT coefficients (repeated calls to F2x\_to\_ZX).

GEN F2xC\_to\_FlxC(GEN x)

GEN FlxC\_to\_F2xC(GEN x)

GEN F2xV\_to\_F2m(GEN v, long n) F2x\_to\_F2v to each polynomial to get an F2m with n rows.

## 7.4 Higher arithmetic over Z: primes, factorization.

### 7.4.1 Pure powers.

long Z\_issquare(GEN n) returns 1 if the t\_INT n is a square, and 0 otherwise. This is tested first modulo small prime powers, then sqrtremi is called.

long Z\_issquareall(GEN n, GEN \*sqrtn) as Z\_issquare. If n is indeed a square, set sqrtn to its integer square root. Uses a fast congruence test mod  $64 \times 63 \times 65 \times 11$  before computing an integer square root.

long Z\_ispow2(GEN x) returns 1 if the t\_INT x is a power of 2, and 0 otherwise.

long uissquare(ulong n) as Z\_issquare, for an ulong operand n.

long uissquareall(ulong n, ulong \*sqrtn) as Z\_issquareall, for an ulong operand n.

ulong usqrt(ulong a) returns the floor of the square root of a.

ulong usqrtn(ulong a, ulong n) returns the floor of the n-th root of a.

long Z\_ispower(GEN x, ulong k) returns 1 if the t\_INT n is a k-th power, and 0 otherwise; assume that  $k > 1$ .

long Z\_ispowerall(GEN x, ulong k, GEN \*pt) as Z\_ispower. If n is indeed a k-th power, set \*pt to its integer k-th root.

long Z\_isanypower(GEN x, GEN \*ptn) returns the maximal  $k \geq 2$  such that the t\_INT  $x = n^k$  is a perfect power, or 0 if no such k exist; in particular ispower(1), ispower(0), ispower(-1) all return 0. If the return value k is not 0 (so that  $x = n^k$ ) and ptn is not NULL, set \*ptn to n.

The following low-level functions are called by `Z_isanypower` but can be directly useful:

`int is_357_power(GEN x, GEN *ptn, ulong *pmask)` tests whether the integer  $x > 0$  is a 3-rd, 5-th or 7-th power. The bits of `*mask` initially indicate which test is to be performed; bit 0: 3-rd, bit 1: 5-th, bit 2: 7-th (e.g. `*pmask = 7` performs all tests). They are updated during the call: if the “ $i$ -th power” bit is set to 0 then  $x$  is not a  $k$ -th power. The function returns 0 (not a 3-rd, 5-th or 7-th power), 3 (3-rd power, not a 5-th or 7-th power), 5 (5-th power, not a 7-th power), or 7 (7-th power); if an  $i$ -th power bit is initially set to 0, we take it at face value and assume  $x$  is not an  $i$ -th power without performing any test. If the return value  $k$  is nonzero, set `*ptn` to  $n$  such that  $x = n^k$ .

`int is_pth_power(GEN x, GEN *ptn, forprime_t *T, ulong cutoff)` let  $x > 0$  be an integer, `cutoff`  $> 0$  and  $T$  be an iterator over primes  $\geq 11$ , we look for the smallest prime  $p$  such that  $x = n^p$  (advancing  $T$  as we go along). The 11 is due to the fact that `is_357_power` and `issquare` are faster than the generic version for  $p < 11$ .

Fail and return 0 when the existence of  $p$  would imply  $2^{\text{cutoff}} > x^{1/p}$ , meaning that a possible  $n$  is so small that it should have been found by trial division; for maximal speed, you should start by a round of trial division, but the cut-off may also be set to 1 for a rigorous result without any trial division.

Otherwise returns the smallest suitable prime power  $p^i$  and set `*ptn` to the  $p^i$ -th root of  $x$  (which is now not a  $p$ -th power). We may immediately recall the function with the same parameters after setting  $x = \text{*ptn}$ : it will start at the next prime.

#### 7.4.2 Factorization.

`GEN Z_factor(GEN n)` factors the `t_INT`  $n$ . The “primes” in the factorization are actually strong pseudoprimes.

`GEN absZ_factor(GEN n)` returns `Z_factor(absi(n))`.

`long Z_issmooth(GEN n, ulong lim)` returns 1 if all the prime factors of the `t_INT`  $n$  are less or equal to  $lim$ .

`GEN Z_issmooth_fact(GEN n, ulong lim)` returns NULL if a prime factor of the `t_INT`  $n$  is  $> lim$ , and returns the factorization of  $n$  otherwise, as a `t_MAT` with `t_VECSMALL` columns (word-size primes and exponents). Neither memory-clean nor suitable for `gerepileupto`.

`GEN Z_factor_until(GEN n, GEN lim)` as `Z_factor`, but stop the factorization process as soon as the unfactored part is smaller than  $lim$ . The resulting factorization matrix only contains the factors found. No other assumptions can be made on the remaining factors.

`GEN Z_factor_limit(GEN n, ulong lim)` trial divide  $n$  by all primes  $p < lim$  in the precomputed list of prime numbers and the `addprimes` prime table. Return the corresponding factorization matrix. The first column of the factorization matrix may contain a single composite, which may or may not be the last entry in presence of a prime table.

If  $lim = 0$ , the effect is the same as setting  $lim = \text{maxprime}() + 1$ : use all precomputed primes.

`GEN absZ_factor_limit(GEN n, ulong all)` returns `Z_factor_limit(absi(n))`.

`GEN absZ_factor_limit_strict(GEN n, ulong all, GEN *pU)`. This function is analogous to `absZ_factor_limit`, with a better interface: trial divide  $n$  by all primes  $p < lim$  in the precomputed list of prime numbers and the `addprimes` prime table. Return the corresponding factorization matrix. In this case, a composite cofactor is *not* included.

If `pU` is not `NULL`, set it to the cofactor, which is either `NULL` (no cofactor) or  $[q, k]$ , where  $k > 0$ , the prime divisors of  $q$  are greater than `all`,  $q$  is not a pure power, and  $q^k$  is the largest power of  $q$  dividing  $n$ . It may happen that  $q$  is prime.

`GEN boundfact(GEN x, ulong lim)` as `Z_factor_limit`, applying to `t_INT` or `t_FRAC` inputs.

`GEN Z_smoother(GEN n, GEN L, GEN *pP, GEN *pE)` given a `t_VEC`  $L$  containing a list of primes and a `t_INT`  $n$ , trial divide  $n$  by the elements of  $L$  and return the cofactor. Return `NULL` if the cofactor is  $\pm 1$ . `*P` and `*E` contain the list of prime divisors found and their exponents, as `t_VECSMALLs`. Neither memory-clean, nor suitable for `gerepileupto`.

`GEN Z_lsmoother(GEN n, GEN L, GEN *pP, GEN *pE)` as `Z_smoother` where  $L$  is a `t_VECSMALL` of small primes and both `*P` and `*E` are given as `t_VECSMALL`.

`GEN Z_factor_listP(GEN N, GEN L)` given a `t_INT`  $N$ , a vector or primes  $L$  containing all prime divisors of  $N$  (and possibly others). Return `factor(N)`. Neither memory-clean, nor suitable for `gerepileupto`.

`GEN factor_pn_1(GEN p, ulong n)` returns the factorization of  $p^n - 1$ , where  $p$  is prime and  $n$  is a positive integer.

`GEN factor_pn_1_limit(GEN p, ulong n, ulong B)` returns a partial factorization of  $p^n - 1$ , where  $p$  is prime and  $n$  is a positive integer. Don't actively search for prime divisors  $p > B$ , but we may find still find some due to Aurifeuillian factorizations. Any entry  $> B^2$  in the output factorization matrix is *a priori* not a prime (but may well be).

`GEN factor_Aurifeuille_prime(GEN p, long n)` an Aurifeuillian factor of  $\phi_n(p)$ , assuming  $p$  prime and an Aurifeuillian factor exists ( $p\zeta_n$  is a square in  $\mathbf{Q}(\zeta_n)$ ).

`GEN factor_Aurifeuille(GEN a, long n)` an Aurifeuillian factor of  $\phi_n(a)$ , assuming  $a$  is a nonzero integer and  $n > 2$ . Returns 1 if no Aurifeuillian factor exists.

`GEN odd_prime_divisors(GEN a)` `t_VEC` of all prime divisors of the `t_INT`  $a$ .

`GEN factoru(ulong n)`, returns the factorization of  $n$ . The result is a 2-component vector  $[P, E]$ , where  $P$  and  $E$  are `t_VECSMALL` containing the prime divisors of  $n$ , and the  $v_p(n)$ .

`GEN factoru_pow(ulong n)`, returns the factorization of  $n$ . The result is a 3-component vector  $[P, E, C]$ , where  $P$ ,  $E$  and  $C$  are `t_VECSMALL` containing the prime divisors of  $n$ , the  $v_p(n)$  and the  $p^{v_p(n)}$ .

`GEN vecfactoru(ulong a, ulong b)`, returns a `t_VEC`  $v$  containing the factorizations (`factoru` format) of  $a, \dots, b$ ; assume that  $b \geq a > 0$ . Uses a sieve with primes up to  $\sqrt{b}$ . For all  $c$ ,  $a \leq c \leq b$ , the factorization of  $c$  is given in  $v[c - a + 1]$ .

`GEN vecfactoroddu(ulong a, ulong b)`, returns a `t_VEC`  $v$  containing the factorizations (`factoru` format) of odd integers in  $a, \dots, b$ ; assume that  $b \geq a > 0$  are odd. Uses a sieve with primes up to  $\sqrt{b}$ . For all odd  $c$ ,  $a \leq c \leq b$ , the factorization of  $c$  is given in  $v[(c - a)/2 + 1]$ .

`GEN vecfactoru_i(ulong a, ulong b)`, private version of `vecfactoru`, not memory clean.

`GEN vecfactoroddu_i(ulong a, ulong b)`, private version of `vecfactoroddu`, not memory clean.

`GEN vecfactorsquarefreeu(ulong a, ulong b)` return a `t_VEC`  $v$  containing the prime divisors of squarefree integers in  $a, \dots, b$ ; assume that  $a \leq b$ . Uses a sieve with primes up to  $\sqrt{b}$ . For all squarefree  $c$ ,  $a \leq c \leq b$ , the prime divisors of  $c$  (as a `t_VECSMALL`) are given in  $v[c - a + 1]$ , and

the other entries are NULL. Note that because of these NULL markers,  $v$  is not a valid GEN, it is not memory clean and cannot be used in garbage collection routines.

GEN `vecfactorsquarefreeu_coprime(ulong a, ulong b, GEN P)` given a *sorted* `t_VECSMALL` of primes  $P$ , return a `t_VEC`  $v$  containing the prime divisors of squarefree integers in  $a, \dots, b$  coprime to the elements of  $P$ ; assume that  $a \leq b$ . Uses a sieve with primes up to  $\sqrt{b}$ . For all squarefree  $c$ ,  $a \leq c \leq b$ , the prime divisors of  $c$  (as a `t_VECSMALL`) are given in  $v[c - a + 1]$ , and the other entries are NULL. Note that because of these NULL markers,  $v$  is not a valid GEN, it is not memory clean and cannot be used in garbage collection routines.

GEN `vecsquarefreeu(ulong a, ulong b)` return a `t_VECSMALL`  $v$  containing the squarefree integers in  $a, \dots, b$ . Assume that  $a \leq b$ . Uses a sieve with primes up to  $\sqrt{b}$ .

ulong `tridiv_bound(GEN n)` returns the trial division bound used by `Z_factor(n)`.

GEN `tridiv_boundu(ulong n)` returns the trial division bound used by `factorun`.

GEN `Z_pollardbrent(GEN N, long n, long seed)` try to factor `t_INT`  $N$  using  $n \geq 1$  rounds of Pollard iterations; *seed* is an integer whose value (mod 8) selects the quadratic polynomial use to generate Pollard's (pseudo)random walk. Returns NULL on failure, else a vector of 2 (possibly 3) integers whose product is  $N$ .

GEN `Z_ECM(GEN N, long n, long seed, ulong B1)` try to factor `t_INT`  $N$  using  $n \geq 1$  rounds of ECM iterations (on 8 to 64 curves simultaneously, depending on the size of  $N$ ); *seed* is an integer whose value selects the curves to be used: increase it by  $64n$  to make sure that a subsequent call with a factor of  $N$  uses a disjoint set of curves. Finally  $B_1 > 7$  determines the computations performed on the curves: we compute  $[k]P$  for some point in  $E(\mathbf{Z}/N\mathbf{Z})$  and  $k = q \prod p^{e_p}$  where  $p^{e_p} \leq B_1$  and  $q \leq B_2 := 110B_1$ ; a higher value of  $B_1$  means higher chances of hitting a factor and more time spent. The computation is deterministic for a given set of parameters. Returns NULL on failure, else a nontrivial factor or  $N$ .

GEN `Q_factor(GEN x)` as `Z_factor`, where  $x$  is a `t_INT` or a `t_FRAC`.

GEN `Q_factor_limit(GEN x, ulong lim)` as `Z_factor_limit`, where  $x$  is a `t_INT` or a `t_FRAC`.

### 7.4.3 Coprime factorization.

Given  $a$  and  $b$  two nonzero integers, let **ppi**( $a, b$ ), **ppo**( $a, b$ ), **ppg**( $a, b$ ), **pple**( $a, b$ ) (powers in  $a$  of primes inside  $b$ , outside  $b$ , greater than those in  $b$ , less than or equal to those in  $b$ ) be the integers defined by

- $v_p(\text{ppi}) = v_p(a)[v_p(b) > 0]$ ,
- $v_p(\text{ppo}) = v_p(a)[v_p(b) = 0]$ ,
- $v_p(\text{ppg}) = v_p(a)[v_p(a) > v_p(b)]$ ,
- $v_p(\text{pple}) = v_p(a)[v_p(a) \leq v_p(b)]$ .

GEN `Z_ppo(GEN a, GEN b)` returns `ppo(a, b)`; shallow function.

ulong `u_ppo(ulong a, ulong b)` returns `ppo(a, b)`.

GEN `Z_ppgle(GEN a, GEN b)` returns `[ppg(a, b), pple(a, b)]`; shallow function.

GEN `Z_ppio(GEN a, GEN b)` returns `[gcd(a, b), ppi(a, b), ppo(a, b)]`; shallow function.

**GEN Z\_cba**(GEN *a*, GEN *b*) fast natural coprime base algorithm. Returns a vector of coprime divisors of *a* and *b* such that both *a* and *b* can be multiplicatively generated from this set. Perfect powers are not removed, use **Z\_isanypower** if needed; shallow function.

**GEN ZV\_cba\_extend**(GEN *P*, GEN *b*) extend a coprime basis *P* by the integer *b*, the result being a coprime basis for  $P \cup \{b\}$ . Perfect powers are not removed; shallow function.

**GEN ZV\_cba**(GEN *v*) given a vector of nonzero integers *v*, return a coprime basis for *v*. Perfect powers are not removed; shallow function.

#### 7.4.4 Checks attached to arithmetic functions.

Arithmetic functions accept arguments of the following kind: a plain positive integer *N* (**t\_INT**), the factorization *fa* of a positive integer (a **t\_MAT** with two columns containing respectively primes and exponents), or a vector [*N*, *fa*]. A few functions accept nonzero integers (e.g. **omega**), and some others arbitrary integers (e.g. **factorint**, ...).

**int is\_Z\_factorpos**(GEN *f*) returns 1 if *f* looks like the factorization of a positive integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that *f* is a two-column matrix all of whose entries are positive integers. It does *not* check that entries in the first column (“primes”) are prime, or even pairwise coprime, nor that they are strictly increasing.

**int is\_Z\_factornon0**(GEN *f*) returns 1 if *f* looks like the factorization of a nonzero integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof, analogous to **is\_Z\_factorpos**. (Entries in the first column need only be nonzero integers.)

**int is\_Z\_factor**(GEN *f*) returns 1 if *f* looks like the factorization of an integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that *f* is a two-column matrix all of whose entries are integers. Entries in the second column (“exponents”) are all positive. Either it encodes the “factorization”  $0^e$ ,  $e > 0$ , or entries in the first column (“primes”) are all nonzero.

**GEN clean\_Z\_factor**(GEN *f*) assuming *f* is the factorization of an integer *n*, return the factorization of  $|n|$ , i.e. remove  $-1$  from the factorization. Shallow function.

**GEN fuse\_Z\_factor**(GEN *f*, GEN *B*) assuming *f* is the factorization of an integer *n*, return **boundfact**(*n*, *B*), i.e. return a factorization where all primary factors for  $|p| \leq B$  are preserved, and all others are “fused” into a single composite integer; if that remainder is trivial, i.e. equal to 1, it is of course not included. Shallow function.

In the following three routines, *f* is the name of an arithmetic function, and *n* a supplied argument. They all raise exceptions if *n* does not correspond to an integer or an integer factorization of the expected shape.

**GEN check\_arith\_pos**(GEN *n*, const char \**f*) check whether *n* is attached to the factorization of a positive integer, and return NULL (plain **t\_INT**) or a factorization extracted from *n* otherwise. May raise an **e\_DOMAIN** ( $n \leq 0$ ) or an **e\_TYPE** exception (other failures).

**GEN check\_arith\_non0**(GEN *n*, const char \**f*) check whether *n* is attached to the factorization of a nonzero integer, and return NULL (plain **t\_INT**) or a factorization extracted from *n* otherwise. May raise an **e\_TYPE** exception.

**GEN check\_arith\_all**(GEN *n*, const char \**f*) is attached to the factorization of an integer, and return NULL (plain **t\_INT**) or a factorization extracted from *n* otherwise.



### 7.4.5 Incremental integer factorization.

Routines attached to the dynamic factorization of an integer  $n$ , iterating over successive prime divisors. This is useful to implement high-level routines allowed to take shortcuts given enough partial information: e.g. `moebius( $n$ )` can be trivially computed if we hit  $p$  such that  $p^2 \mid n$ . For efficiency, trial division by small primes should have already taken place. In any case, the functions below assume that no prime  $< 2^{14}$  divides  $n$ .

`GEN ifac_start(GEN n, int moebius)` schedules a new factorization attempt for the integer  $n$ . If `moebius` is nonzero, the factorization will be aborted as soon as a repeated factor is detected (Moebius mode). The function assumes that  $n > 1$  is a *composite* `t_INT` whose prime divisors satisfy  $p > 2^{14}$  and that one can write to  $n$  in place.

This function stores data on the stack, no `gerepile` call should delete this data until the factorization is complete. Returns `partial`, a data structure recording the partial factorization state.

`int ifac_next(GEN *partial, GEN *p, long *e)` deletes a primary factor  $p^e$  from `partial` and sets `p` (prime) and `e` (exponent), and normally returns 1. Whatever remains in the `partial` structure is now coprime to  $p$ .

Returns 0 if all primary factors have been used already, so we are done with the factorization. In this case `p` is set to `NULL`. If we ran in Moebius mode and the factorization was in fact aborted, we have  $e = 1$ , otherwise  $e = 0$ .

`int ifac_read(GEN part, GEN *k, long *e)` peeks at the next integer to be factored in the list  $k^e$ , where  $k$  is not necessarily prime and can be a perfect power as well, but will be factored by the next call to `ifac_next`. You can remove this factorization from the schedule by calling:

`void ifac_skip(GEN part)` removes the next scheduled factorization.

`int ifac_isprime(GEN n)` given  $n$  whose prime divisors are  $> 2^{14}$ , returns the decision the factoring engine would take about the compositeness of  $n$ : 0 if  $n$  is a proven composite, and 1 if we believe it to be prime; more precisely,  $n$  is a proven prime if `factor_proven` is set, and only a BPSW-pseudoprime otherwise.

### 7.4.6 Integer core, squarefree factorization.

`long Z_issquarefree(GEN n)` returns 1 if the `t_INT`  $n$  is square-free, and 0 otherwise.

`long Z_issquarefree_fact(GEN fa)` same, where `fa` is `factor(n)`.

`long Z_isfundamental(GEN x)` returns 1 if the `t_INT`  $x$  is a fundamental discriminant, and 0 otherwise.

`GEN core(GEN n)` unique squarefree integer  $d$  dividing  $n$  such that  $n/d$  is a square. The core of 0 is defined to be 0.

`GEN core2(GEN n)` return  $[d, f]$  with  $d$  squarefree and  $n = df^2$ .

`GEN corepartial(GEN n, long lim)` as `core`, using `boundfact(n, lim)` to partially factor  $n$ . The result is not necessarily squarefree, but  $p^2 \mid n$  implies  $p > \text{lim}$ .

`GEN core2partial(GEN n, long lim)` as `core2`, using `boundfact(n, lim)` to partially factor  $n$ . The resulting  $d$  is not necessarily squarefree, but  $p^2 \mid n$  implies  $p > \text{lim}$ .

### 7.4.7 Primes, primality and compositeness tests.

#### 7.4.7.1 Chebyshev's $\pi$ function, bounds.

`ulong uprimepi(ulong n)`, returns the number of primes  $p \leq n$  (Chebyshev's  $\pi$  function).

`double primepi_upper_bound(double x)` return a quick upper bound for  $\pi(x)$ , using Dusart bounds.

`GEN gprimepi_upper_bound(GEN x)` as `primepi_upper_bound`, returns a `t_REAL`.

`double primepi_lower_bound(double x)` return a quick lower bound for  $\pi(x)$ , using Dusart bounds.

`GEN gprimepi_lower_bound(GEN x)` as `primepi_lower_bound`, returns a `t_REAL` or `gen_0`.

#### 7.4.7.2 Primes, primes in intervals.

`ulong unextprime(ulong n)`, returns the smallest prime  $\geq n$ . Return 0 if it cannot be represented as an `ulong` ( $n$  bigger than  $2^{64} - 59$  or  $2^{32} - 5$  depending on the word size).

`ulong uprecprime(ulong n)`, returns the largest prime  $\leq n$ . Return 0 if  $n \leq 1$ .

`ulong uprime(long n)` returns the  $n$ -th prime, assuming it fits in an `ulong` (overflow error otherwise).

`GEN prime(long n)` same as `utoi(uprime(n))`.

`GEN primes_zv(long m)` returns the first  $m$  primes, in a `t_VECSMALL`.

`GEN primes(long m)` return the first  $m$  primes, as a `t_VEC` of `t_INTs`.

`GEN primes_interval(GEN a, GEN b)` return the primes in the interval  $[a, b]$ , as a `t_VEC` of `t_INTs`.

`GEN primes_interval_zv(ulong a, ulong b)` return the primes in the interval  $[a, b]$ , as a `t_VECSMALL` of `ulongss`.

`GEN primes_upto_zv(ulong b)` return the primes in the interval  $[2, b]$ , as a `t_VECSMALL` of `ulongss`.

#### 7.4.7.3 Tests.

`int uisprime(ulong p)`, returns 1 if  $p$  is a prime number and 0 otherwise.

`int uisprime_101(ulong p)`, assuming that  $p$  has no divisor  $\leq 101$ , returns 1 if  $p$  is a prime number and 0 otherwise.

`int uisprime_661(ulong p)`, assuming that  $p$  has no divisor  $\leq 661$ , returns 1 if  $p$  is a prime number and 0 otherwise.

`int isprime(GEN n)`, returns 1 if the `t_INT`  $n$  is a (fully proven) prime number and 0 otherwise.

`long isprimeAPRCL(GEN n)`, returns 1 if the `t_INT`  $n$  is a prime number and 0 otherwise, using only the APRCL test — not even trial division or compositeness tests. The workhorse `isprime` should be faster on average, especially if nonprimes are included!

`long isprimeECP(GEN n)`, returns 1 if the `t_INT`  $n$  is a prime number and 0 otherwise, using only the ECP test. The workhorse `isprime` should be faster on average.

`long BPSW_psp(GEN n)`, returns 1 if the `t_INT`  $n$  is a Baillie-Pomerance-Selfridge-Wagstaff pseudoprime, and 0 otherwise (proven composite).

`int BPSW_isprime(GEN x)` assuming  $x$  is a BPSW-pseudoprime, rigorously prove its primality. The function `isprime` is currently implemented as

```
BPSW_psp(x) && BPSW_isprime(x)
```

`long millerrabin(GEN n, long k)` performs  $k$  strong Rabin-Miller compositeness tests on the `t_INT`  $n$ , using  $k$  random bases. This function also caches square roots of  $-1$  that are encountered during the successive tests and stops as soon as three distinct square roots have been produced; we have in principle factored  $n$  at this point, but unfortunately, there is currently no way for the factoring machinery to become aware of it. (It is highly implausible that hard to find factors would be exhibited in this way, though.) This should be slower than `BPSW_psp` for  $k \geq 4$  and we expect it to be less reliable.

`GEN ecpp(GEN N)` returns an ECPP certificate for `t_INT`  $N$ ; underlies `primecert`.

`GEN ecpp0(GEN N, long t)` returns a (potentially) partial ECPP certificate for `t_INT`  $N$  where strong pseudo-primes  $< 2^t$  are included as primes in the certificate. Underlies `primecert` with  $t$  set to the `partial` argument.

`GEN ecppexport(GEN cert, long flag)` export a PARI ECPP certificate to MAGMA or Primo format; underlies `primecertexport`.

`long ecppisvalid(GEN cert)` checks whether a PARI ECPP certificate is valid; underlies `primecertisvalid`.

`long check_ecppcert(GEN cert)` checks whether `cert` looks like a PARI ECPP certificate, (valid or invalid) without doing any computation.

#### 7.4.8 Iterators over primes.

`int forprime_init(forprime_t *T, GEN a, GEN b)` initialize an iterator  $T$  over primes in  $[a, b]$ ; over primes  $\geq a$  if  $b = \text{NULL}$ . Return 0 if the range is known to be empty from the start (as if  $b < a$  or  $b < 0$ ), and return 1 otherwise. Use `forprime_next` to iterate over the prime collection.

`int forprimestep_init(forprime_t *T, GEN a, GEN b, GEN q)` initialize an iterator  $T$  over primes in an arithmetic progression in  $[a, b]$ ; over primes  $\geq a$  if  $b = \text{NULL}$ . The argument  $q$  is either a `t_INT` ( $p \equiv a \pmod{q}$ ) or a `t_INTMOD` `Mod(c, N)` and we restrict to that congruence class. Return 0 if the range is known to be empty from the start (as if  $b < a$  or  $b < 0$ ), and return 1 otherwise. Use `forprime_next` to iterate over the prime collection.

`GEN forprime_next(forprime_t *T)` returns the next prime in the range, assuming that  $T$  was initialized by `forprime_init`.

```
int u_forprime_init(forprime_t *T, ulong a, ulong b)
```

```
ulong u_forprime_next(forprime_t *T)
```

`void u_forprime_restrict(forprime_t *T, ulong c)` let  $T$  an iterator over primes initialized via `u_forprime_init(&T, a, b)`, possibly followed by a number of calls to `u_forprime_next`, and  $a \leq c \leq b$ . Restrict the range of primes considered to  $[a, c]$ .

`int u_forprime_arith_init(forprime_t *T, ulong a, ulong b, ulong c, ulong q)` initialize an iterator over primes in  $[a, b]$ , congruent to  $c$  modulo  $q$ . Subsequent calls to `u_forprime_next` will only return primes congruent to  $c$  modulo  $q$ . Note that unless  $(c, q) = 1$  there will be at most one such prime.

## 7.5 Integral, rational and generic linear algebra.

**7.5.1 ZC / ZV, ZM.** A ZV (resp. a ZM, resp. a ZX) is a `t_VEC` or `t_COL` (resp. `t_MAT`, resp. `t_POL`) with `t_INT` coefficients.

### 7.5.1.1 ZC / ZV.

`void RgV_check_ZV(GEN x, const char *s)` Assuming `x` is a `t_VEC` or `t_COL` raise an error if it is not a ZV (`s` should point to the name of the caller).

`int RgV_is_ZV(GEN x)` Assuming `x` is a `t_VEC` or `t_COL` return 1 if it is a ZV, and 0 otherwise.

`int RgV_is_ZVpos(GEN x)` Assuming `x` is a `t_VEC` or `t_COL` return 1 if it is a ZV with positive entries, and 0 otherwise.

`int RgV_is_ZVnon0(GEN x)` Assuming `x` is a `t_VEC` or `t_COL` return 1 if it is a ZV with nonzero entries, and 0 otherwise.

`int RgV_is_QV(GEN P)` return 1 if the RgV `P` has only `t_INT` and `t_FRAC` coefficients, and 0 otherwise.

`int RgV_is_arithprog(GEN v, GEN *a, GEN *b)` assuming `x` is a `t_VEC` or `t_COL` return 1 if its entries follow an arithmetic progression of the form  $a + b * n$ ,  $n = 0, 1, \dots$  and set `a` and `b`. Else return 0.

`int ZV_equal0(GEN x)` returns 1 if all entries of the ZV `x` are zero, and 0 otherwise.

`int ZV_cmp(GEN x, GEN y)` compare two ZV, which we assume have the same length (lexicographic order).

`int ZV_abscmp(GEN x, GEN y)` compare two ZV, which we assume have the same length (lexicographic order, comparing absolute values).

`int ZV_equal(GEN x, GEN y)` returns 1 if the two ZV are equal and 0 otherwise. A `t_COL` and a `t_VEC` with the same entries are declared equal.

`GEN identity_ZV(long n)` return the `t_VEC`  $[1, 2, \dots, n]$ .

`GEN ZC_add(GEN x, GEN y)` adds `x` and `y`.

`GEN ZC_sub(GEN x, GEN y)` subtracts `x` and `y`.

`GEN ZC_Z_add(GEN x, GEN y)` adds `y` to `x[1]`.

`GEN ZC_Z_sub(GEN x, GEN y)` subtracts `y` to `x[1]`.

`GEN Z_ZC_sub(GEN a, GEN x)` returns the vector  $[a - x_1, -x_2, \dots, -x_n]$ .

`GEN ZC_copy(GEN x)` returns a (`t_COL`) copy of `x`.

`GEN ZC_neg(GEN x)` returns  $-x$  as a `t_COL`.

`void ZV_neg_inplace(GEN x)` negates the ZV `x` in place, by replacing each component by its opposite (the type of `x` remains the same, `t_COL` or `t_COL`). If you want to save even more memory by avoiding the implicit component copies, use `ZV_togglesign`.

`void ZV_togglesign(GEN x)` negates `x` in place, by toggling the sign of its integer components. Universal constants `gen_1`, `gen_m1`, `gen_2` and `gen_m2` are handled specially and will not be corrupted. (We use `togglesign_safe`.)

`GEN ZC_Z_mul(GEN x, GEN y)` multiplies the ZC or ZV  $x$  (which can be a column or row vector) by the  $\mathbf{t\_INT}$   $y$ , returning a ZC.

`GEN ZC_Z_divexact(GEN x, GEN y)` returns  $x/y$  assuming all divisions are exact.

`GEN ZC_divexactu(GEN x, ulong y)` returns  $x/y$  assuming all divisions are exact.

`GEN ZC_Z_div(GEN x, GEN y)` returns  $x/y$ , where the resulting vector has rational entries.

`GEN ZV_ZV_mod(GEN a, GEN b)`. Assuming  $a$  and  $b$  are two ZV of the same length, returns the vector whose  $i$ -th component is `modii(a[i], b[i])`.

`GEN ZV_dotproduct(GEN x, GEN y)` as `RgV_dotproduct` assuming  $x$  and  $y$  have  $\mathbf{t\_INT}$  entries.

`GEN ZV_dotsquare(GEN x)` as `RgV_dotsquare` assuming  $x$  has  $\mathbf{t\_INT}$  entries.

`GEN ZC_lincomb(GEN u, GEN v, GEN x, GEN y)` returns  $ux + vy$ , where  $u, v$  are  $\mathbf{t\_INT}$  and  $x, y$  are ZC or ZV. Return a ZC

`void ZC_lincomb1_inplace(GEN X, GEN Y, GEN v)` sets  $X \leftarrow X + vY$ , where  $v$  is a  $\mathbf{t\_INT}$  and  $X, Y$  are ZC or ZV. (The result has the type of  $X$ .) Memory efficient (e.g. no-op if  $v = 0$ ), but not gerpile-safe.

`void ZC_lincomb1_inplace_i(GEN X, GEN Y, GEN v, long n)` variant of `ZC_lincomb1_inplace`: only update  $X[1], \dots, X[n]$ , assuming that  $n < \lg(X)$ .

`GEN ZC_ZV_mul(GEN x, GEN y, GEN p)` multiplies the ZC  $x$  (seen as a column vector) by the ZV  $y$  (seen as a row vector, assumed to have compatible dimensions).

`GEN ZV_content(GEN x)` returns the GCD of all the components of  $x$ .

`GEN ZV_extgcd(GEN A)` given a vector of  $n$  integers  $A$ , returns  $[d, U]$ , where  $d$  is the content of  $A$  and  $U$  is a matrix in  $\mathrm{GL}_n(\mathbf{Z})$  such that  $AU = [D, 0, \dots, 0]$ .

`GEN ZV_prod(GEN x)` returns the product of all the components of  $x$  (1 for the empty vector).

`GEN ZV_sum(GEN x)` returns the sum of all the components of  $x$  (0 for the empty vector).

`long ZV_max_lg(GEN x)` returns the effective length of the longest entry in  $x$ .

`int ZV_dvd(GEN x, GEN y)` assuming  $x, y$  are two ZVs of the same length, return 1 if  $y[i]$  divides  $x[i]$  for all  $i$  and 0 otherwise. Error if one of the  $y[i]$  is 0.

`GEN ZV_sort(GEN L)` sort the ZV  $L$ . Returns a vector with the same type as  $L$ .

`GEN ZV_sort_shallow(GEN L)` shallow version of `ZV_sort`.

`void ZV_sort_inplace(GEN L)` sort the ZV  $L$ , in place.

`GEN ZV_sort_uniq(GEN L)` sort the ZV  $L$ , removing duplicate entries. Returns a vector with the same type as  $L$ .

`GEN ZV_sort_uniq_shallow(GEN L)` shallow version of `ZV_sort_uniq`.

`long ZV_search(GEN L, GEN y)` look for the  $\mathbf{t\_INT}$   $y$  in the sorted ZV  $L$ . Return an index  $i$  such that  $L[i] = y$ , and 0 otherwise.

`GEN ZV_indexsort(GEN L)` returns the permutation which, applied to the ZV  $L$ , would sort the vector. The result is a  $\mathbf{t\_VECSMALL}$ .

`GEN ZV_union_shallow(GEN x, GEN y)` given two *sorted* ZV (as per `ZV_sort`, returns the union of  $x$  and  $y$ . Shallow function. In case two entries are equal in  $x$  and  $y$ , include the one from  $x$ .

`GEN ZC_union_shallow(GEN x, GEN y)` as `ZV_union_shallow` but return a  $\mathbf{t\_COL}$ .

### 7.5.1.2 ZM.

`void RgM_check_ZM(GEN A, const char *s)` Assuming  $x$  is a `t_MAT` raise an error if it is not a ZM ( $s$  should point to the name of the caller).

`GEN RgM_rescale_to_int(GEN x)` given a matrix  $x$  with real entries (`t_INT`, `t_FRAC` or `t_REAL`), return a ZM which is very close to  $Dx$  for some well-chosen integer  $D$ . More precisely, if the input is exact,  $D$  is the denominator of  $x$ ; else it is a power of 2 chosen so that all inexact entries are correctly rounded to 1 ulp.

`GEN ZM_copy(GEN x)` returns a copy of  $x$ .

`int ZM_equal(GEN A, GEN B)` returns 1 if the two ZM are equal and 0 otherwise.

`int ZM_equal0(GEN A)` returns 1 if the ZM  $A$  is identically equal to 0.

`GEN ZM_add(GEN x, GEN y)` returns  $x + y$  (assumed to have compatible dimensions).

`GEN ZM_sub(GEN x, GEN y)` returns  $x - y$  (assumed to have compatible dimensions).

`GEN ZM_neg(GEN x)` returns  $-x$ .

`void ZM_togglesign(GEN x)` negates  $x$  in place, by toggling the sign of its integer components. Universal constants `gen_1`, `gen_m1`, `gen_2` and `gen_m2` are handled specially and will not be corrupted. (We use `togglesign_safe`.)

`GEN ZM_mul(GEN x, GEN y)` multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

`GEN ZM2_mul(GEN x, GEN y)` multiplies the two-by-two ZM  $x$  and  $y$ .

`GEN ZM_sqr(GEN x)` returns  $x^2$ , where  $x$  is a square ZM.

`GEN ZM_Z_mul(GEN x, GEN y)` multiplies the ZM  $x$  by the `t_INT`  $y$ .

`GEN ZM_ZC_mul(GEN x, GEN y)` multiplies the ZM  $x$  by the ZC  $y$  (seen as a column vector, assumed to have compatible dimensions).

`GEN ZM_ZX_mul(GEN x, GEN T)` returns  $x \times y$ , where  $y$  is `RgX_to_RgC(T, lg(x) - 1)`.

`GEN ZM_diag_mul(GEN d, GEN m)` given a vector  $d$  with integer entries and a ZM  $m$  of compatible dimensions, return `diagonal(d) * m`.

`GEN ZM_mul_diag(GEN m, GEN d)` given a vector  $d$  with integer entries and a ZM  $m$  of compatible dimensions, return `m * diagonal(d)`.

`GEN ZM_multosym(GEN x, GEN y)`

`GEN ZM_transmultosym(GEN x, GEN y)`

`GEN ZM_transmul(GEN x, GEN y)`

`GEN ZMrow_ZC_mul(GEN x, GEN y, long i)` multiplies the  $i$ -th row of ZM  $x$  by the ZC  $y$  (seen as a column vector, assumed to have compatible dimensions). Assumes that  $x$  is nonempty and  $0 < i < \text{lg}(x[1])$ .

`int ZMrow_equal0(GEN V, long i)` returns 1 if the  $i$ -th row of the ZM  $V$  is zero, and 0 otherwise.

`GEN ZV_ZM_mul(GEN x, GEN y)` multiplies the ZV  $x$  by the ZM  $y$ . Returns a `t_VEC`.

`GEN ZM_Z_divexact(GEN x, GEN y)` returns  $x/y$  assuming all divisions are exact.

`GEN ZM_divexactu(GEN x, ulong y)` returns  $x/y$  assuming all divisions are exact.

GEN ZM\_Z\_div(GEN x, GEN y) returns  $x/y$ , where the resulting matrix has rational entries.

GEN ZM\_ZV\_mod(GEN a, GEN b). Assuming  $a$  is a ZM whose columns have the same length as the ZV  $b$ , apply ZV\_ZV\_mod( $a[i], b$ ) to all columns.

GEN ZC\_Q\_mul(GEN x, GEN y) returns  $x*y$ , where  $y$  is a rational number and the resulting  $t\_COL$  has rational entries.

GEN ZM\_Q\_mul(GEN x, GEN y) returns  $x*y$ , where  $y$  is a rational number and the resulting matrix has rational entries.

GEN ZM\_pow(GEN x, GEN n) returns  $x^n$ , assuming  $x$  is a square ZM and  $n \geq 0$ .

GEN ZM\_powu(GEN x, ulong n) returns  $x^n$ , assuming  $x$  is a square ZM and  $n \geq 0$ .

GEN ZM\_det(GEN M) if  $M$  is a ZM, returns the determinant of  $M$ . This is the function underlying `matdet` whenever  $M$  is a ZM.

GEN ZM\_permanent(GEN M) if  $M$  is a ZM, returns its permanent. This is the function underlying `mat-permanent` whenever  $M$  is a ZM. It assumes that the matrix is square of dimension  $< \text{BITS\_IN\_LONG}$ .

GEN ZM\_detmult(GEN M) if  $M$  is a ZM, returns a multiple of the determinant of the lattice generated by its columns. This is the function underlying `detint`.

GEN ZM\_supnorm(GEN x) return the sup norm of the ZM  $x$ .

GEN ZM\_charpoly(GEN M) returns the characteristic polynomial (in variable 0) of the ZM  $M$ .

GEN ZM\_imagecompl(GEN x) returns `matimagecompl(x)`.

long ZM\_rank(GEN x) returns `matrank(x)`.

GEN ZM\_ker(GEN x) returns the primitive part of `matker(x)`; in other words the  $\mathbf{Q}$ -basis vectors are made integral and primitive.

GEN ZM\_indexrank(GEN x) returns `matindexrank(x)`.

GEN ZM\_indeximage(GEN x) returns `gel(ZM_indexrank(x), 2)`.

long ZM\_max\_lg(GEN x) returns the effective length of the longest entry in  $x$ .

GEN ZM\_inv(GEN M, GEN \*pd) if  $M$  is a ZM, return a primitive matrix  $H$  such that  $MH$  is  $d$  times the identity and set `*pd` to  $d$ . Uses a multimodular algorithm up to Hadamard's bound. If you suspect that the denominator is much smaller than  $\det M$ , you may use `ZM_inv_ratlift`.

GEN ZM\_inv\_ratlift(GEN M, GEN \*pd) if  $M$  is a ZM, return a primitive matrix  $H$  such that  $MH$  is  $d$  times the identity and set `*pd` to  $d$ . Uses a multimodular algorithm, attempting rational reconstruction along the way. To be used when you expect that the denominator of  $M^{-1}$  is much smaller than  $\det M$  else use `ZM_inv`.

GEN SL2\_inv\_shallow(GEN M) return the inverse of  $M \in \text{SL}_2(\mathbf{Z})$ . Not gerepile-safe.

GEN ZM\_pseudoinv(GEN M, GEN \*pv, GEN \*pd) if  $M$  is a nonempty ZM, let  $v = [y, z]$  returned by `indexrank` and let  $M_1$  be the corresponding square invertible matrix. Return a primitive left-inverse  $H$  such that  $HM_1$  is  $d$  times the identity and set `*pd` to  $d$ . If `pv` is not NULL, set `*pv` to  $v$ . Not gerepile-safe.

GEN ZM\_gauss(GEN a, GEN b) as `gauss`, where  $a$  and  $b$  coefficients are  $t\_INTs$ .

GEN ZM\_det\_triangular(GEN x) returns the product of the diagonal entries of  $x$  (its determinant if it is indeed triangular).

`int ZM_isidentity(GEN x)` return 1 if the ZM  $x$  is the identity matrix, and 0 otherwise.

`int ZM_isdiagonal(GEN x)` return 1 if the ZM  $x$  is diagonal, and 0 otherwise.

`int ZM_isscalar(GEN x, GEN s)` given a ZM  $x$  and a `t_INT`  $s$ , return 1 if  $x$  is equal to  $s$  times the identity, and 0 otherwise. If  $s$  is NULL, test whether  $x$  is an arbitrary scalar matrix.

`long ZC_is_ei(GEN x)` return  $i$  if the ZC  $x$  has 0 entries, but for a 1 at position  $i$ .

`int ZM_ishnf(GEN x)` return 1 if  $x$  is in HNF form, i.e. is upper triangular with positive diagonal coefficients, and for  $j > i$ ,  $x_{i,i} > x_{i,j} \geq 0$ .

### 7.5.2 QM.

`GEN QM_charpoly_ZX(GEN M)` returns the characteristic polynomial (in variable 0) of the QM  $M$ , assuming that the result has integer coefficients.

`GEN QM_charpoly_ZX_bound(GEN M, long b)` as `QM_charpoly_ZX` assuming that the sup norm of the (integral) result is  $\leq 2^b$ .

`GEN QM_gauss(GEN a, GEN b)` as `gauss`, where  $a$  and  $b$  coefficients are `t_FRACs`.

`GEN QM_gauss_i(GEN a, GEN b, long flag)` as `QM_gauss` if `flag` is 0. Else, no longer assume that  $a$  is left-invertible and return a solution of  $Pax = Pb$  where  $P$  is a row-selection matrix such that  $A = PaQ$  is square invertible of maximal rank, for some column-selection matrix  $Q$ ; in particular,  $x$  is a solution of the original equation  $ax = b$  if and only if a solution exists.

`GEN QM_indexrank(GEN x)` returns `matindexrank(x)`.

`GEN QM_inv(GEN M)` return the inverse of the QM  $M$ .

`long QM_rank(GEN x)` returns `matrank(x)`.

`GEN QM_image(GEN x)` returns an integral matrix with primitive columns generating the image of  $x$ .

`GEN QM_image_shallow(GEN A)` shallow version of the previous function, not suitable for `gerepile`.

### 7.5.3 Qevproj.

`GEN Qevproj_init(GEN M)` let  $M$  be a  $n \times d$  ZM of maximal rank  $d \leq n$ , representing the basis of a  $\mathbf{Q}$ -subspace  $V$  of  $\mathbf{Q}^n$ . Return a projector on  $V$ , to be used by `Qevproj_apply`. The interface details may change in the future, but this function currently returns  $[M, B, D, p]$ , where  $p$  is a `t_VECSMALL` with  $d$  entries such that the submatrix  $A = \text{rowpermute}(M, p)$  is invertible,  $B$  is a ZM and  $d$  a `t_INT` such that  $AB = DId_d$ .

`GEN Qevproj_apply(GEN T, GEN pro)` let  $T$  be an  $n \times n$  QM, stabilizing a  $\mathbf{Q}$ -subspace  $V \subset \mathbf{Q}^n$  of dimension  $d$ , and let `pro` be a projector on that subspace initialized by `Qevproj_init(M)`. Return the  $d \times d$  matrix representing  $T|_V$  on the basis given by the columns of  $M$ .

`GEN Qevproj_apply_vecei(GEN T, GEN pro, long k)` as `Qevproj_apply`, return only the image of the  $k$ -th basis vector  $M[k]$  (still on the basis given by the columns of  $M$ ).

`GEN Qevproj_down(GEN T, GEN pro)` given a ZC (resp. a ZM)  $T$  representing an element (resp. a vector of elements) in the subspace  $V$  return a QC (resp. a QM)  $U$  such that  $T = MU$ .



#### 7.5.4 zv, zm.

GEN identity\_zv(long n) return the t\_VEC SMALL  $[1, 2, \dots, n]$ .

GEN random\_zv(long n) returns a random zv with  $n$  components.

GEN zv\_abs(GEN x) return  $[|x[1]|, \dots, |x[n]|]$  as a zv.

GEN zv\_neg(GEN x) return  $-x$ . No check for overflow is done, which occurs in the fringe case where an entry is equal to  $2^{\text{BITS\_IN\_LONG}-1}$ .

GEN zv\_neg\_inplace(GEN x) negates  $x$  in place and return it. No check for overflow is done, which occurs in the fringe case where an entry is equal to  $2^{\text{BITS\_IN\_LONG}-1}$ .

GEN zm\_zc\_mul(GEN x, GEN y)

GEN zm\_mul(GEN x, GEN y)

GEN zv\_z\_mul(GEN x, long n) return  $nx$ . No check for overflow is done.

long zv\_content(GEN x) returns the gcd of the entries of  $x$ .

long zv\_dotproduct(GEN x, GEN y)

long zv\_prod(GEN x) returns the product of all the components of  $x$  (assumes no overflow occurs).

GEN zv\_prod\_Z(GEN x) returns the product of all the components of  $x$ ; consider all  $x[i]$  as `ulongs`.

long zv\_sum(GEN x) returns the sum of all the components of  $x$  (assumes no overflow occurs).

long zv\_sumpart(GEN v, long n) returns the sum  $v[1] + \dots + v[n]$  (assumes no overflow occurs and  $\lg(v) > n$ ).

int zv\_cmp0(GEN x) returns 1 if all entries of the zv  $x$  are 0, and 0 otherwise.

int zv\_equal(GEN x, GEN y) returns 1 if the two zv are equal and 0 otherwise.

int zv\_equal0(GEN x) returns 1 if all entries are 0, and return 0 otherwise.

long zv\_search(GEN L, long y) look for  $y$  in the sorted zv  $L$ . Return an index  $i$  such that  $L[i] = y$ , and 0 otherwise.

GEN zv\_copy(GEN x) as Flv\_copy.

GEN zm\_transpose(GEN x) as Flm\_transpose.

GEN zm\_copy(GEN x) as Flm\_copy.

GEN zero\_zm(long m, long n) as zero\_Flm.

GEN zero\_zv(long n) as zero\_Flv.

GEN zm\_row(GEN A, long x0) as Flm\_row.

GEN zv\_diagonal(GEN v) return the square zm whose diagonal is given by the entries of  $v$ .

GEN zm\_permanent(GEN M) return the permanent of  $M$ . The function assumes that the matrix is square of dimension  $< \text{BITS\_IN\_LONG}$ .

int zvV\_equal(GEN x, GEN y) returns 1 if the two zvV (vectors of zv) are equal and 0 otherwise.

### 7.5.5 ZMV / zmV (vectors of ZM/zm).

int RgV\_is\_ZMV(GEN x) Assuming  $x$  is a `t_VEC` or `t_COL` return 1 if its components are ZM, and 0 otherwise.

GEN ZMV\_to\_zmV(GEN z)

GEN zmV\_to\_ZMV(GEN z)

GEN ZMV\_to\_FlmV(GEN z, ulong m)

### 7.5.6 QC / QV, QM.

GEN QM\_mul(GEN x, GEN y) multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

GEN QM\_sqr(GEN x) returns the square of  $x$  (assumed to be square).

GEN QM\_QC\_mul(GEN x, GEN y) multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

GEN QM\_det(GEN M) returns the determinant of  $M$ .

GEN QM\_ker(GEN x) returns `matker(x)`.

### 7.5.7 RgC / RgV, RgM.

RgC and RgV routines assume the inputs are VEC or COL of the same dimension. RgM assume the inputs are MAT of compatible dimensions.

#### 7.5.7.1 Matrix arithmetic.

void RgM\_dimensions(GEN x, long \*m, long \*n) sets  $m$ , resp.  $n$ , to the number of rows, resp. columns of the `t_MAT`  $x$ .

GEN RgC\_add(GEN x, GEN y) returns  $x + y$  as a `t_COL`.

GEN RgC\_neg(GEN x) returns  $-x$  as a `t_COL`.

GEN RgC\_sub(GEN x, GEN y) returns  $x - y$  as a `t_COL`.

GEN RgV\_add(GEN x, GEN y) returns  $x + y$  as a `t_VEC`.

GEN RgV\_neg(GEN x) returns  $-x$  as a `t_VEC`.

GEN RgV\_sub(GEN x, GEN y) returns  $x - y$  as a `t_VEC`.

GEN RgM\_add(GEN x, GEN y) return  $x + y$ .

GEN RgM\_neg(GEN x) returns  $-x$ .

GEN RgM\_sub(GEN x, GEN y) returns  $x - y$ .

GEN RgM\_Rg\_add(GEN x, GEN y) assuming  $x$  is a square matrix and  $y$  a scalar, returns the square matrix  $x + y * \text{Id}$ .

GEN RgM\_Rg\_add\_shallow(GEN x, GEN y) as RgM\_Rg\_add with much fewer copies. Not suitable for `gerepileupto`.

GEN RgM\_Rg\_sub(GEN x, GEN y) assuming  $x$  is a square matrix and  $y$  a scalar, returns the square matrix  $x - y * \text{Id}$ .

GEN RgM\_Rg\_sub\_shallow(GEN x, GEN y) as RgM\_Rg\_sub with much fewer copies. Not suitable for `gerepileupto`.

GEN RgC\_Rg\_add(GEN x, GEN y) assuming  $x$  is a nonempty column vector and  $y$  a scalar, returns the vector  $[x_1 + y, x_2, \dots, x_n]$ .

GEN RgC\_Rg\_sub(GEN x, GEN y) assuming  $x$  is a nonempty column vector and  $y$  a scalar, returns the vector  $[x_1 - y, x_2, \dots, x_n]$ .

GEN Rg\_RgC\_sub(GEN a, GEN x) assuming  $x$  is a nonempty column vector and  $a$  a scalar, returns the vector  $[a - x_1, -x_2, \dots, -x_n]$ .

GEN RgC\_Rg\_div(GEN x, GEN y)

GEN RgM\_Rg\_div(GEN x, GEN y) returns  $x/y$  ( $y$  treated as a scalar).

GEN RgC\_Rg\_mul(GEN x, GEN y)

GEN RgV\_Rg\_mul(GEN x, GEN y)

GEN RgM\_Rg\_mul(GEN x, GEN y) returns  $x \times y$  ( $y$  treated as a scalar).

GEN RgV\_RgC\_mul(GEN x, GEN y) returns  $x \times y$ .

GEN RgV\_RgM\_mul(GEN x, GEN y) returns  $x \times y$ .

GEN RgM\_RgC\_mul(GEN x, GEN y) returns  $x \times y$ .

GEN RgM\_RgX\_mul(GEN x, GEN T) returns  $x \times y$ , where  $y$  is  $\text{RgX\_to\_RgC}(T, \lg(x) - 1)$ .

GEN RgM\_mul(GEN x, GEN y) returns  $x \times y$ .

GEN RgM\_ZM\_mul(GEN x, GEN y) returns  $x \times y$  assuming that  $y$  is a ZM.

GEN RgM\_transmul(GEN x, GEN y) returns  $x \sim \times y$ .

GEN RgM\_multosym(GEN x, GEN y) returns  $x \times y$ , assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).

GEN RgM\_transmultosym(GEN x, GEN y) returns  $x \sim \times y$ , assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).

GEN RgMrow\_RgC\_mul(GEN x, GEN y, long i) multiplies the  $i$ -th row of RgM  $x$  by the RgC  $y$  (seen as a column vector, assumed to have compatible dimensions). Assumes that  $x$  is nonempty and  $0 < i < \lg(x[1])$ .

GEN RgM\_mulreal(GEN x, GEN y) returns the real part of  $x \times y$  (whose entries are t\_INT, t\_FRAC, t\_REAL or t\_COMPLEX).

GEN RgM\_sqr(GEN x) returns  $x^2$ .

GEN RgC\_RgV\_mul(GEN x, GEN y) returns  $x \times y$  (the matrix  $(x_i y_j)$ ).

GEN RgC\_RgV\_mulrealsym(GEN x, GEN y) returns the real part of  $x \times y$  (whose entries are t\_INT, t\_FRAC, t\_REAL or t\_COMPLEX), assuming the result is symmetric.

The following two functions are not well defined in general and only provided for convenience in specific cases:

GEN RgC\_RgM\_mul(GEN x, GEN y) returns  $x \times y[1, ]$  if  $y$  is a row matrix  $1 \times n$ , error otherwise.

GEN RgM\_RgV\_mul(GEN x, GEN y) returns  $x \times y[, 1]$  if  $y$  is a column matrix  $n \times 1$ , error otherwise.

GEN RgM\_powers(GEN x, long n) returns  $[x^0, \dots, x^n]$  as a t\_VEC of RgMs.

GEN RgV\_sum(GEN v) sum of the entries of  $v$

GEN RgV\_prod(GEN v) product of the entries of  $v$ , using a divide and conquer strategy

GEN RgV\_sumpart(GEN v, long n) returns the sum  $v[1] + \dots + v[n]$  (assumes that  $\text{lg}(v) > n$ ).

GEN RgV\_sumpart2(GEN v, long m, long n) returns the sum  $v[m] + \dots + v[n]$  (assumes that  $\text{lg}(v) > n$  and  $m > 0$ ). Returns `gen_0` when  $m > n$ .

GEN RgM\_sumcol(GEN v) returns a `t_COL`, sum of the columns of the `t_MAT`  $v$ .

GEN RgV\_dotproduct(GEN x, GEN y) returns the scalar product of  $x$  and  $y$

GEN RgV\_dotsquare(GEN x) returns the scalar product of  $x$  with itself.

GEN RgV\_kill0(GEN v) returns a shallow copy of  $v$  where entries matched by `gequal0` are replaced by `NULL`. The return value is not a valid GEN and must be handled specially. The idea is to pre-treat a vector of coefficients to speed up later linear combinations or scalar products.

GEN gram\_matrix(GEN v) returns the Gram matrix  $(v_i \cdot v_j)$  attached to the entries of  $v$  (matrix, or vector of vectors).

GEN RgV\_polint(GEN X, GEN Y, long v)  $X$  and  $Y$  being two vectors of the same length, returns the polynomial  $T$  in variable  $v$  such that  $T(X[i]) = Y[i]$  for all  $i$ . The special case  $X = \text{NULL}$  corresponds to  $X = [1, 2, \dots, n]$ , where  $n$  is the length of  $Y$ . This is the function underlying `polint` for formal interpolation.

GEN polintspec(GEN X, GEN Y, GEN t, long n, long \*pe) return  $P(t)$  where  $P$  is the Lagrange interpolation polynomial attached to the  $n$  points  $(X[0], Y[0]), \dots, (X[n-1], Y[n-1])$ . If `pe` is not `NULL` and  $t$  is a complex numeric value, `*pe` contains an error estimate for the returned value (Neville's algorithm, see `polinterpolate`). In extrapolation algorithms, e.g., Romberg integration, this function is usually called on actual GEN vectors with offsets:  $x+k$  and  $y+k$  so as to interpolate on  $x[k..k+n-1]$  without having to use `vecslice`. This is the function underlying `polint` for numerical interpolation.

GEN polint\_i(GEN X, GEN Y, GEN t, long \*pe) as `polintspec`, where  $X$  and  $Y$  are actual GEN vectors.

GEN vandermondeinverse(GEN r, GEN T, GEN d, GEN V) Given a vector  $r$  of  $n$  scalars and the `t_POL`  $T = \prod_{i=1}^n (X - r_i)$ , return  $dM^{-1}$ , where  $M = (r_i^{j-1})_{1 \leq i, j \leq n}$  is the van der Monde matrix;  $V$  is `NULL` or a vector containing the  $T'(r_i)$ , as returned by `vandermondeinverseinit`. The demonimator  $d$  may be set to `NULL` (handled as 1). If  $c$  is the  $k$ -column of the result, it is essentially  $d$  times the  $k$ -th Lagrange interpolation polynomial: we have  $\sum_j c_j r_i^{j-1} = d\delta_{i=k}$ . This is the function underlying `RgV_polint` when the base field is not  $\mathbf{Z}/p\mathbf{Z}$ : it uses  $O(n^2)$  scalar operations and is asymptotically slower than variants using multi-evaluation such as `FpV_polint`; it is also accurate over inexact fields.

GEN vandermondeinverseinit(GEN r) Given a vector  $r$  of  $n$  scalars, let  $T$  be the `t_POL`  $T = \prod_{j=1}^n (X - r_j)$ . This function returns the  $T'(r_i)$  computed stably via products of difference: the  $i$ -th entry is  $T'(r_i) = \prod_{j \neq i} (r_i - r_j)$ . It is asymptotically slow (uses  $O(n^2)$  scalar operations, where multi-evaluation achieves quasi-linear running time) but allows accurate computation at low accuracies when  $T$  has large complex coefficients.

### 7.5.7.2 Special shapes.

The following routines check whether matrices or vectors have a special shape, using `gequal1` and `gequal0` to test components. (This makes a difference when components are inexact.)

`int RgV_isscalar(GEN x)` return 1 if all the entries of  $x$  are 0 (as per `gequal0`), except possibly the first one. The name comes from vectors expressing polynomials on the standard basis  $1, T, \dots, T^{n-1}$ , or on `nf.zk` (whose first element is 1).

`int QV_isscalar(GEN x)` as `RgV_isscalar`, assuming  $x$  is a QV (`t_INT` and `t_FRAC` entries only).

`int ZV_isscalar(GEN x)` as `RgV_isscalar`, assuming  $x$  is a ZV (`t_INT` entries only).

`int RgM_isscalar(GEN x, GEN s)` return 1 if  $x$  is the scalar matrix equal to  $s$  times the identity, and 0 otherwise. If  $s$  is NULL, test whether  $x$  is an arbitrary scalar matrix.

`int RgM_isidentity(GEN x)` return 1 if the `t_MAT`  $x$  is the identity matrix, and 0 otherwise.

`int RgM_isdiagonal(GEN x)` return 1 if the `t_MAT`  $x$  is a diagonal matrix, and 0 otherwise.

`long RgC_is_ei(GEN x)` return  $i$  if the `t_COL`  $x$  has 0 entries, but for a 1 at position  $i$ .

`int RgM_is_ZM(GEN x)` return 1 if the `t_MAT`  $x$  has only `t_INT` coefficients, and 0 otherwise.

`long qfiseven(GEN M)` return 1 if the square symmetric `typZM`  $x$  is an even quadratic form (all diagonal coefficients are even), and 0 otherwise.

`int RgM_is_QM(GEN x)` return 1 if the `t_MAT`  $x$  has only `t_INT` or `t_FRAC` coefficients, and 0 otherwise.

`long RgV_isin(GEN v, GEN x)` return the first index  $i$  such that  $v[i] = x$  if it exists, and 0 otherwise. Naive search in linear time, does not assume that  $v$  is sorted.

`long RgV_isin_i(GEN v, GEN x, long n)` return the first index  $i$  *leqn* such that  $v[i] = x$  if it exists, and 0 otherwise. Naive search in linear time, does not assume that  $v$  is sorted. Assume that  $n < \lg(v)$ .

`GEN RgM_diagonal(GEN m)` returns the diagonal of  $m$  as a `t_VEC`.

`GEN RgM_diagonal_shallow(GEN m)` shallow version of `RgM_diagonal`

### 7.5.7.3 Conversion to floating point entries.

`GEN RgC_gtofp(GEN x, GEN prec)` returns the `t_COL` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of  $x$ .

`GEN RgV_gtofp(GEN x, GEN prec)` returns the `t_VEC` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of  $x$ .

`GEN RgC_gtomp(GEN x, long prec)` returns the `t_COL` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of  $x$ .

`GEN RgC_fpnorml2(GEN x, long prec)` returns (a stack-clean variant of)

`gnorml2( RgC_gtofp(x, prec) )`

`GEN RgM_gtofp(GEN x, GEN prec)` returns the `t_MAT` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of  $x$ .

`GEN RgM_gtomp(GEN x, long prec)` returns the `t_MAT` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of  $x$ .

GEN RgM\_fpnorml2(GEN x, long prec) returns (a stack-clean variant of)

```
gnorml2( RgM_gtofp(x, prec) )
```

#### 7.5.7.4 Linear algebra, linear systems.

GEN RgM\_inv(GEN a) returns a left inverse of  $a$  (which needs not be square), or NULL if this turns out to be impossible. The latter happens when the matrix does not have maximal rank (or when rounding errors make it appear so).

GEN RgM\_inv\_upper(GEN a) as RgM\_inv, assuming that  $a$  is a nonempty invertible upper triangular matrix, hence a little faster.

GEN RgM\_RgC\_invimage(GEN A, GEN B) returns a  $\mathbf{t\_COL}$   $X$  such that  $AX = B$  if one such exists, and NULL otherwise.

GEN RgM\_invimage(GEN A, GEN B) returns a  $\mathbf{t\_MAT}$   $X$  such that  $AX = B$  if one such exists, and NULL otherwise.

GEN RgM\_Hadamard(GEN a) returns a upper bound for the absolute value of  $\det(a)$ . The bound is a  $\mathbf{t\_INT}$ .

GEN RgM\_solve(GEN a, GEN b) returns  $a^{-1}b$  where  $a$  is a square  $\mathbf{t\_MAT}$  and  $b$  is a  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$ . Returns NULL if  $a^{-1}$  cannot be computed, see RgM\_inv.

If  $b = \text{NULL}$ , the matrix  $a$  need no longer be square, and we strive to return a left inverse for  $a$  (NULL if it does not exist).

GEN RgM\_solve\_realimag(GEN M, GEN b)  $M$  being a  $\mathbf{t\_MAT}$  with  $r_1 + r_2$  rows and  $r_1 + 2r_2$  columns,  $y$  a  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$  such that the equation  $Mx = y$  makes sense, returns  $x$  under the following simplifying assumptions: the first  $r_1$  rows of  $M$  and  $y$  are real (the  $r_2$  others are complex), and  $x$  is real. This is stabler and faster than calling RgM\_solve( $M, b$ ) over  $\mathbf{C}$ . In most applications,  $M$  approximates the complex embeddings of an integer basis in a number field, and  $x$  is actually rational.

GEN split\_realimag(GEN x, long r1, long r2)  $x$  is a  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$  with  $r_1 + r_2$  rows, whose first  $r_1$  rows have real entries (the  $r_2$  others are complex). Return an object of the same type as  $x$  and  $r_1 + 2r_2$  rows, such that the first  $r_1 + r_2$  rows contain the real part of  $x$ , and the  $r_2$  following ones contain the imaginary part of the last  $r_2$  rows of  $x$ . Called by RgM\_solve\_realimag.

GEN RgM\_det\_triangular(GEN x) returns the product of the diagonal entries of  $x$  (its determinant if it is indeed triangular).

GEN Frobeniusform(GEN V, long n) given the vector  $V$  of elementary divisors for  $M - x\text{Id}$ , where  $M$  is an  $n \times n$  square matrix. Returns the Frobenius form of  $M$ .

int RgM\_QR\_init(GEN x, GEN \*pB, GEN \*pQ, GEN \*pL, long prec) QR-decomposition of a square invertible  $\mathbf{t\_MAT}$   $x$  with real coefficients. Sets  $*pB$  to the vector of squared lengths of the  $x[i]$ ,  $*pL$  to the Gram-Schmidt coefficients and  $*pQ$  to a vector of successive Householder transforms. If  $R$  denotes the transpose of  $L$  and  $Q$  is the result of applying  $*pQ$  to the identity matrix, then  $x = QR$  is the QR decomposition of  $x$ . Returns 0 if  $x$  is not invertible or we hit a precision problem, and 1 otherwise.

int QR\_init(GEN x, GEN \*pB, GEN \*pQ, GEN \*pL, long prec) as RgM\_QR\_init, assuming further that  $x$  has  $\mathbf{t\_INT}$  or  $\mathbf{t\_REAL}$  coefficients.

GEN R\_from\_QR(GEN x, long prec) assuming that  $x$  is a square invertible  $\mathbf{t\_MAT}$  with  $\mathbf{t\_INT}$  or  $\mathbf{t\_REAL}$  coefficients, return the upper triangular  $R$  from the  $QR$  decomposition of  $x$ . Not memory clean. If the matrix is not known to have  $\mathbf{t\_INT}$  or  $\mathbf{t\_REAL}$  coefficients, apply RgM\_gtomp first.

GEN gaussred\_from\_QR(GEN x, long prec) assuming that  $x$  is a square invertible  $\mathbf{t\_MAT}$  with  $\mathbf{t\_INT}$  or  $\mathbf{t\_REAL}$  coefficients, returns `qfgaussred(x~* x)`; this is essentially the upper triangular  $R$  matrix from the  $QR$  decomposition of  $x$ , renormalized to accomodate `qfgaussred` conventions. Not memory clean.

GEN RgM\_gram\_schmidt(GEN e, GEN \*ptB) naive (unstable) Gram-Schmidt orthogonalization of the basis  $(e_i)$  given by the columns of  $\mathbf{t\_MAT}$   $e$ . Return the  $e_i^*$  (as columns of a  $\mathbf{t\_MAT}$ ) and set `*ptB` to the vector of squared lengths  $|e_i^*|^2$ .

GEN RgM\_Babai(GEN M, GEN y) given a  $\mathbf{t\_MAT}$   $M$  of maximal rank  $n$  and a  $\mathbf{t\_COL}$   $y$  of the same dimension, apply Babai's nearest plane algorithm to return an *integral*  $x$  such that  $y - Mx$  has small  $L_2$  norm. This yields an approximate solution to the closest vector problem: if  $M$  is LLL-reduced, then

$$\|y - Mx\|_2 \leq 2(2/\sqrt{3})^n \|y - MX\|_2$$

for all  $X \in \mathbf{Z}^n$ .

### 7.5.8 ZG.

Let  $G$  be a multiplicative group with neutral element  $1_G$  whose multiplication is supported by `gmul` and where equality test is performed using `gidentical`, e.g. a matrix group. The following routines implement basic computations in the group algebra  $\mathbf{Z}[G]$ . All of them are shallow for efficiency reasons. A ZG is either

- a  $\mathbf{t\_INT}$   $n$ , representing  $n[1_G]$
- or a “factorization matrix” with two columns  $[g, e]$ : the first one contains group elements, sorted according to `cmp_universal`, and the second one contains integer “exponents”, representing  $\sum e_i [g_i]$ .

Note that `to_famat` and `to_famat_shallow(g, e)` allow to build the ZG  $e[g]$  from  $e \in \mathbf{Z}$  and  $g \in G$ .

GEN ZG\_normalize(GEN x) given a  $\mathbf{t\_INT}$   $x$  or a factorization matrix *without* assuming that the first column is properly sorted. Return a valid (sorted) ZG. Shallow function.

GEN ZG\_add(GEN x, GEN y) return  $x + y$ ; shallow function.

GEN ZG\_neg(GEN x) return  $-x$ ; shallow function.

GEN ZG\_sub(GEN x, GEN y) return  $x - y$ ; shallow function.

GEN ZG\_mul(GEN x, GEN y) return  $xy$ ; shallow function.

GEN ZG\_G\_mul(GEN x, GEN y) given a ZG  $x$  and  $y \in G$ , return  $xy$ ; shallow function.

GEN G\_ZG\_mul(GEN x, GEN y) given a ZG  $y$  and  $x \in G$ , return  $xy$ ; shallow function.

GEN ZG\_Z\_mul(GEN x, GEN n) given a ZG  $x$  and  $y \in \mathbf{Z}$ , return  $xy$ ; shallow function.

GEN ZGC\_G\_mul(GEN v, GEN x) given  $v$  a vector of ZG and  $x \in G$  return the vector (with the same type as  $v$  with entries  $v[i] \cdot x$ ). Shallow function.

void ZGC\_G\_mul\_inplace(GEN v, GEN x) as `ZGC_G_mul`, modifying  $v$  in place.

GEN ZGC\_Z\_mul(GEN v, GEN n) given  $v$  a vector of ZG and  $n \in Z$  return the vector (with the same type as  $v$  with entries  $n \cdot v[i]$ ). Shallow function.

GEN G\_ZGC\_mul(GEN x, GEN v) given  $v$  a vector of ZG and  $x \in G$  return the vector of  $x \cdot v[i]$ . Shallow function.

GEN ZGCs\_add(GEN x, GEN y) add two sparse vectors of ZG elements (see Sparse linear algebra below).

### 7.5.9 Sparse and blackbox linear algebra.

A sparse column zCs  $v$  is a t\_COL with two components  $C$  and  $E$  which are t\_VECSMALL of the same length, representing  $\sum_i E[i] * e_{C[i]}$ , where  $(e_j)$  is the canonical basis. A sparse matrix (zMs) is a t\_VEC of zCs.

FpCs and FpMs are identical to the above, but  $E[i]$  is now interpreted as a *signed* C long integer representing an element of  $\mathbf{F}_p$ . This is important since  $p$  can be so large that  $p + E[i]$  would not fit in a C long.

RgCs and RgMs are similar, except that the type of the components of  $E$  is now unspecified. Functions handling those later objects must not depend on the type of those components.

F2Ms are t\_VEC of F2Cs. F2Cs are t\_VECSMALL whoses entries are the nonzero coefficients (1).

It is not possible to derive the space dimension (number of rows) from the above data. Thus most functions take an argument nbrow which is the number of rows of the corresponding column/matrix in dense representation.

GEN F2Ms\_to\_F2m(GEN M, long nbrow) convert a F2m to a F2Ms.

GEN F2m\_to\_F2Ms(GEN M) convert a F2m to a F2Ms.

GEN zCs\_to\_ZC(GEN C, long nbrow) convert the sparse vector  $C$  to a dense ZC of dimension nbrow.

GEN zMs\_to\_ZM(GEN M, long nbrow) convert the sparse matrix  $M$  to a dense ZM whose columns have dimension nbrow.

GEN FpMs\_FpC\_mul(GEN M, GEN B, GEN p) multiply the sparse matrix  $M$  (over  $\mathbf{F}_p$ ) by the FpC  $B$ . The result is an FpC, i.e. a dense vector.

GEN zMs\_ZC\_mul(GEN M, GEN B, GEN p) multiply the sparse matrix  $M$  by the ZC  $B$  (over  $\mathbf{Z}$ ). The result is an ZC, i.e. a dense vector.

GEN FpV\_FpMs\_mul(GEN B, GEN M, GEN p) multiply the FpV  $B$  by the sparse matrix  $M$  (over  $\mathbf{F}_p$ ). The result is an FpV, i.e. a dense vector.

GEN ZV\_zMs\_mul(GEN B, GEN M, GEN p) multiply the FpV  $B$  (over  $\mathbf{Z}$ ) by the sparse matrix  $M$ . The result is an ZV, i.e. a dense vector.

void RgMs\_structelim(GEN M, long nbrow, GEN A, GEN \*p\_col, GEN \*p\_row)  $M$  being a RgMs with nbrow rows,  $A$  being a list of row indices, perform structured elimination on  $M$  by removing some rows and columns until the number of effectively present rows is equal to the number of columns. The result is stored in two t\_VECSMALLs, \*p\_col and \*p\_row: \*p\_col is a map from the new columns indices to the old one. \*p\_row is a map from the old rows indices to the new one (0 if removed).



`GEN F2Ms_colelim(GEN M, long nbrow)` returns some subset of the columns of  $M$  as a `t_VECSMALL` of indices, selected such that the dimension of the kernel of the matrix is preserved. The subset is not guaranteed to be minimal.

`GEN F2Ms_ker(GEN M, long nbrow)` returns some kernel vectors of  $M$  using block Lanczos algorithm.

`GEN FpMs_leftkernel_elt(GEN M, long nbrow, GEN p)`  $M$  being a sparse matrix over  $\mathbf{F}_p$ , return a nonzero `FpV`  $X$  such that  $XM$  components are almost all 0.

`GEN FpMs_FpCs_solve(GEN M, GEN B, long nbrow, GEN p)` solve the equation  $MX = B$ , where  $M$  is a sparse matrix and  $B$  is a sparse vector, both over  $\mathbf{F}_p$ . Return either a solution as a `t_COL` (dense vector), the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component, or `NULL` (can happen if  $B$  is not in the image of  $M$ ).

`GEN FpMs_FpCs_solve_safe(GEN M, GEN B, long nbrow, GEN p)` as above, but in the event that  $p$  is not a prime and an impossible division occurs, return `NULL`.

`GEN ZpMs_ZpCs_solve(GEN M, GEN B, long nbrow, GEN p, long e)` solve the equation  $MX = B$ , where  $M$  is a sparse matrix and  $B$  is a sparse vector, both over  $\mathbf{Z}/p^e\mathbf{Z}$ . Return either a solution as a `t_COL` (dense vector), or the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component.

`GEN gen_FpM_Wiedemann(void *E, GEN (*f)(void*, GEN), GEN B, GEN p)` solve the equation  $f(X) = B$  over  $\mathbf{F}_p$ , where  $B$  is a `FpV`, and  $f$  is a blackbox endomorphism, where  $f(E, X)$  computes the value of  $f$  at the (dense) column vector  $X$ . Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`.

`GEN gen_ZpM_Dixon_Wiedemann(void *E, GEN (*f)(void*, GEN), GEN B, GEN p, long e)` solve equation  $f(X) = B$  over  $\mathbf{Z}/p^e\mathbf{Z}$ , where  $B$  is a `ZV`, and  $f$  is a blackbox endomorphism, where  $f(E, X)$  computes the value of  $f$  at the (dense) column vector  $X$ . Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`.

### 7.5.10 Obsolete functions.

The functions in this section are kept for backward compatibility only and will eventually disappear.

`GEN image2(GEN x)` compute the image of  $x$  using a very slow algorithm. Use `image` instead.

## 7.6 Integral, rational and generic polynomial arithmetic.

### 7.6.1 ZX.

`void RgX_check_ZX(GEN x, const char *s)` Assuming  $x$  is a `t_POL` raise an error if it is not a `ZX` ( $s$  should point to the name of the caller).

`GEN ZX_copy(GEN x, GEN p)` returns a copy of  $x$ .

`long ZX_max_lg(GEN x)` returns the effective length of the longest component in  $x$ .

`GEN scalar_ZX(GEN x, long v)` returns the constant `ZX` in variable  $v$  equal to the `t_INT`  $x$ .

`GEN scalar_ZX_shallow(GEN x, long v)` returns the constant `ZX` in variable  $v$  equal to the `t_INT`  $x$ . Shallow function not suitable for `gerepile` and friends.

GEN ZX\_renormalize(GEN x, long l), as `normalizepol`, where  $l = \lg(x)$ , in place.

int ZX\_equal(GEN x, GEN y) returns 1 if the two ZX have the same `degpol` and their coefficients are equal. Variable numbers are not checked.

int ZX\_equal1(GEN x) returns 1 if the ZX  $x$  is equal to 1 and 0 otherwise.

int ZX\_is\_monic(GEN x) returns 1 if the ZX  $x$  is monic and 0 otherwise. The zero polynomial considered not monic.

GEN ZX\_add(GEN x, GEN y) adds  $x$  and  $y$ .

GEN ZX\_sub(GEN x, GEN y) subtracts  $x$  and  $y$ .

GEN ZX\_neg(GEN x) returns  $-x$ .

GEN ZX\_Z\_add(GEN x, GEN y) adds the integer  $y$  to the ZX  $x$ .

GEN ZX\_Z\_add\_shallow(GEN x, GEN y) shallow version of `ZX_Z_add`.

GEN ZX\_Z\_sub(GEN x, GEN y) subtracts the integer  $y$  to the ZX  $x$ .

GEN Z\_ZX\_sub(GEN x, GEN y) subtracts the ZX  $y$  to the integer  $x$ .

GEN ZX\_Z\_mul(GEN x, GEN y) multiplies the ZX  $x$  by the integer  $y$ .

GEN ZX\_mulu(GEN x, ulong y) multiplies  $x$  by the integer  $y$ .

GEN ZX\_shifti(GEN x, long n) shifts all coefficients of  $x$  by  $n$  bits, which can be negative.

GEN ZX\_Z\_divexact(GEN x, GEN y) returns  $x/y$  assuming all divisions are exact.

GEN ZX\_divuexact(GEN x, ulong y) returns  $x/y$  assuming all divisions are exact.

GEN ZX\_remi2n(GEN x, long n) reduces all coefficients of  $x$  to  $n$  bits, using `remi2n`.

GEN ZX\_mul(GEN x, GEN y) multiplies  $x$  and  $y$ .

GEN ZX\_sqr(GEN x, GEN p) returns  $x^2$ .

GEN ZX\_mulspec(GEN a, GEN b, long na, long nb). Internal routine:  $a$  and  $b$  are arrays of coefficients representing polynomials  $\sum_{i=0}^{na-1} a[i]X^i$  and  $\sum_{i=0}^{nb-1} b[i]X^i$ . Returns their product (as a true GEN) in variable 0.

GEN ZX\_sqrspec(GEN a, long na). Internal routine:  $a$  is an array of coefficients representing polynomial  $\sum_{i=0}^{na-1} a[i]X^i$ . Return its square (as a true GEN) in variable 0.

GEN ZX\_rem(GEN x, GEN y) returns the remainder of the Euclidean division of  $x \bmod y$ . Assume that  $x, y$  are two ZX and that  $y$  is monic.

GEN ZX\_mod\_Xnm1(GEN T, ulong n) return  $T$  modulo  $X^n - 1$ . Shallow function.

GEN ZX\_div\_by\_X\_1(GEN T, GEN \*r) return the quotient of  $T$  by  $X - 1$ . If  $r$  is not NULL set it to  $T(1)$ .

GEN ZX\_digits(GEN x, GEN B) returns a vector of ZX  $[c_0, \dots, c_n]$  of degree less than the degree of  $B$  and such that  $x = \sum_{i=0}^n c_i B^i$ . Assume that  $B$  is monic.

GEN ZXV\_ZX\_fromdigits(GEN v, GEN B) where  $v = [c_0, \dots, c_n]$  is a vector of ZX, returns  $\sum_{i=0}^n c_i B^i$ .

GEN ZX\_gcd(GEN x, GEN y) returns a gcd of the ZX  $x$  and  $y$ . Not memory-clean, but suitable for `gerepileupto`.

GEN ZX\_gcd\_all(GEN x, GEN y, GEN \*pX) returns a gcd  $d$  of  $x$  and  $y$ . If  $pX$  is not NULL, set  $*pX$  to a (nonzero) integer multiple of  $x/d$ . If  $x$  and  $y$  are both monic, then  $d$  is monic and  $*pX$  is exactly  $x/d$ . Not memory clean.

GEN ZX\_radical(GEN x) returns the largest squarefree divisor of the ZX  $x$ . Not memory clean.

GEN ZX\_content(GEN x) returns the content of the ZX  $x$ .

long ZX\_val(GEN P) as RgX\_val, but assumes P has  $t\_INT$  coefficients.

long ZX\_valrem(GEN P, GEN \*z) as RgX\_valrem, but assumes P has  $t\_INT$  coefficients.

GEN ZX\_to\_monic(GEN q GEN \*L) given  $q$  a nonzero ZX, returns a monic integral polynomial  $Q$  such that  $Q(x) = Cq(x/L)$ , for some rational  $C$  and positive integer  $L > 0$ . If  $L$  is not NULL, set  $*L$  to  $L$ ; if  $L = 1$ ,  $*L$  is set to `gen_1`. Shallow function.

GEN ZX\_primitive\_to\_monic(GEN q, GEN \*L) as ZX\_to\_monic except  $q$  is assumed to have trivial content, which avoids recomputing it. The result is suboptimal if  $q$  is not primitive ( $L$  larger than necessary), but remains correct. Shallow function.

GEN ZX\_Z\_normalize(GEN q, GEN \*L) a restricted version of ZX\_primitive\_to\_monic, where  $q$  is a *monic* ZX of degree  $> 0$ . Finds the largest integer  $L > 0$  such that  $Q(X) := L^{-\deg q} q(Lx)$  is integral and return  $Q$ ; this is not well-defined if  $q$  is a monomial, in that case, set  $L = 1$  and  $Q = q$ . If  $L$  is not NULL, set  $*L$  to  $L$ . Shallow function.

GEN ZX\_Q\_normalize(GEN q, GEN \*L) a variant of ZX\_Z\_normalize where  $L > 0$  is allowed to be rational, the monic  $Q \in \mathbf{Z}[X]$  has possibly smaller coefficients. Shallow function.

GEN ZX\_Q\_mul(GEN x, GEN y) returns  $x * y$ , where  $y$  is a rational number and the resulting  $t\_POL$  has rational entries.

long ZX\_deflate\_order(GEN P) given a nonconstant ZX  $P$ , returns the largest exponent  $d$  such that  $P$  is of the form  $P(x^d)$ .

long ZX\_deflate\_max(GEN P, long \*d). Given a nonconstant polynomial with integer coefficients  $P$ , sets  $d$  to ZX\_deflate\_order( $P$ ) and returns RgX\_deflate( $P, d$ ). Shallow function.

GEN ZX\_rescale(GEN P, GEN h) returns  $h^{\deg(P)} P(x/h)$ .  $P$  is a ZX and  $h$  is a nonzero integer. Neither memory-clean nor suitable for `gerepileupto`.

GEN ZX\_rescale2n(GEN P, long n) returns  $2^{n \deg(P)} P(x \gg n)$  where  $P$  is a ZX.

GEN ZX\_rescale\_1t(GEN P) returns the monic integral polynomial  $h^{\deg(P)-1} P(x/h)$ , where  $P$  is a nonzero ZX and  $h$  is its leading coefficient. Neither memory-clean nor suitable for `gerepileupto`.

GEN ZX\_translate(GEN P, GEN c) assume  $P$  is a ZX and  $c$  an integer. Returns  $P(X+c)$  (optimized for  $c = \pm 1$ ).

GEN ZX\_affine(GEN P, GEN a, GEN b)  $P$  is a ZX,  $a$  and  $b$  are  $t\_INT$ . Return  $P(aX+b)$  (optimized for  $b = \pm 1$ ). Not memory clean.

GEN ZX\_Z\_eval(GEN P, GEN x) evaluate the ZX  $P$  at the integer  $x$ .

GEN ZX\_unscale(GEN P, GEN h) given a ZX  $P$  and a  $t\_INT$   $h$ , returns  $P(hx)$ . Not memory clean.

GEN ZX\_z\_unscale(GEN P, long h) given a ZX  $P$ , returns  $P(hx)$ . Not memory clean.

GEN ZX\_unscale2n(GEN P, long n) given a ZX  $P$ , returns  $P(x \ll n)$ . Not memory clean.

`GEN ZX_unscale_div(GEN P, GEN h)` given a `ZX P` and a `t_INT h` such that  $h \mid P(0)$ , returns  $P(hx)/h$ . Not memory clean.

`GEN ZX_unscale_divpow(GEN P, GEN h, long k)` given a `ZX P`, a `t_INT h` and  $k > 0$ , returns  $P(hx)/h^k$  assuming the result has integral coefficients. Not memory clean.

`GEN ZX_eval1(GEN P)` returns the integer  $P(1)$ .

`GEN ZX_graeffe(GEN p)` returns the Graeffe transform of  $p$ , i.e. the `ZX q` such that  $p(x)p(-x) = q(x^2)$ .

`GEN ZX_deriv(GEN x)` returns the derivative of  $x$ .

`GEN ZX_resultant(GEN A, GEN B)` returns the resultant of the `ZX A` and `B`.

`GEN ZX_disc(GEN T)` returns the discriminant of the `ZX T`.

`GEN ZX_factor(GEN T)` returns the factorization of the primitive part of  $T$  over  $\mathbf{Q}[X]$  (the content is lost).

`int ZX_is_squarefree(GEN T)` returns 1 if the `ZX T` is squarefree, 0 otherwise.

`long ZX_is_irred(GEN T)` returns 1 if  $T$  is irreducible, and 0 otherwise.

`GEN ZX_squff(GEN T, GEN *E)` write  $T(x)$  as a product  $\prod T_i^{e_i}$  with the  $e_1 < e_2 < \dots$  all distinct and the  $T_i$  pairwise coprime. Return the vector of the  $T_i$ , and set `*E` to the vector of the  $e_i$ , as a `t_VECSMALL`. For efficiency, powers of  $x$  should have been removed from  $T$  using `ZX_valrem`, but the result is also correct if not. Not memory clean.

`GEN ZX_Uspensky(GEN P, GEN ab, long flag, long bitprec)` let  $P$  be a `ZX` polynomial whose real roots are simple and `bitprec` is the relative precision in bits. For efficiency reasons,  $P$  should not only have simple real roots but actually be primitive and squarefree, but the routine neither checks nor enforces this, and it returns correct results in this case as well.

- If `flag` is 0 returns a list of intervals that isolate the real roots of  $P$ . The return value is a column of elements which are either vectors `[a,b]` of rational numbers meaning that there is a single nonrational root in the open interval  $(a,b)$  or elements `x0` such that `x0` is a rational root of  $P$ . Beware that the limits of the open intervals can be roots of the polynomial.

- If `flag` is 1 returns an approximation of the real roots of  $P$ .

- If `flag` is 2 returns the number of roots.

The argument `ab` specify the interval in which the roots are searched. The default interval is  $(-\infty, \infty)$ . If `ab` is an integer or fraction  $a$  then the interval is  $[a, \infty)$ . If `ab` is a vector `[a,b]`, where `t_INT`, `t_FRAC` or `t_INFINITY` are allowed for  $a$  and  $b$ , the interval is  $[a,b]$ .

`long ZX_sturm(GEN P)` number of real roots of the nonconstant squarefree `ZX P`. For efficiency, it is advised to make  $P$  primitive first.

`long ZX_sturmpart(GEN P, GEN ab)` number of real roots of the nonconstant squarefree `ZX P` in the interval specified by `ab`: either `NULL` (no restriction) or a `t_VEC [a,b]` with two real components (of type `t_INT`, `t_FRAC` or `t_INFINITY`). For efficiency, it is advised to make  $P$  primitive first.

`long ZX_sturm_irred(GEN P)` number of real roots of the `ZX P`, assumed irreducible over  $\mathbf{Q}[X]$ . For efficiency, it is advised to make  $P$  primitive first.

`long ZX_realroots_irred(GEN P, long prec)` real roots of the `ZX P`, assumed irreducible over  $\mathbf{Q}[X]$  to precision `prec`. For efficiency, it is advised to make  $P$  primitive first.

### 7.6.2 Resultants.

`GEN ZX_ZXY_resultant(GEN A, GEN B)` under the assumption that  $A$  in  $\mathbf{Z}[Y]$ ,  $B$  in  $\mathbf{Q}[Y][X]$ , and  $R = \text{Res}_Y(A, B) \in \mathbf{Z}[X]$ , returns the resultant  $R$ .

`GEN ZX_compositum_disjoint(GEN A, GEN B)` given two irreducible  $\mathbf{ZX}$  defining linearly disjoint extensions, returns a  $\mathbf{ZX}$  defining their compositum.

`GEN ZX_compositum(GEN A, GEN B, long *lambda)` given two irreducible  $\mathbf{ZX}$ , returns an irreducible  $\mathbf{ZX}$   $C$  defining their compositum and set `lambda` to a small integer  $k$  such that if  $\alpha$  is a root of  $A$  and  $\beta$  is a root of  $B$ , then  $k\alpha + \beta$  is a root of  $C$ .

`GEN ZX_ZXY_rnfequation(GEN A, GEN B, long *lambda)`, assume  $A$  in  $\mathbf{Z}[Y]$ ,  $B$  in  $\mathbf{Q}[Y][X]$ , and  $R = \text{Res}_Y(A, B) \in \mathbf{Z}[X]$ . If `lambda` = `NULL`, returns  $R$  as in `ZY.ZXY_resultant`. Otherwise, `lambda` must point to some integer, e.g. 0 which is used as a seed. The function then finds a small  $\lambda \in \mathbf{Z}$  (starting from `*lambda`) such that  $R_\lambda(X) := \text{Res}_Y(A, B(X + \lambda Y))$  is squarefree, resets `*lambda` to the chosen value and returns  $R_\lambda$ .

### 7.6.3 ZXV.

`GEN ZXV_equal(GEN x, GEN y)` returns 1 if the two vectors of  $\mathbf{ZX}$  are equal, as per `ZX_equal` (variables are not checked to be equal) and 0 otherwise.

`GEN ZXV_Z_mul(GEN x, GEN y)` multiplies the vector of  $\mathbf{ZX}$   $x$  by the integer  $y$ .

`GEN ZXV_remi2n(GEN x, long n)` applies `ZX_remi2n` to all coefficients of  $x$ .

`GEN ZXV_dotproduct(GEN x, GEN y)` as `RgV_dotproduct` assuming  $x$  and  $y$  have  $\mathbf{ZX}$  entries.

### 7.6.4 ZXT.

`GEN ZXT_remi2n(GEN x, long n)` applies `ZX_remi2n` to all leaves of the tree  $x$ .

### 7.6.5 ZXQ.

`GEN ZXQ_mul(GEN x, GEN y, GEN T)` returns  $x * y \bmod T$ , assuming that all inputs are  $\mathbf{ZX}$ s and that  $T$  is monic.

`GEN ZXQ_sqr(GEN x, GEN T)` returns  $x^2 \bmod T$ , assuming that all inputs are  $\mathbf{ZX}$ s and that  $T$  is monic.

`GEN ZXQ_powu(GEN x, ulong n, GEN T)` returns  $x^n \bmod T$ , assuming that all inputs are  $\mathbf{ZX}$ s and that  $T$  is monic.

`GEN ZXQ_powers(GEN x, long n, GEN T)` returns  $[x^0, \dots, x^n] \bmod T$  as a `t_VEC`, assuming that all inputs are  $\mathbf{ZX}$ s and that  $T$  is monic.

`GEN ZXQ_charpoly(GEN A, GEN T, long v)`: let  $T$  and  $A$  be  $\mathbf{ZX}$ s, returns the characteristic polynomial of  $\text{Mod}(A, T)$ . More generally,  $A$  is allowed to be a  $\mathbf{QX}$ , hence possibly has rational coefficients, *assuming* the result is a  $\mathbf{ZX}$ , i.e. the algebraic number  $\text{Mod}(A, T)$  is integral over  $\mathbf{Z}$ .

`GEN ZXQ_minpoly(GEN A, GEN B, long d, ulong bound)` let  $T$  and  $A$  be  $\mathbf{ZX}$ s, returns the minimal polynomial of  $\text{Mod}(A, T)$  assuming it has degree  $d$  and its coefficients are less than  $2^{\text{bound}}$ . More generally,  $A$  is allowed to be a  $\mathbf{QX}$ , hence possibly has rational coefficients, *assuming* the result is a  $\mathbf{ZX}$ , i.e. the algebraic number  $\text{Mod}(A, T)$  is integral over  $\mathbf{Z}$ .

### 7.6.6 ZXn.

GEN ZXn\_mul(GEN x, GEN y, long n) return  $xy \pmod{X^n}$ .

GEN ZXn\_sqr(GEN x, long n) return  $x^2 \pmod{X^n}$ .

GEN eta\_ZXn(long r, long n) return  $\eta(X^r) = \prod_{i>0} (1 - X^{ri}) \pmod{X^n}$ ,  $r > 0$ .

GEN eta\_product\_ZXn(GEN DR, long n): DR =  $[D, R]$  being a vector with two t\_VECSMALL components, return  $\prod_i \eta(X^{d_i})^{r_i}$ . Shallow function.

### 7.6.7 ZXQM.

ZXQM are matrices of ZXQ. All entries must be integers or polynomials of degree strictly less than the degree of  $T$ .

GEN ZXQM\_mul(GEN x, GEN y, GEN T) returns  $x * y \pmod{T}$ , assuming that all inputs are ZXs and that  $T$  is monic.

GEN ZXQM\_sqr(GEN x, GEN T) returns  $x^2 \pmod{T}$ , assuming that all inputs are ZXs and that  $T$  is monic.

### 7.6.8 ZXQX.

GEN ZXQX\_mul(GEN x, GEN y, GEN T) returns  $x * y$ , assuming that all inputs are ZXQXs and that  $T$  is monic.

GEN ZXQX\_ZXQ\_mul(GEN x, GEN y, GEN T) returns  $x * y$ , assuming that  $x$  is a ZXQX,  $y$  is a ZXQ and  $T$  is monic.

GEN ZXQX\_sqr(GEN x, GEN T) returns  $x^2$ , assuming that all inputs are ZXQXs and that  $T$  is monic.

GEN ZXQX\_gcd(GEN x, GEN y, GEN T) returns the gcd of  $x$  and  $y$ , assuming that all inputs are ZXQXs and that  $T$  is monic.

### 7.6.9 ZXX.

void RgX\_check\_ZXX(GEN x, const char \*s) Assuming  $x$  is a t\_POL raise an error if it one of its coefficients is not an integer or a ZX ( $s$  should point to the name of the caller).

GEN ZXX\_renormalize(GEN x, long l), as normalizopol, where  $l = \lg(x)$ , in place.

long ZXX\_max\_lg(GEN x) returns the effective length of the longest component in  $x$ ; assume all coefficients are t\_INT or ZXs.

GEN ZXX\_evalx0(GEN P) returns  $P(X, 0)$ .

GEN ZXX\_Z\_mul(GEN x, GEN y) returns  $xy$ .

GEN ZXX\_Q\_mul(GEN x, GEN y) returns  $x * y$ , where  $y$  is a rational number and the resulting t\_POL has rational entries.

GEN ZXX\_Z\_add\_shallow(GEN x, GEN y) returns  $x + y$ . Shallow function.

GEN ZXX\_Z\_divexact(GEN x, GEN y) returns  $x/y$  assuming all integer divisions are exact.

GEN Kronecker\_to\_ZXX(GEN z, long n, long v) recover  $P(X, Y)$  from its Kronecker form  $P(X, X^{2^n-1})$  (see RgXX\_to\_Kronecker),  $v$  is the variable number corresponding to  $Y$ . Shallow function.

**GEN Kronecker\_to\_ZXQX**(GEN *z*, GEN *T*). Let  $n = \deg T$  and let  $P(X, Y) \in \mathbf{Z}[X, Y]$  lift a polynomial in  $K[Y]$ , where  $K := \mathbf{Z}[X]/(T)$  and  $\deg_X P < 2n - 1$  — such as would result from multiplying minimal degree lifts of two polynomials in  $K[Y]$ . Let  $z = P(t, t^{2n-1})$  be a Kronecker form of  $P$  (see **RgXX\_to\_Kronecker**), this function returns  $Q \in \mathbf{Z}[X, t]$  such that  $Q$  is congruent to  $P(X, t) \pmod{(T(X))}$ ,  $\deg_X Q < n$ . Not stack-clean. Note that  $t$  need not be the same variable as  $Y$ !

**GEN ZXX\_mul\_Kronecker**(GEN *P*, GEN *Q*, long *n*) return **ZX\_mul** applied to the Kronecker forms  $P(X, X^{2n-1})$  and  $Q(X, X^{2n-1})$  of  $P$  and  $Q$ . Not memory clean.

**GEN ZXX\_sqr\_Kronecker**(GEN *P*, long *n*) return **ZX\_sqr** applied to the Kronecker forms  $P(X, X^{2n-1})$  of  $P$ . Not memory clean.

### 7.6.10 QX.

**void RgX\_check\_QX**(GEN *x*, const char \**s*) Assuming *x* is a **t\_POL** raise an error if it is not a QX (*s* should point to the name of the caller).

**GEN QX\_mul**(GEN *x*, GEN *y*)

**GEN QX\_sqr**(GEN *x*)

**GEN QX\_ZX\_rem**(GEN *x*, GEN *y*) *y* is assumed to be monic.

**GEN QX\_gcd**(GEN *x*, GEN *y*) returns a gcd of the QX *x* and *y*.

**GEN QX\_disc**(GEN *T*) returns the discriminant of the QX *T*.

**GEN QX\_factor**(GEN *T*) as **ZX\_factor**.

**GEN QX\_resultant**(GEN *A*, GEN *B*) returns the resultant of the QX *A* and *B*.

**GEN QX\_complex\_roots**(GEN *p*, long *l*) returns the complex roots of the QX *p* at accuracy *l*, where real roots are returned as **t\_REALs**. More efficient when *p* is irreducible and primitive. Special case of **cleanroots**.

### 7.6.11 QXQ.

**GEN QXQ\_norm**(GEN *A*, GEN *B*) *A* being a QX and *B* being a ZX, returns the norm of the algebraic number  $A \bmod B$ , using a modular algorithm. To ensure that *B* is a ZX, one may replace it by **Q\_primpart**(*B*), which of course does not change the norm.

If *A* is not a ZX — it has a denominator —, but the result is nevertheless known to be an integer, it is much more efficient to call **QXQ\_intnorm** instead.

**GEN QXQ\_intnorm**(GEN *A*, GEN *B*) *A* being a QX and *B* being a ZX, returns the norm of the algebraic number  $A \bmod B$ , *assuming* that the result is an integer, which is for instance the case is  $A \bmod B$  is an algebraic integer, in particular if *A* is a ZX. To ensure that *B* is a ZX, one may replace it by **Q\_primpart**(*B*) (which of course does not change the norm).

If the result is not known to be an integer, you must use **QXQ\_norm** instead, which is slower.

**GEN QXQ\_mul**(GEN *A*, GEN *B*, GEN *T*) returns the product of *A* and *B* modulo *T* where both *A* and *B* are a QX and *T* is a monic ZX.

**GEN QXQ\_sqr**(GEN *A*, GEN *T*) returns the square of *A* modulo *T* where *A* is a QX and *T* is a monic ZX.

**GEN QXQ\_inv**(GEN *A*, GEN *B*) returns the inverse of *A* modulo *B* where *A* is a QX and *B* is a ZX. Should you need this for a QX *B*, just use

`QXQ_inv(A, Q_primpart(B));`

But in all cases where modular arithmetic modulo  $B$  is desired, it is much more efficient to replace  $B$  by `Q_primpart(B)` once and for all.

`GEN QXQ_div(GEN A, GEN B, GEN T)` returns  $A/B$  modulo  $T$  where  $A$  and  $B$  are `QX` and  $T$  is a `ZX`. Use this function when the result is expected to be of the same size as  $B^{-1} \bmod T$  or smaller. Otherwise, it will be faster to use `QXQ_mul(A, QXQ_inv(B, T), T)`.

`GEN QXQ_charpoly(GEN A, GEN T, long v)` where  $A$  is a `QX` and  $T$  is a `ZX`, returns the characteristic polynomial of  $\text{Mod}(A, T)$ . If the result is known to be a `ZX`, then calling `ZXQ_charpoly` will be faster.

`GEN QXQ_powers(GEN x, long n, GEN T)` returns  $[x^0, \dots, x^n]$  as `RgXQ_powers` would, but in a more efficient way when  $x$  has a huge integer denominator (we start by removing that denominator). Assume that  $x$  is a `QX` and  $T$  is a `ZX`. Meant to precompute powers of algebraic integers in  $\mathbf{Q}[t]/(T)$ .

`GEN QXQ_reverse(GEN f, GEN T)` as `RgXQ_reverse`, assuming  $f$  is a `QX`.

`GEN QX_ZXQV_eval(GEN f, GEN nV, GEN dV)` as `RgX_RgXQV_eval`, except that  $f$  is assumed to be a `QX`,  $V$  is given implicitly by a numerator `nV` (`ZV`) and denominator `dV` (a positive `t_INT` or `NULL` for trivial denominator). Not memory clean, but suitable for `gerepileupto`.

`GEN QXV_QXQ_eval(GEN v, GEN a, GEN T)`  $v$  is a vector of `QX`s (possibly scalars, i.e. rational numbers, for convenience),  $a$  and  $T$  both `QX`. Return the vector of evaluations at  $a$  modulo  $T$ . Not memory clean, nor suitable for `gerepileupto`.

`GEN QXY_QXQ_evalx(GEN P, GEN a, GEN T)`  $P(X, Y)$  is a `t_POL` with `QX` coefficients (possibly scalars, i.e. rational numbers, for convenience),  $a$  and  $T$  both `QX`. Return the `QX`  $P(a \bmod T, Y)$ . Not memory clean, nor suitable for `gerepileupto`.

#### 7.6.12 QXQX.

`GEN QXQX_mul(GEN x, GEN y, GEN T)` where  $T$  is a monic `ZX`.

`GEN QXQX_QXQ_mul(GEN x, GEN y, GEN T)` where  $T$  is a monic `ZX`.

`GEN QXQX_sqr(GEN x, GEN T)` where  $T$  is a monic `ZX`

`GEN QXQX_powers(GEN x, long n, GEN T)` where  $T$  is a monic `ZX`

`GEN nfgcd(GEN P, GEN Q, GEN T, GEN den)` given  $P$  and  $Q$  in  $\mathbf{Z}[X, Y]$ ,  $T$  monic irreducible in  $\mathbf{Z}[Y]$ , returns the primitive  $d$  in  $\mathbf{Z}[X, Y]$  which is a gcd of  $P, Q$  in  $K[X]$ , where  $K$  is the number field  $\mathbf{Q}[Y]/(T)$ . If not `NULL`, `den` is a multiple of the integral denominator of the (monic) gcd of  $P, Q$  in  $K[X]$ .

`GEN nfgcd_all(GEN P, GEN Q, GEN T, GEN den, GEN *Pnew)` as `nfgcd`. If `Pnew` is not `NULL`, set `*Pnew` to a nonzero integer multiple of  $P/d$ . If  $P$  and  $Q$  are both monic, then  $d$  is monic and `*Pnew` is exactly  $P/d$ . Not memory clean if the gcd is 1 (in that case `*Pnew` is set to  $P$ ).

`GEN QXQX_gcd(GEN x, GEN y, GEN T)` returns the gcd of  $x$  and  $y$ , assuming that  $x$  and  $y$  are `QXQX`s and that  $T$  is a monic `ZX`.

`GEN QXQX_homogenous_evalpow(GEN P, GEN a, GEN B, GEN T)` Evaluate the homogenous polynomial associated to the univariate polynomial  $P$  on  $(a, b)$  where  $B$  is the vector of powers of  $b$  with exponents 0 to the degree of  $P$  (`QXQ_powers(b, degpol(P), T)`).



### 7.6.13 QXQM.

QXQM are matrices of QXQ. All entries must be `t_INT`, `t_FRAC` or polynomials of degree strictly less than the degree of  $T$ , which must be a monic ZX.

`GEN QXQM_mul(GEN x, GEN y, GEN T)` returns  $x * y \bmod T$ .

`GEN QXQM_sqr(GEN x, GEN T)` returns  $x^2 \bmod T$ .

### 7.6.14 zx.

`GEN zero_zx(long sv)` returns a zero zx in variable  $v$ .

`GEN polx_zx(long sv)` returns the variable  $v$  as degree 1 Flx.

`GEN zx_renormalize(GEN x, long l)`, as `Flx_renormalize`, where  $l = \lg(x)$ , in place.

`GEN zx_shift(GEN T, long n)` return  $T$  multiplied by  $x^n$ , assuming  $n \geq 0$ .

`long zx_lval(GEN f, long p)` return the valuation of  $f$  at  $p$ .

`GEN zx_z_divexact(GEN x, long y)` return  $x/y$  assuming all divisions are exact.

### 7.6.15 RgX.

#### 7.6.15.1 Tests.

`long RgX_degree(GEN x, long v)`  $x$  being a `t_POL` and  $v \geq 0$ , returns the degree in  $v$  of  $x$ . Error if  $x$  is not a polynomial in  $v$ .

`int RgX_isscalar(GEN x)` return 1 if all the coefficients of  $x$  of degree  $> 0$  are 0 (as per `gequal0`).

`int RgX_is_rational(GEN P)` return 1 if the RgX  $P$  has only rational coefficients (`t_INT` and `t_FRAC`), and 0 otherwise.

`int RgX_is_QX(GEN P)` return 1 if the RgX  $P$  has only `t_INT` and `t_FRAC` coefficients, and 0 otherwise.

`int RgX_is_ZX(GEN P)` return 1 if the RgX  $P$  has only `t_INT` coefficients, and 0 otherwise.

`int RgX_is_monomial(GEN x)` returns 1 (true) if  $x$  is a nonzero monomial in its main variable, 0 otherwise.

`long RgX_equal(GEN x, GEN y)` returns 1 if the `t_POLs`  $x$  and  $y$  have the same `degpol` and their coefficients are equal (as per `gequal`). Variable numbers are not checked. Note that this is more stringent than `gequal(x,y)`, which only checks whether  $x - y$  satisfies `gequal0`; in particular, they may have different apparent degrees provided the extra leading terms are 0.

`long RgX_equal_var(GEN x, GEN y)` returns 1 if  $x$  and  $y$  have the same variable number and `RgX_equal(x,y)` is 1.

### 7.6.15.2 Coefficients, blocks.

`GEN RgX_coeff(GEN P, long n)` return the coefficient of  $x^n$  in  $P$ , defined as `gen_0` if  $n < 0$  or  $n > \text{degpol}(P)$ . Shallow function.

`int RgX_blocks(GEN P, long n, long m)` writes  $P(X) = a_0(X) + X^n * a_1(X) * X^n + \dots + X^{n*(m-1)} a_{m-1}(X)$ , where the  $a_i$  are polynomial of degree at most  $n - 1$  (except possibly for the last one) and returns  $[a_0(X), a_1(X), \dots, a_{m-1}(X)]$ . Shallow function.

`void RgX_even_odd(GEN p, GEN *pe, GEN *po)` write  $p(X) = E(X^2) + XO(X^2)$  and set `*pe = E`, `*po = 0`. Shallow function.

`GEN RgX_splitting(GEN P, long k)` write  $P(X) = a_0(X^k) + X a_1(X^k) + \dots + X^{k-1} a_{k-1}(X^k)$  and return  $[a_0(X), a_1(X), \dots, a_{k-1}(X)]$ . Shallow function.

`GEN RgX_copy(GEN x)` returns (a deep copy of)  $x$ .

`GEN RgX_renormalize(GEN x)` remove leading terms in  $x$  which are equal to (necessarily inexact) zeros.

`GEN RgX_renormalize_lg(GEN x, long lx)` as `setlg(x, lx)` followed by `RgX_renormalize(x)`. Assumes that  $lx \leq \lg(x)$ .

`GEN RgX_recip(GEN P)` returns the reverse of the polynomial  $P$ , i.e.  $X^{\deg P} P(1/X)$ .

`GEN RgX_recip_shallow(GEN P)` shallow function of `RgX_recip`.

`GEN RgX_recip_i(GEN P)` shallow function of `RgX_recip`, where we further assume that  $P(0) \neq 0$ , so that the degree of the output is the degree of  $P$ .

`long rfracrecip(GEN *a, GEN *b)` let `*a` and `*b` be such that their quotient  $F$  is a `t_RFRAC` in variable  $X$ . Write  $F(1/X) = X^v A/B$  where  $A$  and  $B$  are coprime to  $X$  and  $v$  in  $\mathbf{Z}$ . Set `*a` to  $A$ , `*b` to  $B$  and return  $v$ .

`GEN RgX_deflate(GEN P, long d)` assuming  $P$  is a polynomial of the form  $Q(X^d)$ , return  $Q$ . Shallow function, not suitable for `gerepileupto`.

`long RgX_deflate_order(GEN P)` given a nonconstant polynomial  $P$ , returns the largest exponent  $d$  such that  $P$  is of the form  $P(x^d)$  (use `gequal0` to check whether coefficients are 0).

`long RgX_deflate_max(GEN P, long *d)` given a nonconstant polynomial  $P$ , sets  $d$  to `RgX_deflate_order(P)` and returns `RgX_deflate(P,d)`. Shallow function.

`long rfrac_deflate_order(GEN F)` as `RgX_deflate_order` where  $F$  is a nonconstant `t_RFRAC`.

`long rfrac_deflate_max(GEN F, long *d)` as `RgX_deflate_max` where  $F$  is a nonconstant `t_RFRAC`.

`GEN rfrac_deflate(GEN F, long m)` as `RgX_deflate` where  $F$  is a `t_RFRAC`.

`GEN RgX_inflate(GEN P, long d)` return  $P(X^d)$ . Shallow function, not suitable for `gerepileupto`.

`GEN RgX_rescale_to_int(GEN x)` given a polynomial  $x$  with real entries (`t_INT`, `t_FRAC` or `t_REAL`), return a `ZX` which is very close to  $Dx$  for some well-chosen integer  $D$ . More precisely, if the input is exact,  $D$  is the denominator of  $x$ ; else it is a power of 2 chosen so that all inexact entries are correctly rounded to 1 ulp.

GEN RgX\_homogenize(GEN P, long v) Return the homogenous polynomial associated to  $P$  in the secondary variable  $v$ , that is  $y^d * P(x/y)$  where  $d$  is the degree of  $P$ ,  $x$  is the variable of  $P$ , and  $y$  is the variable with number  $v$ .

GEN RgX\_homogenous\_evalpow(GEN P, GEN a, GEN B) Evaluate the homogenous polynomial associated to the univariate polynomial  $P$  on  $(a,b)$  where  $B$  is the vector of powers of  $b$  with exponents 0 to the degree of  $P$  (gpowers(b,degpol(P))).

GEN RgXX\_to\_Kronecker(GEN P, long n) Assuming  $P(X,Y)$  is a polynomial of degree in  $X$  strictly less than  $n$ , returns  $P(X, X^{2*n-1})$ , the Kronecker form of  $P$ . Shallow function.

GEN RgXX\_to\_Kronecker\_spec(GEN Q, long lQ, long n) return RgXX\_to\_Kronecker( $P, n$ ), where  $P$  is the polynomial  $\sum_{i=0}^{lQ-1} Q[i]x^i$ . To be used when splitting the coefficients of genuine polynomials into blocks. Shallow function.

### 7.6.15.3 Shifts, valuations.

GEN RgX\_shift(GEN x, long n) returns  $x * t^n$  if  $n \geq 0$ , and  $x \backslash t^{-n}$  otherwise.

GEN RgX\_shift\_shallow(GEN x, long n) as RgX\_shift, but shallow (coefficients are not copied).

GEN RgX\_rotate\_shallow(GEN P, long k, long p) returns  $P * X^k \pmod{X^p - 1}$ , assuming the degree of  $P$  is strictly less than  $p$ , and  $k \geq 0$ .

void RgX\_shift\_inplace\_init(long v)  $v \geq 0$ , prepare for a later call to RgX\_shift\_inplace. Reserves  $v$  words on the stack.

GEN RgX\_shift\_inplace(GEN x, long v)  $v \geq 0$ , assume that RgX\_shift\_inplace\_init( $v$ ) has been called (reserving  $v$  words on the stack), immediately followed by a t\_POL  $x$ . Return RgX\_shift( $x, v$ ) by shifting  $x$  in place. To be used as follows

```
RgX_shift_inplace_init(v);
av = avma;
...
x = gerepileupto(av, ...); /* a t_POL */
return RgX_shift_inplace(x, v);
```

long RgX\_valrem(GEN P, GEN \*pz) returns the valuation  $v$  of the t\_POL  $P$  with respect to its main variable  $X$ . Check whether coefficients are 0 using isexactzero. Set \*pz to RgX\_shift\_shallow( $P, -v$ ).

long RgX\_val(GEN P) returns the valuation  $v$  of the t\_POL  $P$  with respect to its main variable  $X$ . Check whether coefficients are 0 using isexactzero.

long RgX\_valrem\_inexact(GEN P, GEN \*z) as RgX\_valrem, using gequal0 instead of isexactzero.

long RgXV\_maxdegree(GEN V) returns the maximum of the degrees of the components of the vector of t\_POLs  $V$ .

#### 7.6.15.4 Basic arithmetic.

GEN RgX\_add(GEN x, GEN y) adds  $x$  and  $y$ .

GEN RgX\_sub(GEN x, GEN y) subtracts  $x$  and  $y$ .

GEN RgX\_neg(GEN x) returns  $-x$ .

GEN RgX\_Rg\_add(GEN y, GEN x) returns  $x + y$ .

GEN RgX\_Rg\_add\_shallow(GEN y, GEN x) returns  $x + y$ ; shallow function.

GEN Rg\_RgX\_sub(GEN x, GEN y)

GEN RgX\_Rg\_sub(GEN y, GEN x) returns  $x - y$

GEN RgX\_Rg\_mul(GEN y, GEN x) multiplies the RgX  $y$  by the scalar  $x$ .

GEN RgX\_muls(GEN y, long s) multiplies the RgX  $y$  by the long  $s$ .

GEN RgX\_mul2n(GEN y, long n) multiplies the RgX  $y$  by  $2^n$ .

GEN RgX\_Rg\_div(GEN y, GEN x) divides the RgX  $y$  by the scalar  $x$ .

GEN RgX\_divs(GEN y, long s) divides the RgX  $y$  by the long  $s$ .

GEN RgX\_Rg\_divexact(GEN x, GEN y) exact division of the RgX  $y$  by the scalar  $x$ .

GEN RgX\_Rg\_eval\_bk(GEN f, GEN x) returns  $f(x)$  using Brent and Kung algorithm. (Use `poleval` for Horner algorithm.)

GEN RgX\_RgV\_eval(GEN f, GEN V) as `RgX_Rg_eval_bk(f, x)`, assuming  $V$  was output by `gpowers(x, n)` for some  $n \geq 1$ .

GEN RgXV\_RgV\_eval(GEN f, GEN V) apply `RgX_RgV_eval_bk(, V)` to all the components of the vector  $f$ .

GEN RgX\_normalize(GEN x) divides  $x$  by its leading coefficient. If the latter is 1,  $x$  itself is returned, not a copy. Leading coefficients equal to 0 are stripped, e.g.

$$0.*t^3 + \text{Mod}(0,3)*t^2 + 2*t$$

is normalized to  $t$ .

GEN RgX\_mul(GEN x, GEN y) multiplies the two `t_POL` (in the same variable)  $x$  and  $y$ . Detect the coefficient ring and use an appropriate algorithm.

GEN RgX\_mul\_i(GEN x, GEN y) multiplies the two `t_POL` (in the same variable)  $x$  and  $y$ . Do not detect the coefficient ring. Use a generic Karatsuba algorithm.

GEN RgX\_mul\_normalized(GEN A, long a, GEN B, long b) returns  $(X^a + A)(X^b + B) - X^{(a+b)}$ , where we assume that  $\deg A < a$  and  $\deg B < b$  are polynomials in the same variable  $X$ .

GEN RgX\_sqr(GEN x) squares the `t_POL`  $x$ . Detect the coefficient ring and use an appropriate algorithm.

GEN RgX\_sqr\_i(GEN x) squares the `t_POL`  $x$ . Do not detect the coefficient ring. Use a generic Karatsuba algorithm.

GEN RgXV\_prod(GEN V),  $V$  being a vector of RgX, returns their product.

GEN RgX\_divrem(GEN x, GEN y, GEN \*r) by default, returns the Euclidean quotient and store the remainder in *r*. Three special values of *r* change that behavior • NULL: do not store the remainder, used to implement RgX\_div,

- ONLY\_REM: return the remainder, used to implement RgX\_rem,
- ONLY\_DIVIDES: return the quotient if the division is exact, and NULL otherwise.

In the generic case, the remainder is created after the quotient and can be disposed of individually with a cgiv(*r*).

GEN RgX\_div(GEN x, GEN y)

GEN RgX\_div\_by\_X\_x(GEN A, GEN a, GEN \*r) returns the quotient of the RgX A by  $(X - a)$ , and sets *r* to the remainder A(*a*).

GEN RgX\_rem(GEN x, GEN y)

GEN RgX\_pseudodivrem(GEN x, GEN y, GEN \*ptr) compute a pseudo-quotient *q* and pseudo-remainder *r* such that  $\text{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$ . Return *q* and set \*ptr to *r*.

GEN RgX\_pseudorem(GEN x, GEN y) return the remainder in the pseudo-division of *x* by *y*.

GEN RgXQX\_pseudorem(GEN x, GEN y, GEN T) return the remainder in the pseudo-division of *x* by *y* over  $R[X]/(T)$ .

int ZXQX\_dvd(GEN x, GEN y, GEN T) let *T* be a monic irreducible ZX, let *x, y* be t\_POL whose coefficients are either t\_INTs or ZX in the same variable as *T*. Assume further that the leading coefficient of *y* is an integer. Return 1 if  $y|x$  in  $(\mathbf{Z}[Y]/(T))[X]$ , and 0 otherwise.

GEN RgXQX\_pseudodivrem(GEN x, GEN y, GEN T, GEN \*ptr) compute a pseudo-quotient *q* and pseudo-remainder *r* such that  $\text{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$  in  $R[X]/(T)$ . Return *q* and set \*ptr to *r*.

GEN RgX\_mulXn(GEN a, long n) returns  $a * X^n$ . This may be a t\_FRAC if  $n < 0$  and the valuation of *a* is not large enough.

GEN RgX\_addmulXn(GEN a, GEN b, long n) returns  $a + b * X^n$ , assuming that  $n > 0$ .

GEN RgX\_addmulXn\_shallow(GEN a, GEN b, long n) shallow variant of RgX\_addmulXn.

GEN RgX\_digits(GEN x, GEN B) returns a vector of RgX  $[c_0, \dots, c_n]$  of degree less than the degree of *B* and such that  $x = \sum_{i=0}^n c_i B^i$ .

#### 7.6.15.5 Internal routines working on coefficient arrays.

These routines operate on coefficient blocks which are invalid GENs A GEN argument *a* or *b* in routines below is actually a coefficient arrays representing the polynomials  $\sum_{i=0}^{na-1} a[i]X^i$  and  $\sum_{i=0}^{nb-1} b[i]X^i$ . Note that *a*[0] and *b*[0] contain coefficients and not the mandatory GEN codeword. This allows to implement divide-and-conquer methods directly, without needing to allocate wrappers around coefficient blocks.

GEN RgX\_mulspec(GEN a, GEN b, long na, long nb). Internal routine: given two coefficient arrays representing polynomials, return their product (as a true GEN) in variable 0.

GEN RgX\_sqrspec(GEN a, long na). Internal routine: given a coefficient array representing a polynomial *r* return its square (as a true GEN) in variable 0.

GEN RgX\_addspec(GEN x, GEN y, long nx, long ny) given two coefficient arrays representing polynomials, return their sum (as a true GEN) in variable 0.

GEN RgX\_addspec\_shallow(GEN x, GEN y, long nx, long ny) shallow variant of RgX\_addspec.

### 7.6.15.6 GCD, Resultant.

GEN RgX\_gcd(GEN x, GEN y) returns the GCD of x and y, assumed to be t\_POLs in the same variable.

GEN RgX\_gcd\_simple(GEN x, GEN y) as RgX\_gcd using a standard extended Euclidean algorithm. Usually slower than RgX\_gcd.

GEN RgX\_extgcd(GEN x, GEN y, GEN \*u, GEN \*v) returns  $d = \text{GCD}(x, y)$ , and sets \*u, \*v to the Bezout coefficients such that  $*ux + *vy = d$ . Uses a generic subresultant algorithm.

GEN RgX\_extgcd\_simple(GEN x, GEN y, GEN \*u, GEN \*v) as RgX\_extgcd using a standard extended Euclidean algorithm. Usually slower than RgX\_extgcd.

GEN RgX\_halfgcd(GEN x, GEN y) assuming x and y are t\_POLs in the same variable, returns a 2-components t\_VEC  $[M, V]$  where  $M$  is a  $2 \times 2$  t\_MAT and  $V$  a 2-component t\_COL, both with t\_POL entries, such that  $M \cdot [x, y] = V$  and such that if  $V = [a, b]$ , then  $\deg a \geq \lceil \max(\deg x, \deg y)/2 \rceil > \deg b$ .

GEN RgX\_chinese\_coprime(GEN x, GEN y, GEN Tx, GEN Ty, GEN Tz) returns an RgX, congruent to x mod Tx and to y mod Ty. Assumes Tx and Ty are coprime, and Tz = Tx \* Ty or NULL (in which case it is computed within).

GEN RgX\_disc(GEN x) returns the discriminant of the t\_POL x with respect to its main variable.

GEN RgX\_resultant\_all(GEN x, GEN y, GEN \*sol) returns resultant(x,y). If sol is not NULL, sets it to the last nonconstant remainder in the polynomial remainder sequence if it exists and to gen\_0 otherwise (e.g. one polynomial has degree 0).

### 7.6.15.7 Other operations.

GEN RgX\_gtofp(GEN x, GEN prec) returns the polynomial obtained by applying

gtofp(gel(x,i), prec)

to all coefficients of x.

GEN RgX\_fpnorml2(GEN x, long prec) returns (a stack-clean variant of)

gnorml2( RgX\_gtofp(x, prec) )

GEN RgX\_deriv(GEN x) returns the derivative of x with respect to its main variable.

GEN RgX\_integ(GEN x) returns the primitive of x vanishing at 0, with respect to its main variable.

GEN RgX\_rescale(GEN P, GEN h) returns  $h^{\deg(P)} P(x/h)$ . P is an RgX and h is nonzero. (Leaves small objects on the stack. Suitable but inefficient for gerepileupto.)

GEN RgX\_unscale(GEN P, GEN h) returns  $P(hx)$ . (Leaves small objects on the stack. Suitable but inefficient for gerepileupto.)

GEN RgXV\_unscale(GEN v, GEN h) apply RgX\_unscale to a vector of RgX.

GEN RgX\_translate(GEN P, GEN c) assume c is a scalar or a polynomials whose main variable has lower priority than the main variable X of P. Returns  $P(X + c)$  (optimized for  $c = \pm 1$ ).

GEN RgX\_affine(GEN P, GEN a, GEN b) Return  $P(aX + b)$  (optimized for  $b = \pm 1$ ). Not memory clean.

### 7.6.15.8 Function related to modular forms.

GEN RgX\_act\_Gl2Q(GEN g, long k) let  $R$  be a commutative ring and  $g = [a, b; c, d]$  be in  $\text{GL}_2(\mathbf{Q})$ ,  $g$  acts (on the left) on homogeneous polynomials of degree  $k - 2$  in  $V := R[X, Y]_{k-2}$  via

$$g \cdot P := P(dX - cY, -bX + aY) = (\det g)^{k-2} P((X, Y) \cdot g^{-1}).$$

This function returns the matrix in  $M_{k-1}(R)$  of  $P \mapsto g \cdot P$  in the basis  $(X^{k-2}, \dots, Y^{k-2})$  of  $V$ .

GEN RgX\_act\_ZGl2Q(GEN z, long k) let  $G := \text{GL}_2(\mathbf{Q})$ , acting on  $R[X, Y]_{k-2}$  and  $z \in \mathbf{Z}[G]$ . Return the matrix giving  $P \mapsto z \cdot P$  in the basis  $(X^{k-2}, \dots, Y^{k-2})$ .

### 7.6.16 RgXn.

GEN RgXn\_red\_shallow(GEN x, long n) return  $x \% t^n$ , where  $n \geq 0$ . Shallow function.

GEN RgXn\_recip\_shallow(GEN P) returns  $X^n P(1/X)$ . Shallow function.

GEN RgXn\_mul(GEN a, GEN b, long n) returns  $ab$  modulo  $X^n$ , where  $a, b$  are two  $\mathbf{t\_POL}$  in the same variable  $X$  and  $n \geq 0$ . Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

GEN RgXn\_sqr(GEN a, long n) returns  $a^2$  modulo  $X^n$ , where  $a$  is a  $\mathbf{t\_POL}$  in the variable  $X$  and  $n \geq 0$ . Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

GEN RgX\_mulhigh\_i(GEN f, GEN g, long n) return the Euclidean quotient of  $f(x) * g(x)$  by  $x^n$  (high product). Uses RgXn\_mul applied to the reciprocal polynomials of  $f$  and  $g$ . Not suitable for gerepile.

GEN RgX\_sqrhigh\_i(GEN f, long n) return the Euclidean quotient of  $f(x)^2$  by  $x^n$  (high product). Uses RgXn\_sqr applied to the reciprocal polynomial of  $f$ . Not suitable for gerepile.

GEN RgXn\_inv(GEN a, long n) returns  $a^{-1}$  modulo  $X^n$ , where  $a$  is a  $\mathbf{t\_POL}$  in the variable  $X$  and  $n \geq 0$ . Uses Newton-Raphson algorithm.

GEN RgXn\_inv\_i(GEN a, long n) as RgXn\_inv without final garbage collection (suitable for gerepileupto).

GEN RgXn\_div(GEN a, GEN b, long n) returns  $a/b$  modulo  $X^n$ , where  $a$  and  $b$  are  $\mathbf{t\_POL}$ s in the variable  $X$  and  $n \geq 0$ . Uses Newton-Raphson/Karp-Markstein algorithm.

GEN RgXn\_div\_i(GEN a, GEN b, long n) as RgXn\_div without final garbage collection (suitable for gerepileupto).

GEN RgXn\_powers(GEN x, long m, long n) returns  $[x^0, \dots, x^m]$  modulo  $X^n$  as a  $\mathbf{t\_VEC}$  of RgXns.

GEN RgXn\_powu(GEN x, ulong m, long n) returns  $x^m$  modulo  $X^n$ .

GEN RgXn\_powu\_i(GEN x, ulong m, long n) as RgXn\_powu, not memory clean.

GEN RgXn\_sqrt(GEN a, long n) returns  $a^{1/2}$  modulo  $X^n$ , where  $a$  is a  $\mathbf{t\_POL}$  in the variable  $X$  and  $n \geq 0$ . Assume that  $a = 1 \bmod X$ . Uses Newton algorithm.

GEN RgXn\_exp(GEN a, long n) returns  $\exp(a)$  modulo  $X^n$ , assuming  $a = 0 \bmod X$ .

GEN RgXn\_expint(GEN f, long n) return  $\exp(F)$  where  $F$  is the primitive of  $f$  that vanishes at 0.

GEN RgXn\_eval(GEN Q, GEN x, long n) special case of RgX\_RgXQ\_eval, when the modulus is a monomial: returns  $Q(x)$  modulo  $t^n$ , where  $x \in R[t]$ .

GEN RgX\_RgXn\_eval(GEN f, GEN x, long n) returns  $f(x)$  modulo  $X^n$ .

GEN RgX\_RgXnV\_eval(GEN f, GEN V, long n) as RgX\_RgXn\_eval(f, x, n), assuming  $V$  was output by RgXn\_powers(x, m, n) for some  $m \geq 1$ .

GEN RgXn\_reverse(GEN f, long n) assuming that  $f = ax \bmod x^2$  with  $a$  invertible, returns a  $\mathbf{t\_POL}$   $g$  of degree  $< n$  such that  $(g \circ f)(x) = x \bmod x^n$ .

#### 7.6.17 RgXnV.

GEN RgXnV\_red\_shallow(GEN x, long n) apply RgXn\_red\_shallow to all the components of the vector  $x$ .

#### 7.6.18 RgXQ.

GEN RgXQ\_mul(GEN y, GEN x, GEN T) computes  $xy \bmod T$

GEN RgXQ\_sqr(GEN x, GEN T) computes  $x^2 \bmod T$

GEN RgXQ\_inv(GEN x, GEN T) return the inverse of  $x \bmod T$ .

GEN RgXQ\_pow(GEN x, GEN n, GEN T) computes  $x^n \bmod T$

GEN RgXQ\_powu(GEN x, ulong n, GEN T) computes  $x^n \bmod T$ ,  $n$  being an ulong.

GEN RgXQ\_powers(GEN x, long n, GEN T) returns  $[x^0, \dots, x^n]$  as a  $\mathbf{t\_VEC}$  of RgXQs.

GEN RgXQ\_matrix\_pow(GEN y, long n, long m, GEN P) returns RgXQ\_powers(y, m-1, P), as a matrix of dimension  $n \geq \deg P$ .

GEN RgXQ\_norm(GEN x, GEN T) returns the norm of  $\text{Mod}(x, T)$ .

GEN RgXQ\_trace(GEN x, GEN T) returns the trace of  $\text{Mod}(x, T)$ .

GEN RgXQ\_charpoly(GEN x, GEN T, long v) returns the characteristic polynomial of  $\text{Mod}(x, T)$ , in variable  $v$ .

GEN RgXQ\_minpoly(GEN x, GEN T, long v) returns the minimal polynomial of  $\text{Mod}(x, T)$ , in variable  $v$ .

GEN RgX\_RgXQ\_eval(GEN f, GEN x, GEN T) returns  $f(x)$  modulo  $T$ .

GEN RgX\_RgXQV\_eval(GEN f, GEN V, GEN T) as RgX\_RgXQ\_eval(f, x, T), assuming  $V$  was output by RgXQ\_powers(x, n, T) for some  $n \geq 1$ .

int RgXQ\_ratlift(GEN x, GEN T, long amax, long bmax, GEN \*P, GEN \*Q) Assuming that  $\text{amax} + \text{bmax} < \deg T$ , attempts to recognize  $x$  as a rational function  $a/b$ , i.e. to find  $\mathbf{t\_POL}$ s  $P$  and  $Q$  such that

- $P \equiv Qx \bmod T$ ,
- $\deg P \leq \text{amax}$ ,  $\deg Q \leq \text{bmax}$ ,
- $\gcd(T, P) = \gcd(P, Q)$ .

If unsuccessful, the routine returns 0 and leaves  $P, Q$  unchanged; otherwise it returns 1 and sets  $P$  and  $Q$ .

GEN RgXQ\_reverse(GEN f, GEN T) returns a  $\mathbf{t\_POL}$   $g$  of degree  $< n = \deg T$  such that  $T(x)$  divides  $(g \circ f)(x) - x$ , by solving a linear system. Low-level function underlying **modreverse**: it returns a lift of  $[\text{modreverse}(f, T)]$ ; faster than the high-level function since it needs not compute the characteristic polynomial of  $f \bmod T$  (often already known in applications). In the trivial case where  $n \leq 1$ , returns a scalar, not a constant  $\mathbf{t\_POL}$ .



### 7.6.19 RgXQV, RgXQC.

GEN RgXQC\_red(GEN z, GEN T) z a vector whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying grem coefficientwise) in a t\_COL.

GEN RgXQV\_red(GEN z, GEN T) z a vector whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying grem coefficientwise) in a t\_VEC.

GEN RgXQV\_RgXQ\_mul(GEN z, GEN x, GEN T) z multiplies the RgXQV z by the scalar (RgXQ) x.

GEN RgXQV\_factorback(GEN L, GEN e, GEN T) returns  $\prod_i L_i^{e_i} \bmod T$  where L is a vector of RgXQs and e a vector of t\_INTs.

### 7.6.20 RgXQM.

GEN RgXQM\_red(GEN z, GEN T) z a matrix whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying grem coefficientwise).

GEN RgXQM\_mul(GEN x, GEN y, GEN T)

### 7.6.21 RgXQX.

GEN RgXQX\_red(GEN z, GEN T) z a t\_POL whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying grem coefficientwise).

GEN RgXQX\_mul(GEN x, GEN y, GEN T)

GEN RgXQX\_RgXQ\_mul(GEN x, GEN y, GEN T) multiplies the RgXQX y by the scalar (RgXQ) x.

GEN RgXQX\_sqr(GEN x, GEN T)

GEN RgXQX\_powers(GEN x, long n, GEN T)

GEN RgXQX\_divrem(GEN x, GEN y, GEN T, GEN \*pr)

GEN RgXQX\_div(GEN x, GEN y, GEN T)

GEN RgXQX\_rem(GEN x, GEN y, GEN T)

GEN RgXQX\_translate(GEN P, GEN c, GEN T) assume the main variable X of P has higher priority than the main variable Y of T and c. Return a lift of  $P(X + \text{Mod}(c(Y), T(Y)))$ .

GEN Kronecker\_to\_mod(GEN z, GEN T)  $z \in R[X]$  represents an element  $P(X, Y)$  in  $R[X, Y] \bmod T(Y)$  in Kronecker form, i.e.  $z = P(X, X^{2*n-1})$

Let R be some commutative ring,  $n = \deg T$  and let  $P(X, Y) \in R[X, Y]$  lift a polynomial in  $K[Y]$ , where  $K := R[X]/(T)$  and  $\deg_X P < 2n - 1$  — such as would result from multiplying minimal degree lifts of two polynomials in  $K[Y]$ . Let  $z = P(t, t^{2*n-1})$  be a Kronecker form of P, this function returns the image of  $P(X, t)$  in  $K[t]$ , with t\_POLMOD coefficients. Not stack-clean. Note that t need not be the same variable as Y!

## Chapter 8:

### Black box algebraic structures

The generic routines like gmul or gadd allow handling objects belonging to a fixed list of basic types, with some natural polymorphism (you can mix rational numbers and polynomials, etc.), at

the expense of efficiency and sometimes of clarity when the recursive structure becomes complicated, e.g. a few levels of `t_POLMODs` attached to different polynomials and variable numbers for quotient structures. This is the only possibility in GP.

On the other hand, the Level 2 Kernel allows dedicated routines to handle efficiently objects of a very specific type, e.g. polynomials with coefficients in the same finite field. This is more efficient, but involves a lot of code duplication since polymorphism is no longer possible.

A third and final option, still restricted to library programming, is to define an arbitrary algebraic structure (currently groups, fields, rings, algebras and  $\mathbf{Z}_p$ -modules) by providing suitable methods, then using generic algorithms. For instance naive Gaussian pivoting applies over all base fields and need only be implemented once. The difference with the first solution is that we no longer depend on the way functions like `gmul` or `gadd` will guess what the user is trying to do. We can then implement independently various groups / fields / algebras in a clean way.

## 8.1 Black box groups.

A black box group is defined by a `bb_group` struct, describing methods available to handle group elements:

```
struct bb_group
{
    GEN (*mul)(void*, GEN, GEN);
    GEN (*pow)(void*, GEN, GEN);
    GEN (*rand)(void*);
    ulong (*hash)(GEN);
    int (*equal)(GEN, GEN);
    int (*equal1)(GEN);
    GEN (*easylog)(void *E, GEN, GEN, GEN);
};
```

`mul(E,x,y)` returns the product  $xy$ .

`pow(E,x,n)` returns  $x^n$  ( $n$  integer, possibly negative or zero).

`rand(E)` returns a random element in the group.

`hash(x)` returns a hash value for  $x$  (`hash_GEN` is suitable for this field).

`equal(x,y)` returns one if  $x = y$  and zero otherwise.

`equal1(x)` returns one if  $x$  is the neutral element in the group, and zero otherwise.

`easylog(E,a,g,o)` (optional) returns either NULL or the discrete logarithm  $n$  such that  $g^n = a$ , the element  $g$  being of order  $o$ . This provides a short-cut in situation where a better algorithm than the generic one is known.

A group is thus described by a `struct bb_group` as above and auxiliary data typecast to `void*`. The following functions operate on black box groups:

`GEN gen_Shanks_log(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp)`  
 Generic baby-step/giant-step algorithm (Shanks's method). Assuming that  $g$  has order  $N$ , compute an integer  $k$  such that  $g^k = x$ . Return `cgetg(1, t_VEC)` if there are no solutions. This requires  $O(\sqrt{N})$  group operations and uses an auxiliary table containing  $O(\sqrt{N})$  group elements.

The above is useful for a one-shot computation. If many discrete logs are desired: `GEN gen_Shanks_init(GEN g, long n, void *E, const struct bb_group *grp)` return an auxiliary data structure  $T$  required to compute a discrete log in base  $g$ . Compute and store all powers  $g^i$ ,  $i < n$ .

`GEN gen_Shanks(GEN T, GEN x, ulong N, void *E, const struct bb_group *grp)` Let  $T$  be computed by `gen_Shanks_init(g, n, ...)`. Return  $k < nN$  such that  $g^k = x$  or NULL if no such index exist. It uses  $O(N)$  operation in the group and fast table lookups (in time  $O(\log n)$ ). The interface is such that the function may be used when the order of the base  $g$  is unknown, and hence compute it given only an upper bound  $B$  for it: e.g. choose  $n, N$  such that  $nN \geq B$  and compute the discrete log  $l$  of  $g^{-1}$  in base  $g$ , then use `gen_order` with multiple  $N = l + 1$ .

`GEN gen_Pollard_log(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp)` Generic Pollard rho algorithm. Assuming that  $g$  has order  $N$ , compute an integer  $k$  such that  $g^k = x$ . This requires  $O(\sqrt{N})$  group operations in average and  $O(1)$  storage. Will enter an infinite loop if there are no solutions.

`GEN gen_plog(GEN x, GEN g, GEN N, void *E, const struct bb_group)` Assuming that  $g$  has prime order  $N$ , compute an integer  $k$  such that  $g^k = x$ , using either `gen_Shanks_log` or `gen_Pollard_log`. Return `cgetg(1, t_VEC)` if there are no solutions.

`GEN gen_Shanks_sqrtn(GEN a, GEN n, GEN N, GEN *zetan, void *E, const struct bb_group *grp)` returns one solution of  $x^n = a$  in a black box cyclic group of order  $N$ . Return NULL if no solution exists. If `zetan` is not NULL it is set to an element of exact order  $n$ . This function uses `gen_plog` for all prime divisors of  $\gcd(n, N)$ .

`GEN gen_PH_log(GEN a, GEN g, GEN N, void *E, const struct bb_group *grp)` returns an integer  $k$  such that  $g^k = x$ , assuming that the order of  $g$  divides  $N$ , using Pohlig-Hellman algorithm. Return `cgetg(1, t_VEC)` if there are no solutions. This calls `gen_plog` repeatedly for all prime divisors  $p$  of  $N$ .

In the following functions the integer parameter `ord` can be given in all the formats recognized for the argument of arithmetic functions, i.e. either as a positive `t_INT`  $N$ , or as its factorization matrix  $faN$ , or (preferred) as a pair  $[N, faN]$ .

`GEN gen_order(GEN x, GEN ord, void *E, const struct bb_group *grp)` computes the order of  $x$ ; `ord` is a multiple of the order, for instance the group order.

`GEN gen_factored_order(GEN x, GEN ord, void *E, const struct bb_group *grp)` returns a pair  $[o, F]$ , where  $o$  is the order of  $x$  and  $F$  is the factorization of  $o$ ; `ord` is as in `gen_order`.

`GEN gen_gener(GEN ord, void *E, const struct bb_group *grp)` returns a random generator of the group, assuming it is of order exactly `ord`.

`GEN get_arith_Z(GEN ord)` given `ord` as above in one of the formats recognized for arithmetic functions, i.e. a positive `t_INT`  $N$ , its factorization  $faN$ , or the pair  $[N, faN]$ , return  $N$ .

`GEN get_arith_ZZM(GEN ord)` given `ord` as above, return the pair  $[N, faN]$ . This may require factoring  $N$ .

`GEN gen_select_order(GEN v, void *E, const struct bb_group *grp)` Let  $v$  be a vector of possible orders for the group; try to find the true order by checking orders of random points. This will not terminate if there is an ambiguity.

### 8.1.1 Black box groups with pairing.

These functions handle groups of rank at most 2 equipped with a family of bilinear pairings which behave like the Weil pairing on elliptic curves over finite field. In the descriptions below, the function `pairorder(E, P, Q, m, F)` must return the order of the  $m$ -pairing of  $P$  and  $Q$ , both of order dividing  $m$ , where  $F$  is the factorization matrix of a multiple of  $m$ .

`GEN gen_ellgroup(GEN o, GEN d, GEN *pt_m, void *E, const struct bb_group *grp, GEN pairorder(void *E, GEN P, GEN Q, GEN m, GEN F))` returns the elementary divisors  $[d_1, d_2]$  of the group, assuming it is of order exactly  $o > 1$ , and that  $d_2$  divides  $d$ . If  $d_2 = 1$  then  $[o]$  is returned, otherwise `m=*pt_m` is set to the order of the pairing required to verify a generating set which is to be used with `gen_ellgens`. For the parameter  $o$ , all formats recognized by arithmetic functions are allowed, preferably a factorization matrix or a pair  $[n, \text{factor}(n)]$ .

`GEN gen_ellgens(GEN d1, GEN d2, GEN m, void *E, const struct bb_group *grp, GEN pairorder(void *E, GEN P, GEN Q, GEN m, GEN F))` the parameters  $d_1, d_2, m$  being as returned by `gen_ellgroup`, returns a pair of generators  $[P, Q]$  such that  $P$  is of order  $d_1$  and the  $m$ -pairing of  $P$  and  $Q$  is of order  $m$ . (Note:  $Q$  needs not be of order  $d_2$ ). For the parameter  $d_1$ , all formats recognized by arithmetic functions are allowed, preferably a factorization matrix or a pair  $[n, \text{factor}(n)]$ .

### 8.1.2 Functions returning black box groups.

`const struct bb_group * get_Flxq_star(void **E, GEN T, ulong p)`

`const struct bb_group * get_FpXQ_star(void **E, GEN T, GEN p)` returns a pointer to the black box group  $(\mathbf{F}_p[x]/(T))^*$ .

`const struct bb_group * get_FpE_group(void **pE, GEN a4, GEN a6, GEN p)` returns a pointer to a black box group and set `*pE` to the necessary data for computing in the group  $E(\mathbf{F}_p)$  where  $E$  is the elliptic curve  $E : y^2 = x^3 + a_4x + a_6$ , with  $a_4$  and  $a_6$  in  $\mathbf{F}_p$ .

`const struct bb_group * get_FpXQE_group(void **pE, GEN a4, GEN a6, GEN T, GEN p)` returns a pointer to a black box group and set `*pE` to the necessary data for computing in the group  $E(\mathbf{F}_p[X]/(T))$  where  $E$  is the elliptic curve  $E : y^2 = x^3 + a_4x + a_6$ , with  $a_4$  and  $a_6$  in  $\mathbf{F}_p[X]/(T)$ .

`const struct bb_group * get_FlxqE_group(void **pE, GEN a4, GEN a6, GEN T, ulong p)` idem for small  $p$ .

`const struct bb_group * get_F2xqE_group(void **pE, GEN a2, GEN a6, GEN T)` idem for  $p = 2$ .

## 8.2 Black box fields.

A black box field is defined by a `bb_field` struct, describing methods available to handle field elements:

```
struct bb_field
{
    GEN (*red)(void *E ,GEN);
    GEN (*add)(void *E ,GEN, GEN);
    GEN (*mul)(void *E ,GEN, GEN);
    GEN (*neg)(void *E ,GEN);
    GEN (*inv)(void *E ,GEN);
    int (*equal0)(GEN);
    GEN (*s)(void *E, long);
};
```

In contrast of black box group, elements can have non canonical forms, and only `red` is required to return a canonical form. For instance a black box implementation of finite fields, all methods except `red` may return arbitrary representatives in  $\mathbf{Z}[X]$  of the correct congruence class modulo  $(p, T(X))$ .

`red(E,x)` returns the canonical form of  $x$ .

`add(E,x,y)` returns the sum  $x + y$ .

`mul(E,x,y)` returns the product  $xy$ .

`neg(E,x)` returns  $-x$ .

`inv(E,x)` returns the inverse of  $x$ .

`equal0(x)`  $x$  being in canonical form, returns one if  $x = 0$  and zero otherwise.

`s(n)`  $n$  being a small signed integer, returns  $n$  times the unit element.

A field is thus described by a `struct bb_field` as above and auxiliary data typecast to `void*`. The following functions operate on black box fields:

```
GEN gen_Gauss(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_Gauss_pivot(GEN x, long *rr, void *E, const struct bb_field *ff)
GEN gen_det(GEN a, void *E, const struct bb_field *ff)
GEN gen_ker(GEN x, long deplin, void *E, const struct bb_field *ff)
GEN gen_matcolinvimage(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_matcolmul(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_matid(long n, void *E, const struct bb_field *ff)
GEN gen_matinvimage(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_matmul(GEN a, GEN b, void *E, const struct bb_field *ff)
```

### 8.2.1 Functions returning black box fields.

```
const struct bb_field * get_Fp_field(void **pE, GEN p)
const struct bb_field * get_Fq_field(void **pE, GEN T, GEN p)
const struct bb_field * get_Flxq_field(void **pE, GEN T, ulong p)
const struct bb_field * get_F2xq_field(void **pE, GEN T)
const struct bb_field * get_nf_field(void **pE, GEN nf)
```

## 8.3 Black box algebra.

A black box algebra is defined by a `bb_algebra` struct, describing methods available to handle algebra elements:

```
struct bb_algebra
{
    GEN (*red)(void *E, GEN x);
    GEN (*add)(void *E, GEN x, GEN y);
    GEN (*sub)(void *E, GEN x, GEN y);
    GEN (*mul)(void *E, GEN x, GEN y);
    GEN (*sqr)(void *E, GEN x);
    GEN (*one)(void *E);
    GEN (*zero)(void *E);
};
```

In contrast with black box groups, elements can have non canonical forms, but only `add` is allowed to return a non canonical form.

`red(E,x)` returns the canonical form of  $x$ .

`add(E,x,y)` returns the sum  $x + y$ .

`sub(E,x,y)` returns the difference  $x - y$ .

`mul(E,x,y)` returns the product  $xy$ .

`sqr(E,x)` returns the square  $x^2$ .

`one(E)` returns the unit element.

`zero(E)` returns the zero element.

An algebra is thus described by a `struct bb_algebra` as above and auxiliary data typecast to `void*`. The following functions operate on black box algebra:

`GEN gen_bkeval(GEN P, long d, GEN x, int use_sqr, void *E, const struct bb_algebra *ff, GEN cmul(void *E, GEN P, long a, GEN x))`  $x$  being an element of the black box algebra, and  $P$  some black box polynomial of degree  $d$  over the base field, returns  $P(x)$ . The function `cmul(E,P,a,y)` must return the coefficient of degree  $a$  of  $P$  multiplied by  $y$ . `cmul` is allowed to return a non canonical form; it is also allowed to return `NULL` instead of an exact 0.

The flag `use_sqr` has the same meaning as for `gen_powers`. This implements an algorithm of Brent and Kung (1978).

GEN gen\_bkeval\_powers(GEN P, long d, GEN V, void \*E, const struct bb\_algebra \*ff, GEN cmul(void \*E, GEN P, long a, GEN x)) as gen\_RgX\_bkeval assuming  $V$  was output by gen\_powers( $x, l, E, ff$ ) for some  $l \geq 1$ . For optimal performance,  $l$  should be computed by brent\_kung\_optpow.

long brent\_kung\_optpow(long d, long n, long m) returns the optimal parameter  $l$  for the evaluation of  $n/m$  polynomials of degree  $d$ . Fractional values can be used if the evaluations are done with different accuracies, and thus have different weights.

### 8.3.1 Functions returning black box algebras.

const struct bb\_algebra \* get\_FpX\_algebra(void \*\*E, GEN p, long v) return the algebra of polynomials over  $\mathbf{F}_p$  in variable  $v$ .

const struct bb\_algebra \* get\_FpXQ\_algebra(void \*\*E, GEN T, GEN p) return the algebra  $\mathbf{F}_p[X]/(T(X))$ .

const struct bb\_algebra \* get\_FpXQX\_algebra(void \*\*E, GEN T, GEN p, long v) return the algebra of polynomials over  $\mathbf{F}_p[X]/(T(X))$  in variable  $v$ .

const struct bb\_algebra \* get\_FlxqXQ\_algebra(void \*\*E, GEN S, GEN T, ulong p) return the algebra  $\mathbf{F}_p[X, Y]/(S(X, Y), T(X))$  (for ulong  $p$ ).

const struct bb\_algebra \* get\_FpXQXQ\_algebra(void \*\*E, GEN S, GEN T, GEN p) return the algebra  $\mathbf{F}_p[X, Y]/(S(X, Y), T(X))$ .

const struct bb\_algebra \* get\_Rg\_algebra(void) return the generic algebra.

## 8.4 Black box ring.

A black box ring is defined by a bb\_ring struct, describing methods available to handle ring elements:

```
struct bb_ring
{
    GEN (*add)(void *E, GEN x, GEN y);
    GEN (*mul)(void *E, GEN x, GEN y);
    GEN (*sqr)(void *E, GEN x);
};
```

add(E,x,y) returns the sum  $x + y$ .

mul(E,x,y) returns the product  $xy$ .

sqr(E,x) returns the square  $x^2$ .

GEN gen\_fromdigits(GEN v, GEN B, void \*E, struct bb\_ring \*r) where  $B$  is a ring element and  $v = [c_0, \dots, c_{n-1}]$  a vector of ring elements, return  $\sum_{i=0}^n c_i B^i$  using binary splitting.

GEN gen\_digits(GEN x, GEN B, long n, void \*E, struct bb\_ring \*r, GEN (\*div)(void \*E, GEN x, GEN y, GEN \*r))

(Require the ring to be Euclidean)

div(E,x,y,&r) performs the Euclidean division of  $x$  by  $y$  in the ring  $R$ , returning the quotient  $q$  and setting  $r$  to the residue so that  $x = qy + r$  holds. The residue must belong to a fixed set of representatives of  $R/(y)$ .

The argument  $x$  being a ring element, `gen_digits` returns a vector of ring elements  $[c_0, \dots, c_{n-1}]$  such that  $x = \sum_{i=0}^n c_i B^i$ . Furthermore for all  $i \neq n-1$ , the elements  $c_i$  belonging to the fixed set of representatives of  $R/(B)$ .

## 8.5 Black box free $\mathbf{Z}_p$ -modules.

(Very experimental)

`GEN gen_ZpX_Dixon(GEN F, GEN V, GEN q, GEN p, long N, void *E, GEN lin(void *E, GEN F, GEN z, GEN q), GEN invl(void *E, GEN z))`

Let  $F$  be a `ZpXT` representing the coefficients of some abstract linear mapping  $f$  over  $\mathbf{Z}_p[X]$  seen as a free  $\mathbf{Z}_p$ -module, let  $V$  be an element of  $\mathbf{Z}_p[X]$  and let  $q = p^N$ . Return  $y \in \mathbf{Z}_p[X]$  such that  $f(y) = V \pmod{p^N}$  assuming the following holds for  $n \leq N$ :

- $\text{lin}(E, \text{FpX\_red}(F, p^n), z, p^n) \equiv f(z) \pmod{p^n}$
- $f(\text{invl}(E, z)) \equiv z \pmod{p}$

The rationale for the argument  $F$  being that it allows `gen_ZpX_Dixon` to reduce it to the required  $p$ -adic precision.

`GEN gen_ZpX_Newton(GEN x, GEN p, long n, void *E, GEN eval(void *E, GEN a, GEN q), GEN invd(void *E, GEN b, GEN v, GEN q, long N))`

Let  $x$  be an element of  $\mathbf{Z}_p[X]$  seen as a free  $\mathbf{Z}_p$ -module, and  $f$  some differentiable function over  $\mathbf{Z}_p[X]$  such that  $f(x) \equiv 0 \pmod{p}$ . Return  $y$  such that  $f(y) \equiv 0 \pmod{p^n}$ , assuming the following holds for all  $a, b \in \mathbf{Z}_p[X]$  and  $M \leq N$ :

- $v = \text{eval}(E, a, p^N)$  is a vector of elements of  $\mathbf{Z}_p[X]$ ,
- $w = \text{invd}(E, b, v, p^M, M)$  is an element in  $\mathbf{Z}_p[X]$ ,
- $v[1] \equiv f(a) \pmod{p^N \mathbf{Z}_p[X]}$ ,
- $df_a(w) \equiv b \pmod{p^M \mathbf{Z}_p[X]}$

and  $df_a$  denotes the differential of  $f$  at  $a$ . Motivation: `eval` allows to evaluate  $f$  and `invd` allows to invert its differential. Frequently, data useful to compute the differential appear as a subproduct of computing the function. The vector  $v$  allows `eval` to provide these to `invd`. The implementation of `invd` will generally involves the use of the function `gen_ZpX_Dixon`.

`GEN gen_ZpM_Newton(GEN x, GEN p, long n, void *E, GEN eval(void *E, GEN a, GEN q), GEN invd(void *E, GEN b, GEN v, GEN q, long N))` as above, with polynomials replaced by matrices.



## Chapter 9:

### Operations on general PARI objects

#### 9.1 Assignment.

It is in general easier to use a direct conversion, e.g. `y = stoi(s)`, than to allocate a target of correct type and sufficient size, then assign to it:

```
GEN y = cgeti(3); affsi(s, y);
```

These functions can still be moderately useful in complicated garbage collecting scenarios but you will be better off not using them.

`void gaffsg(long s, GEN x)` assigns the `long s` into the object `x`.

`void gaffect(GEN x, GEN y)` assigns the object `x` into the object `y`. Both `x` and `y` must be scalar types. Type conversions (e.g. from `t_INT` to `t_REAL` or `t_INTMOD`) occur if legitimate.

`int is_universal_constant(GEN x)` returns 1 if `x` is a global PARI constant you should never assign to (such as `gen_1`), and 0 otherwise.

#### 9.2 Conversions.

##### 9.2.1 Scalars.

`double rtodbl(GEN x)` applied to a `t_REAL x`, converts `x` into a `double` if possible.

`GEN dbltor(double x)` converts the `double x` into a `t_REAL`.

`long dblexpo(double x)` returns `expo(dbltor(x))`, but faster and without cluttering the stack.

`ulong dblmantissa(double x)` returns the most significant word in the mantissa of `dbltor(x)`.

`int gisdouble(GEN x)` if `x` is a real number (not necessarily a `t_REAL`), return 1 if `x` can be converted to a `double`, 0 otherwise.

`double gtodouble(GEN x)` if `x` is a real number (not necessarily a `t_REAL`), converts `x` into a `double` if possible.

`long gtos(GEN x)` converts the `t_INT x` to a small integer if possible, otherwise raise an exception. This function is similar to `itos`, slightly slower since it checks the type of `x`.

`ulong gtou(GEN x)` converts the non-negative `t_INT x` to an unsigned small integer if possible, otherwise raise an exception. This function is similar to `itou`, slightly slower since it checks the type of `x`.

`double dbllog2r(GEN x)` assuming that `x` is a nonzero `t_REAL`, returns an approximation to `log2(|x|)`.

`double dblmodulus(GEN x)` return an approximation to `|x|`.

`long gtolong(GEN x)` if  $x$  is an integer (not necessarily a `t_INT`), converts  $x$  into a `long` if possible.

`GEN fractor(GEN x, long l)` applied to a `t_FRAC`  $x$ , converts  $x$  into a `t_REAL` of length `prec`.

`GEN quadtofp(GEN x, long l)` applied to a `t_QUAD`  $x$ , converts  $x$  into a `t_REAL` or `t_COMPLEX` depending on the sign of the discriminant of  $x$ , to precision `l BITS_IN_LONG`-bit words.

`GEN upper_to_cx(GEN x, long *prec)` valid for a `t_COMPLEX` or `t_QUAD` belonging to the upper half-plane. If a `t_QUAD`, convert it to `t_COMPLEX` using accuracy `*prec`. If  $x$  is inexact, sets `*prec` to the precision of  $x$ .

`GEN cxtotfp(GEN x, long prec)` converts the `t_COMPLEX`  $x$  to a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtotfp`).

`GEN cxcompotor(GEN x, long prec)` converts the `t_INT`, `t_REAL` or `t_FRAC`  $x$  to a `t_REAL` of length `prec`. These are all the real types which may occur as components of a `t_COMPLEX`; special case of `gtotfp` (introduced so that the latter is not recursive and can thus be inlined).

`GEN cxtoreal(GEN x)` converts the complex (`t_INT`, `t_REAL`, `t_FRAC` or `t_COMPLEX`)  $x$  to a real number if its imaginary part is 0. Shallow function.

converts the `t_COMPLEX`  $x$  to a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtotfp`).

`GEN gtotfp(GEN x, long prec)` converts the complex number  $x$  (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of precision `prec`; not necessarily of *length* `prec`: a real 0 may be given as `real_0(...)`. If the result is a `t_COMPLEX` extra care is taken so that its modulus really has accuracy `prec`: there is a problem if the real part of the input is an exact 0; indeed, converting it to `real_0(prec)` would be wrong if the imaginary part is tiny, since the modulus would then become equal to 0, as in  $1.E-100 + 0.E-28 = 0.E-28$ .

`GEN gtomp(GEN z, long prec)` converts the real number  $x$  (`t_INT`, `t_REAL`, `t_FRAC`, real `t_QUAD`) to either a `t_INT` or a `t_REAL` of precision `prec`. Not memory clean if  $x$  is a `t_INT`: we return  $x$  itself and not a copy.

`GEN gcvtotp(GEN x, GEN p, long l)` converts  $x$  into a `t_PADIC` of precision  $l$ . Works componentwise on recursive objects, e.g. `t_POL` or `t_VEC`. Converting 0 yields  $O(p^l)$ ; converting a nonzero number yield a result well defined modulo  $p^{v_p(x)+l}$ .

`GEN cvtotp(GEN x, GEN p, long l)` as `gcvtotp`, assuming that  $x$  is a scalar.

`GEN cvtop2(GEN x, GEN y)`  $y$  being a  $p$ -adic, converts the scalar  $x$  to a  $p$ -adic of the same accuracy. Shallow function.

`GEN cvstop2(long s, GEN y)`  $y$  being a  $p$ -adic, converts the scalar  $s$  to a  $p$ -adic of the same accuracy. Shallow function.

`GEN gprec(GEN x, long l)` returns a copy of  $x$  whose precision is changed to  $l$  digits. The precision change is done recursively on all components of  $x$ . Digits means *decimal*,  $p$ -adic and  $X$ -adic digits for `t_REAL`, `t_SER`, `t_PADIC` components, respectively.

`GEN gprec_w(GEN x, long l)` returns a shallow copy of  $x$  whose `t_REAL` components have their precision changed to  $l$  words. This is often more useful than `gprec`.

`GEN gprec_wtrunc(GEN x, long l)` returns a shallow copy of  $x$  whose `t_REAL` components have their precision *truncated* to  $l$  words. Contrary to `gprec_w`, this function may never increase the precision of  $x$ .

GEN `gprec_wensure`(GEN `x`, long `l`) returns a shallow copy of `x` whose `t_REAL` components have their precision *increased* to at least `l words`. Contrary to `gprec_w`, this function may never decrease the precision of `x`.

The following functions are obsolete and kept for backward compatibility only:

GEN `precision0`(GEN `x`, long `n`)  
 GEN `bitprecision0`(GEN `x`, long `n`)

### 9.2.2 Modular objects / lifts.

GEN `gmodulo`(GEN `x`, GEN `y`) creates the object **Mod**(`x`,`y`) on the PARI stack, where `x` and `y` are either both `t_INTs`, and the result is a `t_INTMOD`, or `x` is a scalar or a `t_POL` and `y` a `t_POL`, and the result is a `t_POLMOD`.

GEN `gmodulgs`(GEN `x`, long `y`) same as **gmodulo** except `y` is a long.

GEN `gmodulsg`(long `x`, GEN `y`) same as **gmodulo** except `x` is a long.

GEN `gmodulss`(long `x`, long `y`) same as **gmodulo** except both `x` and `y` are longs.

GEN `lift_shallow`(GEN `x`) shallow version of `lift`

GEN `liftall_shallow`(GEN `x`) shallow version of `liftall`

GEN `liftint_shallow`(GEN `x`) shallow version of `liftint`

GEN `liftpol_shallow`(GEN `x`) shallow version of `liftpol`

GEN `centerlift0`(GEN `x`, long `v`) DEPRECATED, kept for backward compatibility only: use either `lift0(x,v)` or `centerlift(x)`.

### 9.2.3 Between polynomials and coefficient arrays.

GEN `gtopoly`(GEN `x`, long `v`) converts or truncates the object `x` into a `t_POL` with main variable number `v`. A common application would be the conversion of coefficient vectors (coefficients are given by decreasing degree). E.g. `[2,3]` goes to `2*v + 3`

GEN `gtopolyrev`(GEN `x`, long `v`) converts or truncates the object `x` into a `t_POL` with main variable number `v`, but vectors are converted in reverse order compared to `gtopoly` (coefficients are given by increasing degree). E.g. `[2,3]` goes to `3*v + 2`. In other words the vector represents a polynomial in the basis  $(1, v, v^2, v^3, \dots)$ .

GEN `normalizpol`(GEN `x`) applied to an unnormalized `t_POL` `x` (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes `x` correctly in place and returns `x`. For internal use. Normalizing means deleting all leading *exact* zeroes (as per `isexactzero`), except if the polynomial turns out to be 0, in which case we try to find a coefficient `c` which is a nonrational zero, and return the constant polynomial `c`. (We do this so that information about the base ring is not lost.)

GEN `normalizpol_lg`(GEN `x`, long `l`) applies `normalizpol` to `x`, pretending that `lg(x)` is `l`, which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `normalizpol(x)`.

GEN `normalizpol_approx`(GEN `x`, long `lx`) as `normalizpol_lg`, with the difference that we just delete all leading zeroes (as per `gequal0`). This rougher normalization is used when we have no other choice, for instance before attempting a Euclidean division by `x`.

The following routines do *not* copy coefficients on the stack (they only move pointers around), hence are very fast but not suitable for `gerepile` calls. Recall that an `RgV` (resp. an `RgX`, resp. an `RgM`) is a `t_VEC` or `t_COL` (resp. a `t_POL`, resp. a `t_MAT`) with arbitrary components. Similarly, an `RgXV` is a `t_VEC` or `t_COL` with `RgX` components, etc.

`GEN RgV_to_RgX(GEN x, long v)` converts the `RgV` `x` to a (normalized) polynomial in variable `v` (as `gtopolyrev`, without copy).

`GEN RgV_to_RgX_reverse(GEN x, long v)` converts the `RgV` `x` to a (normalized) polynomial in variable `v` (as `gtopoly`, without copy).

`GEN RgX_to_RgC(GEN x, long N)` converts the `t_POL` `x` to a `t_COL` `v` with `N` components. Coefficients of `x` are listed by increasing degree, so that `y[i]` is the coefficient of the term of degree  $i - 1$  in `x`.

`GEN Rg_to_RgC(GEN x, long N)` as `RgX_to_RgV`, except that other types than `t_POL` are allowed for `x`, which is then considered as a constant polynomial.

`GEN RgM_to_RgXV(GEN x, long v)` converts the `RgM` `x` to a `t_VEC` of `RgX`, by repeated calls to `RgV_to_RgX`.

`GEN RgM_to_RgXV_reverse(GEN x, long v)` converts the `RgM` `x` to a `t_VEC` of `RgX`, by repeated calls to `RgV_to_RgX_reverse`.

`GEN RgV_to_RgM(GEN v, long N)` converts the vector `v` to a `t_MAT` with `N` rows, by repeated calls to `Rg_to_RgV`.

`GEN RgXV_to_RgM(GEN v, long N)` converts the vector of `RgX` `v` to a `t_MAT` with `N` rows, by repeated calls to `RgX_to_RgV`.

`GEN RgM_to_RgXX(GEN x, long v, long w)` converts the `RgM` `x` into a `t_POL` in variable `v`, whose coefficients are `t_POLs` in variable `w`. This is a shortcut for

`RgV_to_RgX( RgM_to_RgXV(x, w), v );`

There are no consistency checks with respect to variable priorities: the above is an invalid object if `varncmp(v, w) ≥ 0`.

`GEN RgXX_to_RgM(GEN x, long N)` converts the `t_POL` `x` with `RgX` (or constant) coefficients to a matrix with `N` rows.

`long RgXY_degreeex(GEN P)` return the degree of `P` with respect to the secondary variable.

`GEN RgXY_derivx(GEN P)` return the derivative of `P` with respect to the secondary variable.

`GEN RgXY_swap(GEN P, long n, long w)` converts the bivariate polynomial  $P(u, v)$  (a `t_POL` with `t_POL` or scalar coefficients) to  $P(\text{pol\_x}[w], u)$ , assuming `n` is an upper bound for  $\deg_v(P)$ .

`GEN RgXY_swapspec(GEN C, long n, long w, long lP)` as `RgXY_swap` where the coefficients of `P` are given by `gel(C, 0), ..., gel(C, lP-1)`.

`GEN RgX_to_ser(GEN x, long l)` convert the `t_POL` `x` to a *shallow* `t_SER` of length  $l \geq 2$ . Unless the polynomial is an exact zero, the coefficient of lowest degree  $T^d$  of the result is not an exact zero (as per `isexactzero`). The remainder is  $O(T^{d+l-2})$ .

`GEN RgX_to_ser_inexact(GEN x, long l)` convert the `t_POL` `x` to a *shallow* `t_SER` of length  $l \geq 2$ . Unless the polynomial is zero, the coefficient of lowest degree  $T^d$  of the result is not zero (as per `equal0`). The remainder is  $O(T^{d+l-2})$ .

GEN `RgV_to_ser`(GEN `x`, long `v`, long `l`) convert the `t_VEC` `x`, to a *shallow* `t_SER` of length  $l \geq 2$ .

GEN `rfrac_to_ser`(GEN `F`, long `l`) applied to a `t_RFRAC` `F`, creates a `t_SER` of length  $l \geq 2$  congruent to `F`. Not memory-clean but suitable for `gerepileupto`.

GEN `rfrac_to_ser_i`(GEN `F`, long `l`) internal variant of `rfrac_to_ser`, neither memory-clean nor suitable for `gerepileupto`.

GEN `rfracrecip_to_ser_absolute`(GEN `F`, long `d`) applied to a `t_RFRAC` `F`, creates the `t_SER`  $F(1/t) + O(t^d)$ . Note that we use absolute and not relative precision here.

GEN `gtoser`(GEN `s`, long `v`, long `d`). This function is deprecated, kept for backward compatibility: it follows the semantic of `Ser(s,v)`, with `d = seriesprecision` implied and is hard to use as a general conversion function. Use `gtoser_prec` instead.

It converts the object `s` into a `t_SER` with main variable number `v` and  $d > 0$  significant terms, but the argument `d` is sometimes ignored. More precisely

- if `s` is a scalar (with respect to variable `v`), we return a constant power series with  $d$  significant terms;
- if `s` is a `t_POL` in variable `v`, it is truncated to  $d$  terms if needed;
- if `s` is a vector, the coefficients of the vector are understood to be the coefficients of the power series starting from the constant term (as in `Polrev`), and the precision  $d$  is *ignored*;
- if `s` is already a power series in `v`, we return a copy, and the precision  $d$  is again *ignored*.

GEN `gtoser_prec`(GEN `s`, long `v`, long `d`) this function is a variant of `gtoser` following the semantic of `Ser(s,v,d)`: the precision  $d$  is always taken into account.

GEN `gtocol`(GEN `x`) converts the object `x` into a `t_COL`

GEN `gtomat`(GEN `x`) converts the object `x` into a `t_MAT`.

GEN `gtovec`(GEN `x`) converts the object `x` into a `t_VEC`.

GEN `gtovecsmall`(GEN `x`) converts the object `x` into a `t_VECSMALL`.

GEN `normalizeser`(GEN `x`) applied to an unnormalized `t_SER` `x` (i.e. type `t_SER` with all coefficients correctly set except that `x[2]` might be zero), normalizes `x` correctly in place. Returns `x`. For internal use.

GEN `serchop0`(GEN `s`) given a `t_SER` of the form  $x^v s(x)$ , with  $s(0) \neq 0$ , return  $x^v (s - s(0))$ . Shallow function.

GEN `serchop_i`(GEN `x`, long `n`) returns a shallow copy of `t_SER` `x` with all terms of degree strictly less than  $n$  removed. Shallow version of `serchop`.

## 9.3 Constructors.

### 9.3.1 Clean constructors.

GEN `zeropadic`(GEN `p`, long `n`) creates a 0 `t_PADIC` equal to  $O(p^n)$ .

GEN `zeroser`(long `v`, long `n`) creates a 0 `t_SER` in variable `v` equal to  $O(X^n)$ .

GEN `scalarser`(GEN `x`, long `v`, long `prec`) creates a constant `t_SER` in variable `v` and precision `prec`, whose constant coefficient is (a copy of) `x`, in other words  $x + O(v^{\text{prec}})$ . Assumes that `prec`  $\geq 0$ .

GEN `pol_0`(long `v`) Returns the constant polynomial 0 in variable `v`.

GEN `pol_1`(long `v`) Returns the constant polynomial 1 in variable `v`.

GEN `pol_x`(long `v`) Returns the monomial of degree 1 in variable `v`.

GEN `pol_xn`(long `n`, long `v`) Returns the monomial of degree `n` in variable `v`; assume that `n`  $\geq 0$ .

GEN `pol_xnall`(long `n`, long `v`) Returns the Laurent monomial of degree `n` in variable `v`; `n`  $< 0$  is allowed.

GEN `pol_x_powers`(long `N`, long `v`) returns the powers of `pol_x(v)`, of degree 0 to `N`  $- 1$ , in a vector with `N` components.

GEN `scalarpol`(GEN `x`, long `v`) creates a constant `t_POL` in variable `v`, whose constant coefficient is (a copy of) `x`.

GEN `deg1pol`(GEN `a`, GEN `b`, long `v`) creates the degree 1 `t_POL`  $a\text{pol}_x(v) + b$

GEN `zeropol`(long `v`) is identical `pol_0`.

GEN `zerocol`(long `n`) creates a `t_COL` with `n` components set to `gen_0`.

GEN `zerovec`(long `n`) creates a `t_VEC` with `n` components set to `gen_0`.

GEN `zerovec_block`(long `n`) as `zerovec` but return a clone.

GEN `col_ei`(long `n`, long `i`) creates a `t_COL` with `n` components set to `gen_0`, but for the `i`-th one which is set to `gen_1` (`i`-th vector in the canonical basis).

GEN `vec_ei`(long `n`, long `i`) creates a `t_VEC` with `n` components set to `gen_0`, but for the `i`-th one which is set to `gen_1` (`i`-th vector in the canonical basis).

GEN `trivial_fact`(void) returns the trivial (empty) factorization `Mat([ ]~, [ ]~)`

GEN `prime_fact`(GEN `x`) returns the factorization `Mat([x]~, [1]~)`

GEN `Rg_col_ei`(GEN `x`, long `n`, long `i`) creates a `t_COL` with `n` components set to `gen_0`, but for the `i`-th one which is set to `x`.

GEN `vecsmall_ei`(long `n`, long `i`) creates a `t_VECSMALL` with `n` components set to 0, but for the `i`-th one which is set to 1 (`i`-th vector in the canonical basis).

GEN `scalarcol`(GEN `x`, long `n`) creates a `t_COL` with `n` components set to `gen_0`, but the first one which is set to a copy of `x`. (The name comes from `RgV_isscalar`.)

GEN `mkintmodu`(ulong `x`, ulong `y`) creates the `t_INTMOD` `Mod(x, y)`. The inputs must satisfy  $x < y$ .

GEN `zeromat(long m, long n)` creates a `t_MAT` with  $m \times n$  components set to `gen_0`. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns. To fully allocate a matrix initialized with zero entries, use `zeromacopy`.

GEN `zeromacopy(long m, long n)` creates a `t_MAT` with  $m \times n$  components set to `gen_0`.

GEN `matid(long n)` identity matrix in dimension  $n$  (with components `gen_1` and `gen_0`).

GEN `scalarmat(GEN x, long n)` scalar matrix,  $x$  times the identity.

GEN `scalarmat_s(long x, long n)` scalar matrix, `stoi(x)` times the identity.

GEN `vecrange(GEN a, GEN b)` returns the `t_VEC`  $[a..b]$ .

GEN `vecrangess(long a, long b)` returns the `t_VEC`  $[a..b]$ .

See also next section for analogs of the following functions:

GEN `mkfracss(long x, long y)` creates the `t_FRAC`  $x/y$ . Assumes that  $y > 1$  and  $(x, y) = 1$ .

GEN `sstoQ(long x, long y)` returns the `t_INT` or `t_FRAC`  $x/y$ ; no assumptions.

GEN `uutoQ(ulong x, ulong y)` returns the `t_INT` or `t_FRAC`  $x/y$ ; no assumptions.

void `Qtoss(GEN q, long *n, long *d)` given a `t_INT` or `t_FRAC`  $q$ , set  $n$  and  $d$  such that  $q = n/d$  with  $d \geq 1$  and  $(n, d) = 1$ . Overflow error if numerator or denominator do not fit into a long integer.

GEN `mkfraccopy(GEN x, GEN y)` creates the `t_FRAC`  $x/y$ . Assumes that  $y > 1$  and  $(x, y) = 1$ .

GEN `mkrfraccopy(GEN x, GEN y)` creates the `t_RFRAC`  $x/y$ . Assumes that  $y$  is a `t_POL`,  $x$  a compatible type whose variable has lower or same priority, with  $(x, y) = 1$ .

GEN `mkcolcopy(GEN x)` creates a 1-dimensional `t_COL` containing  $x$ .

GEN `mkmatcopy(GEN x)` creates a 1-by-1 `t_MAT` wrapping the `t_COL`  $x$ .

GEN `mkveccopy(GEN x)` creates a 1-dimensional `t_VEC` containing  $x$ .

GEN `mkvec2copy(GEN x, GEN y)` creates a 2-dimensional `t_VEC` equal to  $[x, y]$ .

GEN `mkcols(long x)` creates a 1-dimensional `t_COL` containing `stoi(x)`.

GEN `mkcol2s(long x, long y)` creates a 2-dimensional `t_COL` containing  $[\text{stoi}(x), \text{stoi}(y)]$ .

GEN `mkcol3s(long x, long y, long z)` creates a 3-dimensional `t_COL` containing  $[\text{stoi}(x), \text{stoi}(y), \text{stoi}(z)]$ .

GEN `mkcol4s(long x, long y, long z, long t)` creates a 4-dimensional `t_COL` containing  $[\text{stoi}(x), \text{stoi}(y), \text{stoi}(z), \text{stoi}(t)]$ .

GEN `mkvecs(long x)` creates a 1-dimensional `t_VEC` containing `stoi(x)`.

GEN `mkvec2s(long x, long y)` creates a 2-dimensional `t_VEC` containing  $[\text{stoi}(x), \text{stoi}(y)]$ .

GEN `mkmat22s(long a, long b, long c, long d)` creates the 2 by 2 `t_MAT` with successive rows  $[\text{stoi}(a), \text{stoi}(b)]$  and  $[\text{stoi}(c), \text{stoi}(d)]$ .

GEN `mkvec3s(long x, long y, long z)` creates a 3-dimensional `t_VEC` containing  $[\text{stoi}(x), \text{stoi}(y), \text{stoi}(z)]$ .

GEN `mkvec4s(long x, long y, long z, long t)` creates a 4-dimensional `t_VEC` containing  $[\text{stoi}(x), \text{stoi}(y), \text{stoi}(z), \text{stoi}(t)]$ .

GEN `mkvecsmall(long x)` creates a 1-dimensional `t_VECSMALL` containing `x`.

GEN `mkvecsmall2(long x, long y)` creates a 2-dimensional `t_VECSMALL` containing `[x, y]`.

GEN `mkvecsmall3(long x, long y, long z)` creates a 3-dimensional `t_VECSMALL` containing `[x, y, z]`.

GEN `mkvecsmall4(long x, long y, long z, long t)` creates a 4-dimensional `t_VECSMALL` containing `[x, y, z, t]`.

GEN `mkvecsmall5(long x, long y, long z, long t, long u)` creates a 5-dimensional `t_VECSMALL` containing `[x, y, z, t, u]`.

GEN `mkvecsmalln(long n, ...)` returns the `t_VECSMALL` whose  $n$  coefficients (`long`) follow.  
*Warning:* since this is a variadic function, C type promotion is not performed on the arguments by the compiler, thus you have to make sure that all the arguments are of type `long`, in particular integer constants need to be written with the L suffix: `mkvecsmalln(2, 1L, 2L)` is correct, but `mkvecsmalln(2, 1, 2)` is not.

### 9.3.2 Unclean constructors.

Contrary to the policy of general PARI functions, the functions in this subsection do *not* copy their arguments, nor do they produce an object a priori suitable for `gerepileupto`. In particular, they are faster than their clean equivalent (which may not exist). *If* you restrict their arguments to universal objects (e.g `gen_0`), then the above warning does not apply.

GEN `mkcomplex(GEN x, GEN y)` creates the `t_COMPLEX`  $x + iy$ .

GEN `mulcxI(GEN x)` creates the `t_COMPLEX`  $ix$ . The result in general contains data pointing back to the original  $x$ . Use `gcop` if this is a problem. But in most cases, the result is to be used immediately, before  $x$  is subject to garbage collection.

GEN `mulcxmI(GEN x)`, as `mulcxI`, but returns  $-ix$ .

GEN `mulcxpowIs(GEN x, long k)`, as `mulcxI`, but returns  $x \cdot i^k$ .

GEN `mkquad(GEN n, GEN x, GEN y)` creates the `t_QUAD`  $x + yw$ , where  $w$  is a root of  $n$ , which is of the form `quadpoly(D)`.

GEN `quadpoly_i(GEN D)` creates the canonical quadratic polynomial of discriminant  $D$ . Assume that the `t_INT`  $D$  is congruent to 0, 1 mod 4 and not a square.

GEN `mkfrac(GEN x, GEN y)` creates the `t_FRAC`  $x/y$ . Assumes that  $y > 1$  and  $(x, y) = 1$ .

GEN `mkrfrac(GEN x, GEN y)` creates the `t_RFRAC`  $x/y$ . Assumes that  $y$  is a `t_POL`,  $x$  a compatible type whose variable has lower or same priority, with  $(x, y) = 1$ .

GEN `mkcol(GEN x)` creates a 1-dimensional `t_COL` containing `x`.

GEN `mkcol2(GEN x, GEN y)` creates a 2-dimensional `t_COL` equal to `[x, y]`.

GEN `mkcol3(GEN x, GEN y, GEN z)` creates a 3-dimensional `t_COL` equal to `[x, y, z]`.

GEN `mkcol4(GEN x, GEN y, GEN z, GEN t)` creates a 4-dimensional `t_COL` equal to `[x, y, z, t]`.

GEN `mkcol5(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5)` creates the 5-dimensional `t_COL` equal to `[a1, a2, a3, a4, a5]`.

GEN `mkcol6(GEN x, GEN y, GEN z, GEN t, GEN u, GEN v)` creates the 6-dimensional column vector `[x, y, z, t, u, v]`.



GEN mkintmod(GEN x, GEN y) creates the  $t\_INTMOD \text{Mod}(x, y)$ . The inputs must be  $t\_INTs$  satisfying  $0 \leq x < y$ .

GEN mkpolmod(GEN x, GEN y) creates the  $t\_POLMOD \text{Mod}(x, y)$ . The input must satisfy  $\deg x < \deg y$  with respect to the main variable of the  $t\_POL$   $y$ .  $x$  may be a scalar.

GEN mkmat(GEN x) creates a 1-column  $t\_MAT$  with column  $x$  (a  $t\_COL$ ).

GEN mkmat2(GEN x, GEN y) creates a 2-column  $t\_MAT$  with columns  $x, y$  ( $t\_COLs$  of the same length).

GEN mkmat22(GEN a, GEN b, GEN c, GEN d) creates the 2 by 2  $t\_MAT$  with successive rows  $[a, b]$  and  $[c, d]$ .

GEN mkmat3(GEN x, GEN y, GEN z) creates a 3-column  $t\_MAT$  with columns  $x, y, z$  ( $t\_COLs$  of the same length).

GEN mkmat4(GEN x, GEN y, GEN z, GEN t) creates a 4-column  $t\_MAT$  with columns  $x, y, z, t$  ( $t\_COLs$  of the same length).

GEN mkmat5(GEN x, GEN y, GEN z, GEN t, GEN u) creates a 5-column  $t\_MAT$  with columns  $x, y, z, t, u$  ( $t\_COLs$  of the same length).

GEN mkvec(GEN x) creates a 1-dimensional  $t\_VEC$  containing  $x$ .

GEN mkvec2(GEN x, GEN y) creates a 2-dimensional  $t\_VEC$  equal to  $[x, y]$ .

GEN mkvec3(GEN x, GEN y, GEN z) creates a 3-dimensional  $t\_VEC$  equal to  $[x, y, z]$ .

GEN mkvec4(GEN x, GEN y, GEN z, GEN t) creates a 4-dimensional  $t\_VEC$  equal to  $[x, y, z, t]$ .

GEN mkvec5(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5) creates the 5-dimensional  $t\_VEC$  equal to  $[a_1, a_2, a_3, a_4, a_5]$ .

GEN mkqfb(GEN a, GEN b, GEN c, GEN D) creates  $t\_QFB$  equal to  $Qfb(a, b, c)$ , assuming that  $D = b^2 - 4ac$ .

GEN mkerr(long n) returns a  $t\_ERROR$  with error code  $n$  (enum `err_list`).

It is sometimes useful to return such a container whose entries are not universal objects, but nonetheless suitable for `gerepileupto`. If the entries can be computed at the time the result is returned, the following macros achieve this effect:

GEN retmkvec(GEN x) returns a vector containing the single entry  $x$ , where the vector root is created just before the function argument  $x$  is evaluated. Expands to

```
{
  GEN res = cgetg(2, t_VEC);
  gel(res, 1) = x; /* or rather, the expansion of  $x$  */
  return res;
}
```

For instance, the `retmkvec(gcopy(x))` returns a clean object, just like `return mkveccopy(x)` would.

GEN retmkvec2(GEN x, GEN y) returns the 2-dimensional  $t\_VEC$   $[x, y]$ .

GEN retmkvec3(GEN x, GEN y, GEN z) returns the 3-dimensional  $t\_VEC$   $[x, y, z]$ .

`GEN retmkvec4(GEN x, GEN y, GEN z, GEN t)` returns the 4-dimensional `t_VEC` `[x,y,z,t]`.  
`GEN retmkvec5(GEN x, GEN y, GEN z, GEN t, GEN u)` returns the 5-dimensional row vector `[x,y,z,t,u]`.  
`GEN retconst_vec(long n, GEN x)` returns the  $n$ -dimensional `t_VEC` whose entries are constant and all equal to  $x$ .  
`GEN retmkcol(GEN x)` returns the 1-dimensional `t_COL` `[x]` .  
`GEN retmkcol2(GEN x, GEN y)` returns the 2-dimensional `t_COL` `[x,y]` .  
`GEN retmkcol3(GEN x, GEN y, GEN z)` returns the 3-dimensional `t_COL` `[x,y,z]` .  
`GEN retmkcol4(GEN x, GEN y, GEN z, GEN t)` returns the 4-dimensional `t_COL` `[x,y,z,t]` .  
`GEN retmkcol5(GEN x, GEN y, GEN z, GEN t, GEN u)` returns the 5-dimensional column vector `[x,y,z,t,u]` .  
`GEN retmkcol6(GEN x, GEN y, GEN z, GEN t, GEN u, GEN v)` returns the 6-dimensional column vector `[x,y,z,t,u,v]` .  
`GEN retconst_col(long n, GEN x)` returns the  $n$ -dimensional `t_COL` whose entries are constant and all equal to  $x$ .  
`GEN retmkmat(GEN x)` returns the 1-column `t_MAT` with column  $x$ .  
`GEN retmkmat2(GEN x, GEN y)` returns the 2-column `t_MAT` with columns  $x, y$ .  
`GEN retmkmat3(GEN x, GEN y, GEN z)` returns the 3-dimensional `t_MAT` with columns  $x, y, z$ .  
`GEN retmkmat4(GEN x, GEN y, GEN z, GEN t)` returns the 4-dimensional `t_MAT` with columns  $x, y, z, t$ .  
`GEN retmkmat5(GEN x, GEN y, GEN z, GEN t, GEN u)` returns the 5-dimensional `t_MAT` with columns  $x, y, z, t, u$ .  
`GEN retmkcomplex(GEN x, GEN y)` returns the `t_COMPLEX`  $x + I*y$ .  
`GEN retmkfrac(GEN x, GEN y)` returns the `t_FRAC`  $x / y$ . Assume  $x$  and  $y$  are coprime and  $y > 1$ .  
`GEN retmkrffrac(GEN x, GEN y)` returns the `t_RFRAC`  $x / y$ . Assume  $x$  and  $y$  are coprime and more generally that the rational function cannot be simplified.  
`GEN retmkintmod(GEN x, GEN y)` returns the `t_INTMOD` `Mod(x, y)`.  
`GEN retmkquad(GEN n, GEN a, GEN b)`.  
`GEN retmkpolmod(GEN x, GEN y)` returns the `t_POLMOD` `Mod(x, y)`.  
`GEN mkintn(long n, ...)` returns the nonnegative `t_INT` whose development in base  $2^{32}$  is given by the following  $n$  32bit-words (`unsigned int`).  
`mkintn(3, a2, a1, a0);`  
returns  $a_2 2^{64} + a_1 2^{32} + a_0$ .  
`GEN mkpoln(long n, ...)` Returns the `t_POL` whose  $n$  coefficients (`GEN`) follow, in order of decreasing degree.  
`mkpoln(3, gen_1, gen_2, gen_0);`

returns the polynomial  $X^2 + 2X$  (in variable 0, use `setvarn` if you want other variable numbers). Beware that  $n$  is the number of coefficients, hence *one more* than the degree.

GEN `mkvecn(long n, ...)` returns the `t_VEC` whose  $n$  coefficients (GEN) follow.

GEN `mkcoln(long n, ...)` returns the `t_COL` whose  $n$  coefficients (GEN) follow.

GEN `scalarcoll_shallow(GEN x, long n)` creates a `t_COL` with  $n$  components set to `gen_0`, but the first one which is set to a shallow copy of  $x$ . (The name comes from `RgV_isscalar`.)

GEN `scalarmat_shallow(GEN x, long n)` creates an  $n \times n$  scalar matrix whose diagonal is set to shallow copies of the scalar  $x$ .

GEN `RgX_sylvestermatrix(GEN f, GEN g)` return the Sylvester matrix attached to the two `t_POL` in the same variable  $f$  and  $g$ .

GEN `diagonal_shallow(GEN x)` returns a diagonal matrix whose diagonal is given by the vector  $x$ . Shallow function.

GEN `scalarpol_shallow(GEN a, long v)` returns the degree 0 `t_POL`  $a\text{pol\_x}(v)^0$ .

GEN `deg1pol_shallow(GEN a, GEN b, long v)` returns the degree 1 `t_POL`  $a\text{pol\_x}(v) + b$

GEN `deg2pol_shallow(GEN a, GEN b, GEN c, long v)` returns the degree 2 `t_POL`  $ax^2 + bx + c$  where  $x = \text{pol\_x}(v)$ .

GEN `zeropadic_shallow(GEN p, long n)` returns a (shallow) 0 `t_PADIC` equal to  $O(\mathfrak{p}^n)$ .

### 9.3.3 From roots to polynomials.

GEN `deg1_from_roots(GEN L, long v)` given a vector  $L$  of scalars, returns the vector of monic linear polynomials in variable  $v$  whose roots are the  $L[i]$ , i.e. the  $x - L[i]$ .

GEN `roots_from_deg1(GEN L)` given a vector  $L$  of monic linear polynomials, return their roots, i.e. the  $-L[i](0)$ .

GEN `roots_to_pol(GEN L, long v)` given a vector of scalars  $L$ , returns the monic polynomial in variable  $v$  whose roots are the  $L[i]$ . Leaves some garbage on stack, but suitable for `gerepileupto`.

GEN `roots_to_pol_r1(GEN L, long v, long r1)` as `roots_to_pol` assuming the first  $r_1$  roots are “real”, and the following ones are representatives of conjugate pairs of “complex” roots. So if  $L$  has  $r_1 + r_2$  elements, we obtain a polynomial of degree  $r_1 + 2r_2$ . In most applications, the roots are indeed real and complex, but the implementation assumes only that each “complex” root  $z$  introduces a quadratic factor  $X^2 - \text{trace}(z)X + \text{norm}(z)$ . Leaves some garbage on stack, but suitable for `gerepileupto`.

## 9.4 Integer parts.

GEN `gfloor`(GEN `x`) creates the floor of `x`, i.e. the (true) integral part.

GEN `gfrac`(GEN `x`) creates the fractional part of `x`, i.e. `x` minus the floor of `x`.

GEN `gceil`(GEN `x`) creates the ceiling of `x`.

GEN `ground`(GEN `x`) rounds towards  $+\infty$  the components of `x` to the nearest integers.

GEN `grndtoi`(GEN `x`, long `*e`) same as `ground`, but in addition sets `*e` to the binary exponent of  $x - \text{ground}(x)$ . If this is positive, then significant bits are lost in the rounded result. This kind of situation raises an error message in `ground` but not in `grndtoi`. The parameter `e` can be set to NULL if an error estimate is not needed, for a minor speed up.

GEN `gtrunc`(GEN `x`) truncates `x`. This is the false integer part if `x` is a real number (i.e. the unique integer closest to `x` among those between 0 and `x`). If `x` is a `t_SER`, it is truncated to a `t_POL`; if `x` is a `t_RFRAC`, this takes the polynomial part.

GEN `gtrunc2n`(GEN `x`, long `n`) creates the floor of  $2^n x$ , this is only implemented for `t_INT`, `t_REAL`, `t_FRAC` and `t_COMPLEX` of those.

GEN `gcvttoi`(GEN `x`, long `*e`) analogous to `grndtoi` for `t_REAL` inputs except that rounding is replaced by truncation. Also applies componentwise for vector or matrix inputs; otherwise, sets `*e` to `-HIGHEXPOBIT` (infinite real accuracy) and return `gtrunc(x)`.

## 9.5 Valuation and shift.

GEN `gshift`[`z`](GEN `x`, long `n`[, GEN `z`]) yields the result of shifting (the components of) `x` left by `n` (if `n` is nonnegative) or right by  $-\text{n}$  (if `n` is negative). Applies only to `t_INT` and vectors/matrices of such. For other types, it is simply multiplication by  $2^n$ .

GEN `gmul2n`[`z`](GEN `x`, long `n`[, GEN `z`]) yields the product of `x` and  $2^n$ . This is different from `gshift` when `n` is negative and `x` is a `t_INT`: `gshift` truncates, while `gmul2n` creates a fraction if necessary.

long `gvaluation`(GEN `x`, GEN `p`) returns the greatest exponent  $e$  such that  $p^e$  divides `x`, when this makes sense.

long `gval`(GEN `x`, long `v`) returns the highest power of the variable number `v` dividing the `t_POL` `x`.

## 9.6 Comparison operators.

### 9.6.1 Generic.

`long gcmp(GEN x, GEN y)` comparison of  $x$  with  $y$ : returns 1 ( $x > y$ ), 0 ( $x = y$ ) or  $-1$  ( $x < y$ ). Two `t_STR` are compared using the standard lexicographic ordering; a `t_STR` cannot be compared to any non-string type. If neither  $x$  nor  $y$  is a `t_STR`, their allowed types are `t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` with positive discriminant (use the canonical embedding  $w \rightarrow \sqrt{D}/2$  or  $w \rightarrow (1 + \sqrt{D})/2$ ) or `t_INFINITY`. Use `cmp_universal` to compare arbitrary GENs.

`long lexcmp(GEN x, GEN y)` comparison of  $x$  with  $y$  for the lexicographic ordering; when comparing objects of different lengths whose components are all equal up to the smallest of their length, consider that the longest is largest. Consider scalars as 1-component vectors. Return `gcmp(x, y)` if both arguments are scalars.

`int gequalX(GEN x)` return 1 (true) if  $x$  is a variable (monomial of degree 1 with `t_INT` coefficients equal to 1 and 0), and 0 otherwise

`long gequal(GEN x, GEN y)` returns 1 (true) if  $x$  is equal to  $y$ , 0 otherwise. A priori, this makes sense only if  $x$  and  $y$  have the same type, in which case they are recursively compared componentwise. When the types are different, a `true` result means that  $x - y$  was successfully computed and that `gequal0` found it equal to 0. In particular

`gequal(cgetg(1, t_VEC), gen_0)`

is true, and the relation is not transitive. E.g. an empty `t_COL` and an empty `t_VEC` are not equal but are both equal to `gen_0`.

`long gidentical(GEN x, GEN y)` returns 1 (true) if  $x$  is identical to  $y$ , 0 otherwise. In particular, the types and length of  $x$  and  $y$  must be equal. This test is much stricter than `gequal`, in particular, `t_REAL` with different accuracies are tested different. This relation is transitive.

`GEN gmax(GEN x, GEN y)` returns a copy of the maximum of  $x$  and  $y$ , compared using `gcmp`.

`GEN gmin(GEN x, GEN y)` returns a copy of the minimum of  $x$  and  $y$ , compared using `gcmp`.

`GEN gmax_shallow(GEN x, GEN y)` shallow version of `gmax`.

`GEN gmin_shallow(GEN x, GEN y)` shallow version of `gmin`.

### 9.6.2 Comparison with a small integer.

`int isexactzero(GEN x)` returns 1 (true) if  $x$  is exactly equal to 0 (including `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. This includes recursive objects, for instance vectors, whose components are 0.

`GEN gisexactzero(GEN x)` returns `NULL` unless  $x$  is exactly equal to 0 (as per `isexactzero`). When  $x$  is an exact zero return the attached scalar zero as a `t_INT` (`gen_0`), a `t_INTMOD` (`Mod(0,N)` for the largest possible  $N$ ) or a `t_FFELT`.

`int isrationalzero(GEN x)` returns 1 (true) if  $x$  is equal to an integer 0 (excluding `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. Contrary to `isintzero`, this includes recursive objects, for instance vectors, whose components are 0.

`int ismpzzero(GEN x)` returns 1 (true) if  $x$  is a `t_INT` or a `t_REAL` equal to 0.

`int isintzero(GEN x)` returns 1 (true) if  $x$  is a `t_INT` equal to 0.

`int isint1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 1.

`int isintm1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to  $-1$ .

`int equali1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to 1, and return 0 (false) otherwise.

`int equalim1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to  $-1$ , and return 0 (false) otherwise.

`int is_pm1(GEN x)`. Assuming that `x` is a *nonzero* `t_INT`, return 1 (true) if `x` is equal to  $-1$  or 1, and return 0 (false) otherwise.

`int gequal0(GEN x)` returns 1 (true) if `x` is equal to 0, 0 (false) otherwise.

`int gequal1(GEN x)` returns 1 (true) if `x` is equal to 1, 0 (false) otherwise.

`int gequalm1(GEN x)` returns 1 (true) if `x` is equal to  $-1$ , 0 (false) otherwise.

`long gcmpsg(long s, GEN x)`

`long gcmpgs(GEN x, long s)` comparison of `x` with the `long s`.

`GEN gmaxsg(long s, GEN x)`

`GEN gmaxgs(GEN x, long s)` returns the largest of `x` and the `long s` (converted to `GEN`)

`GEN gminsg(long s, GEN x)`

`GEN gmings(GEN x, long s)` returns the smallest of `x` and the `long s` (converted to `GEN`)

`long gequalsg(long s, GEN x)`

`long gequalgs(GEN x, long s)` returns 1 (true) if `x` is equal to the `long s`, 0 otherwise.

## 9.7 Miscellaneous Boolean functions.

`int isrationalzeroscalar(GEN x)` equivalent to, but faster than,

`is_scalar_t(typ(x)) && isrationalzero(x)`

`int isinexact(GEN x)` returns 1 (true) if `x` has an inexact component, and 0 (false) otherwise.

`int isinexactreal(GEN x)` return 1 if `x` has an inexact `t_REAL` component, and 0 otherwise.

`int isrealappr(GEN x, long e)` applies (recursively) to complex inputs; returns 1 if `x` is approximately real to the bit accuracy `e`, and 0 otherwise. This means that any `t_COMPLEX` component must have imaginary part `t` satisfying `gexpo(t) < e`.

`int isint(GEN x, GEN *n)` returns 0 (false) if `x` does not round to an integer. Otherwise, returns 1 (true) and set `n` to the rounded value.

`int issmall(GEN x, long *n)` returns 0 (false) if `x` does not round to a small integer (suitable for `itos`). Otherwise, returns 1 (true) and set `n` to the rounded value.

`long iscomplex(GEN x)` returns 1 (true) if `x` is a complex number (of component types embeddable into the reals) but is not itself real, 0 if `x` is a real (not necessarily of type `t_REAL`), or raises an error if `x` is not embeddable into the complex numbers.

### 9.7.1 Obsolete.

The following less convenient comparison functions and Boolean operators were used by the historical GP interpreter. They are provided for backward compatibility only and should not be used:

```
GEN gle(GEN x, GEN y)
GEN glt(GEN x, GEN y)
GEN gge(GEN x, GEN y)
GEN ggt(GEN x, GEN y)
GEN geq(GEN x, GEN y)
GEN gne(GEN x, GEN y)
GEN gor(GEN x, GEN y)
GEN gand(GEN x, GEN y)
GEN gnot(GEN x, GEN y)
```

## 9.8 Sorting.

### 9.8.1 Basic sort.

`GEN sort(GEN x)` sorts the vector `x` in ascending order using a mergesort algorithm, and `gcmp` as the underlying comparison routine (returns the sorted vector). This routine copies all components of `x`, use `gen_sort_inplace` for a more memory-efficient function.

`GEN lexsort(GEN x)`, as `sort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

`GEN vecsort(GEN x, GEN k)`, as `sort`, but sorts the vector `x` in ascending *lexicographic* order, according to the entries of the `t_VECSMALL` `k`. For example, if `k = [2, 1, 3]`, sorting will be done with respect to the second component, and when these are equal, with respect to the first, and when these are equal, with respect to the third.

### 9.8.2 Indirect sorting.

`GEN indexsort(GEN x)` as `sort`, but only returns the permutation which, applied to `x`, would sort the vector. The result is a `t_VECSMALL`.

`GEN indexlexsort(GEN x)`, as `indexsort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

`GEN indexvecsort(GEN x, GEN k)`, as `vecsort`, but only returns the permutation that would sort the vector `x`.

`long vecindexmin(GEN x)` returns the index for a minimal element of `x` (`t_VEC`, `t_COL` or `t_VECSMALL`).

`long vecindexmax(GEN x)` returns the index for a maximal element of `x` (`t_VEC`, `t_COL` or `t_VECSMALL`).

**9.8.3 Generic sort and search.** The following routines allow to use an arbitrary comparison function `int (*cmp)(void* data, GEN x, GEN y)`, such that `cmp(data,x,y)` returns a negative result if  $x < y$ , a positive one if  $x > y$  and 0 if  $x = y$ . The `data` argument is there in case your `cmp` requires additional context.

`GEN gen_sort(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, as `sort`, with an explicit comparison routine.

`GEN gen_sort_shallow(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, shallow variant of `gen_sort`.

`GEN gen_sort_uniq(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, as `gen_sort`, removing duplicate entries.

`GEN gen_indexsort(GEN x, void *data, int (*cmp)(void*, GEN, GEN))`, as `indexsort`.

`GEN gen_indexsort_uniq(GEN x, void *data, int (*cmp)(void*, GEN, GEN))`, as `indexsort`, removing duplicate entries.

`void gen_sort_inplace(GEN x, void *data, int (*cmp)(void*, GEN, GEN), GEN *perm)`  
sort `x` in place, without copying its components. If `perm` is not `NULL`, it is set to the permutation that would sort the original `x`.

`GEN gen_setminus(GEN A, GEN B, int (*cmp)(GEN, GEN))` given two sorted vectors  $A$  and  $B$ , returns the vector of elements of  $A$  not belonging to  $B$ .

`GEN sort_factor(GEN y, void *data, int (*cmp)(void *, GEN, GEN))`: assuming `y` is a factorization matrix, sorts its rows in place (no copy is made) according to the comparison function `cmp` applied to its first column.

`GEN merge_sort_uniq(GEN x, GEN y, void *data, int (*cmp)(void *, GEN, GEN))` assuming `x` and `y` are sorted vectors, with respect to the `cmp` comparison function, return a sorted concatenation, with duplicates removed. Shallow function.

`GEN setunion_i(GEN x, GEN y)` shallow version of `setunion`, a simple alias for

`merge_sort_uniq(x,y, (void*)cmp_universal, cmp_nodata)`

`GEN merge_factor(GEN fx, GEN fy, void *data, int (*cmp)(void *, GEN, GEN))` let `fx` and `fy` be factorization matrices for  $X$  and  $Y$  sorted with respect to the comparison function `cmp` (see `sort_factor`), returns the factorization of  $X * Y$ .

`long gen_search(GEN v, GEN y, void *data, int (*cmp)(void*, GEN, GEN))`.

Let  $v$  be a vector sorted according to `cmp(data,a,b)`; look for an index  $i$  such that  $v[i]$  is equal to  $y$ . If  $y$  is found, return  $i$  (not necessarily the first occurrence in case of multisets), else return  $-i$  where  $i$  is the index where  $y$  should be inserted.

`long tablesearch(GEN T, GEN x, int (*cmp)(GEN, GEN))` is a faster implementation for the common case `gen_search(T,x,cmp,cmp_nodata)` when we have no need to insert missing elements; return 0 in case  $x$  is not found.



### 9.8.4 Further useful comparison functions.

`int cmp_universal(GEN x, GEN y)` a somewhat arbitrary universal comparison function, devoid of sensible mathematical meaning. It is transitive, and returns 0 if and only if `gidentical(x,y)` is true. Useful to sort and search vectors of arbitrary data.

`int cmp_nodata(void *data, GEN x, GEN y)`. This function is a hack used to pass an existing basic comparison function lacking the `data` argument, i.e. with prototype `int (*cmp)(GEN x, GEN y)`. Instead of `gen_sort(x, NULL, cmp)` which may or may not work depending on how your compiler handles typecasts between incompatible function pointers, one should use `gen_sort(x, (void*)cmp, cmp_nodata)`.

Here are a few basic comparison functions, to be used with `cmp_nodata`:

`int ZV_cmp(GEN x, GEN y)` compare two ZV, which we assume have the same length (lexicographic order).

`int cmp_Flx(GEN x, GEN y)` compare two Flx, which we assume have the same main variable (lexicographic order).

`int cmp_RgX(GEN x, GEN y)` compare two polynomials, which we assume have the same main variable (lexicographic order). The coefficients are compared using `gcmp`.

`int cmp_prime_over_p(GEN x, GEN y)` compare two prime ideals, which we assume divide the same prime number. The comparison is ad hoc but orders according to increasing residue degrees.

`int cmp_prime_ideal(GEN x, GEN y)` compare two prime ideals in the same  $nf$ . Orders by increasing primes, breaking ties using `cmp_prime_over_p`.

`int cmp_padic(GEN x, GEN y)` compare two `t_PADIC` (for the same prime  $p$ ).

Finally a more elaborate comparison function:

`int gen_cmp_RgX(void *data, GEN x, GEN y)` compare two polynomials, ordering first by increasing degree, then according to the coefficient comparison function:

```
int (*cmp_coeff)(GEN,GEN) = (int (*)(GEN,GEN)) data;
```

## 9.9 Division.

`GEN gdivgu(GEN x, ulong u)` return  $x/u$ .

`GEN gdivgunextu(GEN x, ulong u)` return  $x/(u(u+1))$ . If  $u(u+1)$  does not fit into an `ulong`, it is created and left on the stack for efficiency.

`GEN divrunextu(GEN x, ulong i)` as `gdivgunextu` for a `t_REAL`  $x$ .

## 9.10 Divisibility, Euclidean division.

**GEN gdivexact**(GEN *x*, GEN *y*) returns the quotient  $x/y$ , assuming *y* divides *x*. Not stack clean if  $y = 1$  (we return *x*, not a copy).

**int gdvd**(GEN *x*, GEN *y*) returns 1 (true) if *y* divides *x*, 0 otherwise.

**GEN gdiventres**(GEN *x*, GEN *y*) creates a 2-component vertical vector whose components are the true Euclidean quotient and remainder of *x* and *y*.

**GEN gdivent[z]**(GEN *x*, GEN *y*[, GEN *z*]) yields the true Euclidean quotient of *x* and the **t\_INT** or **t\_POL** *y*, as per the  $\backslash$  GP operator.

**GEN gdiventsg**(long *s*, GEN *y*[, GEN *z*]), as **gdivent** except that *x* is a long.

**GEN gdiventgs[z]**(GEN *x*, long *s*[, GEN *z*]), as **gdivent** except that *y* is a long.

**GEN gmod[z]**(GEN *x*, GEN *y*[, GEN *z*]) yields the remainder of *x* modulo the **t\_INT** or **t\_POL** *y*, as per the % GP operator. A **t\_REAL** or **t\_FRAC** *y* is also allowed, in which case the remainder is the unique real  $r$  such that  $0 \leq r < |y|$  and  $y = qx + r$  for some (in fact unique) integer  $q$ .

**GEN gmodsg**(long *s*, GEN *y*[, GEN *z*]) as **gmod**, except *x* is a long.

**GEN gmodgs**(GEN *x*, long *s*[, GEN *z*]) as **gmod**, except *y* is a long.

**GEN gdivmod**(GEN *x*, GEN *y*, GEN *\*r*) If *r* is not equal to **NULL** or **ONLY\_REM**, creates the (false) Euclidean quotient of *x* and *y*, and puts (the address of) the remainder into *\*r*. If *r* is equal to **NULL**, do not create the remainder, and if *r* is equal to **ONLY\_REM**, create and output only the remainder. The remainder is created after the quotient and can be disposed of individually with a **cgiv(r)**.

**GEN poldivrem**(GEN *x*, GEN *y*, GEN *\*r*) same as **gdivmod** but specifically for **t\_POLs** *x* and *y*, not necessarily in the same variable. Either of *x* and *y* may also be scalars, treated as polynomials of degree 0.

**GEN gdeuc**(GEN *x*, GEN *y*) creates the Euclidean quotient of the **t\_POLs** *x* and *y*. Either of *x* and *y* may also be scalars, treated as polynomials of degree 0.

**GEN grem**(GEN *x*, GEN *y*) creates the Euclidean remainder of the **t\_POL** *x* divided by the **t\_POL** *y*. Either of *x* and *y* may also be scalars, treated as polynomials of degree 0.

**GEN gdivround**(GEN *x*, GEN *y*) if *x* and *y* are real (**t\_INT**, **t\_REAL**, **t\_FRAC**), return the rounded Euclidean quotient of *x* and *y* as per the  $\backslash/$  GP operator. Operate componentwise if *x* is a **t\_COL**, **t\_VEC** or **t\_MAT**. Otherwise as **gdivent**.

**GEN centermod\_i**(GEN *x*, GEN *y*, GEN *y2*), as **centermodii**, componentwise.

**GEN centermod**(GEN *x*, GEN *y*), as **centermod\_i**, except that *y2* is computed (and left on the stack for efficiency).

**GEN ginvmod**(GEN *x*, GEN *y*) creates the inverse of *x* modulo *y* when it exists. *y* must be of type **t\_INT** (in which case *x* is of type **t\_INT**) or **t\_POL** (in which case *x* is either a scalar type or a **t\_POL**).

## 9.11 GCD, content and primitive part.

### 9.11.1 Generic.

`GEN resultant(GEN x, GEN y)` creates the resultant of the `t_POLs` `x` and `y` computed using Sylvester's matrix (inexact inputs), a modular algorithm (inputs in  $\mathbf{Q}[X]$ ) or the subresultant algorithm, as optimized by Lazard and Ducos. Either of `x` and `y` may also be scalars (treated as polynomials of degree 0)

`GEN ggcd(GEN x, GEN y)` creates the GCD of `x` and `y`.

`GEN glcm(GEN x, GEN y)` creates the LCM of `x` and `y`.

`GEN gbezout(GEN x, GEN y, GEN *u, GEN *v)` returns the GCD of `x` and `y`, and puts (the addresses of) objects `u` and `v` such that  $ux + vy = \text{gcd}(x, y)$  into `*u` and `*v`.

`GEN subresext(GEN x, GEN y, GEN *U, GEN *V)` returns the resultant of `x` and `y`, and puts (the addresses of) polynomials `u` and `v` such that  $ux + vy = \text{Res}(x, y)$  into `*U` and `*V`.

`GEN content(GEN x)` returns the GCD of all the components of `x`.

`GEN primitive_part(GEN x, GEN *c)` sets `c` to `content(x)` and returns the primitive part  $x / c$ . A trivial content is set to `NULL`.

`GEN primpart(GEN x)` as above but the content is lost. (For efficiency, the content remains on the stack.)

`GEN denom_i(GEN x)` shallow version of `denom`.

`GEN numer_i(GEN x)` shallow version of `numer`.

### 9.11.2 Over the rationals.

`long Q_pval(GEN x, GEN p)` valuation at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`.

`long Q_lval(GEN x, ulong p)` same for `ulong p`.

`long Q_pvalrem(GEN x, GEN p, GEN *r)` returns the valuation  $e$  at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`. The quotient  $x/p^e$  is returned in `*r`.

`long Q_lvalrem(GEN x, ulong p, GEN *r)` same for `ulong p`.

`GEN Q_abs(GEN x)` absolute value of the `t_INT` or `t_FRAC` `x`.

`GEN Qdivii(GEN x, GEN y)`, assuming  $x$  and  $y$  are both of type `t_INT`, return the quotient  $x/y$  as a `t_INT` or `t_FRAC`; marginally faster than `gdiv`.

`GEN Qdivis(GEN x, long y)`, assuming  $x$  is an `t_INT`, return the quotient  $x/y$  as a `t_INT` or `t_FRAC`; marginally faster than `gdiv`.

`GEN Qdiviu(GEN x, ulong y)`, assuming  $x$  is an `t_INT`, return the quotient  $x/y$  as a `t_INT` or `t_FRAC`; marginally faster than `gdiv`.

`GEN Q_abs_shallow(GEN x)`  $x$  being a `t_INT` or a `t_FRAC`, returns a shallow copy of  $|x|$ , in particular returns  $x$  itself when  $x \geq 0$ , and `gneg(x)` otherwise.

`GEN Q_gcd(GEN x, GEN y)` gcd of the `t_INT` or `t_FRAC` `x` and `y`.

In the following functions, arguments belong to a  $M \otimes_{\mathbf{Z}} \mathbf{Q}$  for some natural  $\mathbf{Z}$ -module  $M$ , e.g. multivariate polynomials with integer coefficients (or vectors/matrices recursively built from such

objects), and an element of  $M$  is said to be *integral*. We are interested in contents, denominators, etc. with respect to this canonical integral structure; in particular, contents belong to  $\mathbf{Q}$ , denominators to  $\mathbf{Z}$ . For instance the  $\mathbf{Q}$ -content of  $(1/2)xy$  is  $(1/2)$ , and its  $\mathbf{Q}$ -denominator is 2, whereas `content` would return  $y/2$  and `denom` 1.

`GEN Q_content(GEN x)` the  $\mathbf{Q}$ -content of  $x$ .

`GEN Z_content(GEN x)` as `Q_content` but assume that all rationals are in fact `t_INTs` and return `NULL` when the content is 1. This function returns as soon as the content is found to equal 1.

`GEN Q_content_safe(GEN x)` as `Q_content`, returning `NULL` when the  $\mathbf{Q}$ -content is not defined (e.g. for a `t_REAL` or `t_INTMOD` component).

`GEN Q_denom(GEN x)` the  $\mathbf{Q}$ -denominator of  $x$ . Shallow function. Raises an `e_TYPE` error out when the notion is meaningless, e.g. for a `t_REAL` or `t_INTMOD` component.

`GEN Q_denom_safe(GEN x)` the  $\mathbf{Q}$ -denominator of  $x$ . Shallow function. Return `NULL` when the notion is meaningless.

`GEN Q_primitive_part(GEN x, GEN *c)` sets  $c$  to the  $\mathbf{Q}$ -content of  $x$  and returns  $x / c$ , which is integral.

`GEN Q_primpart(GEN x)` as above but the content is lost. (For efficiency, the content remains on the stack.)

`GEN vec_Q_primpart(GEN x)` as above component-wise. Applied to a `t_MAT`, the result has primitive columns.

`GEN row_Q_primpart(GEN x)` as above, applied to the rows of a `t_MAT`, so that the result has primitive rows. Not `gerepile`-safe.

`GEN Q_remove_denom(GEN x, GEN *ptd)` sets  $d$  to the  $\mathbf{Q}$ -denominator of  $x$  and returns  $x * d$ , which is integral. Shallow function.

`GEN Q_div_to_int(GEN x, GEN c)` returns  $x / c$ , assuming  $c$  is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

`GEN Q_mul_to_int(GEN x, GEN c)` returns  $x * c$ , assuming  $c$  is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

`GEN Q_muli_to_int(GEN x, GEN d)` returns  $x * c$ , assuming  $c$  is a `t_INT` and the result is integral.

`GEN mul_content(GEN cx, GEN cy)`  $cx$  and  $cy$  are as set by `primitive_part`: either a `GEN` or `NULL` representing the trivial content 1. Returns their product (either a `GEN` or `NULL`).

`GEN div_content(GEN cx, GEN cy)`  $cx$  and  $cy$  are as set by `primitive_part`: either a `GEN` or `NULL` representing the trivial content 1. Returns their quotient (either a `GEN` or `NULL`).

`GEN inv_content(GEN c)`  $c$  is as set by `primitive_part`: either a `GEN` or `NULL` representing the trivial content 1. Returns its inverse (either a `GEN` or `NULL`).

`GEN mul_denom(GEN dx, GEN dy)`  $dx$  and  $dy$  are as set by `Q_remove_denom`: either a `t_INT` or `NULL` representing the trivial denominator 1. Returns their product (either a `t_INT` or `NULL`).

## 9.12 Generic arithmetic operators.

### 9.12.1 Unary operators.

GEN `gneg[z](GEN x[, GEN z])` yields  $-x$ .

GEN `gneg_i(GEN x)` shallow function yielding  $-x$ .

GEN `gabs[z](GEN x[, GEN z])` yields  $|x|$ .

GEN `gsqr(GEN x)` creates the square of  $x$ .

GEN `ginv(GEN x)` creates the inverse of  $x$ .

### 9.12.2 Binary operators.

Let “*op*” be a binary operation among

*op*=**add**: addition ( $x + y$ ).

*op*=**sub**: subtraction ( $x - y$ ).

*op*=**mul**: multiplication ( $x * y$ ).

*op*=**div**: division ( $x / y$ ).

The names and prototypes of the functions corresponding to *op* are as follows:

GEN `gop(GEN x, GEN y)`

GEN `gopgs(GEN x, long s)`

GEN `gopgu(GEN x, ulong u)`

GEN `gopsg(long s, GEN y)`

GEN `gopug(ulong u, GEN y)`

Explicitly

GEN `gadd(GEN x, GEN y)`, GEN `gaddgs(GEN x, long s)`, GEN `gaddsg(long s, GEN x)`

GEN `gmul(GEN x, GEN y)`, GEN `gmulgs(GEN x, long s)`, GEN `gmulsg(long s, GEN x)`, GEN `gmulgu(GEN x, ulong u)`, GEN `gmulug(GEN x, ulong u)`,

GEN `gsub(GEN x, GEN y)`, GEN `gsubgs(GEN x, long s)`, GEN `gsubsg(long s, GEN x)`

GEN `gdiv(GEN x, GEN y)`, GEN `gdivgs(GEN x, long s)`, GEN `gdivsg(long s, GEN x)`, GEN `gdivgu(GEN x, ulong u)`,

GEN `gpow(GEN x, GEN y, long l)` creates  $x^y$ . If  $y$  is a `t_INT`, return `powgi(x,y)` (the precision  $l$  is not taken into account). Otherwise, the result is  $\exp(y * \log(x))$  where exact arguments are converted to floats of precision  $l$  in case of need; if there is no need, for instance if  $x$  is a `t_REAL`,  $l$  is ignored. Indeed, if  $x$  is a `t_REAL`, the accuracy of  $\log x$  is determined from the accuracy of  $x$ , it is no problem to multiply by  $y$ , even if it is an exact type, and the accuracy of the exponential is determined, exactly as in the case of the initial  $\log x$ .

GEN `gpowgs(GEN x, long n)` creates  $x^n$  using binary powering. To treat the special case  $n = 0$ , we consider `gpowgs` as a series of `gmul`, so we follow the rule of returning result which is as exact as possible given the input. More precisely, we return

- `gen_1` if  $x$  has type `t_INT`, `t_REAL`, `t_FRAC`, or `t_PADIC`
- `Mod(1,N)` if  $x$  is a `t_INTMOD` modulo  $N$ .
- `gen_1` for `t_COMPLEX`, `t_QUAD` unless one component is a `t_INTMOD`, in which case we return `Mod(1, N)` for a suitable  $N$  (the gcd of the moduli that appear).
- `FF_1(x)` for a `t_FFELT`.
- `qfb_1(x)` for `t_QFB`.
- the identity permutation for `t_VECSMALL`.
- `Rg_get_1(x)` otherwise

Of course, the only practical use of this routine for  $n = 0$  is to obtain the multiplicative neutral element in the base ring (or to treat marginal cases that should be special cased anyway if there is the slightest doubt about what the result should be).

`GEN powgi(GEN x, GEN y)` creates  $x^y$ , where  $y$  is a `t_INT`, using left-shift binary powering. The case where  $y = 0$  (as all cases where  $y$  is small) is handled by `gpowgs(x, 0)`.

`GEN gpowers(GEN x, long n)` returns the vector  $[1, x, \dots, x^n]$ .

`GEN grootsof1(long n, long prec)` returns the vector  $[1, x, \dots, x^{n-1}]$ , where  $x$  is the  $n$ -th root of unity  $\exp(2i\pi/n)$ .

`GEN gsqrpowers(GEN x, long n)` returns the vector  $[x, x^4, \dots, x^{n^2}]$ .

In addition we also have the obsolete forms:

```
void gaddz(GEN x, GEN y, GEN z)
```

```
void gsubz(GEN x, GEN y, GEN z)
```

```
void gmulz(GEN x, GEN y, GEN z)
```

```
void gdivz(GEN x, GEN y, GEN z)
```

### 9.13 Generic operators: product, powering, factorback.

To describe the following functions, we use the following private typedefs to simplify the description:

```
typedef (*F0)(void *);
typedef (*F1)(void *, GEN);
typedef (*F2)(void *, GEN, GEN);
```

They correspond to generic functions with one and two arguments respectively (the `void*` argument provides some arbitrary evaluation context).

`GEN gen_product(GEN v, void *D, F2 op)` Given two objects  $x, y$ , assume that `op(D, x, y)` implements an associative binary operator. If  $v$  has  $k$  entries, return

$$v[1] \text{ op } v[2] \text{ op } \dots \text{ op } v[k];$$

returns `gen_1` if  $k = 0$  and a copy of  $v[1]$  if  $k = 1$ . Use divide and conquer strategy. Leave some garbage on stack, but suitable for `gerepileupto` if `mul` is.

GEN `gen_pow`(GEN `x`, GEN `n`, void `*D`, F1 `sqr`, F2 `mul`)  $n > 0$  a `t_INT`, returns  $x^n$ ; `mul`(`D`, `x`, `y`) implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul`(`D`, `x`, `x`).

GEN `gen_powu`(GEN `x`, ulong `n`, void `*D`, F1 `sqr`, F2 `mul`)  $n > 0$ , returns  $x^n$ . See `gen_pow`.

GEN `gen_pow_i`(GEN `x`, GEN `n`, void `*E`, F1 `sqr`, F2 `mul`) internal variant of `gen_pow`, not memory-clean.

GEN `gen_powu_i`(GEN `x`, ulong `n`, void `*E`, F1 `sqr`, F2 `mul`) internal variant of `gen_powu`, not memory-clean.

GEN `gen_pow_fold`(GEN `x`, GEN `n`, void `*D`, F1 `sqr`, F1 `msqr`) variant of `gen_pow`, where `mul` is replaced by `msqr`, with `msqr`(`D`, `y`) returning  $xy^2$ . In particular `D` must implicitly contain `x`.

GEN `gen_pow_fold_i`(GEN `x`, GEN `n`, void `*E`, F1 `sqr`, F1 `msqr`) internal variant of the function `gen_pow_fold`, not memory-clean.

GEN `gen_powu_fold`(GEN `x`, ulong `n`, void `*D`, F1 `sqr`, F1 `msqr`), see `gen_pow_fold`.

GEN `gen_powu_fold_i`(GEN `x`, ulong `n`, void `*E`, F1 `sqr`, F1 `msqr`) see `gen_pow_fold_i`.

GEN `gen_pow_init`(GEN `x`, GEN `n`, long `k`, void `*E`, GEN (`*sqr`)(void\*, GEN), GEN (`*mul`)(void\*, GEN, GEN)) Return a table `R` that can be used with `gen_pow_table` to compute the powers of `x` up to `n`. The table is of size  $2^k \log_2(n)$ .

GEN `gen_pow_table`(GEN `R`, GEN `n`, void `*E`, GEN (`*one`)(void\*), GEN (`*mul`)(void\*, GEN, GEN))

Return  $x^n$ , where `R` is as given by `gen_pow_init`(`x`,`m`,`k`,`E`,`sqr`,`mul`) for some integer  $m \geq n$ .

GEN `gen_powers`(GEN `x`, long `n`, long `usesqr`, void `*D`, F1 `sqr`, F2 `mul`, F0 `one`) returns  $[x^0, \dots, x^n]$  as a `t_VEC`; `mul`(`D`, `x`, `y`) implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul`(`D`, `x`, `x`); `one` returns the monoid unit. The flag `usesqr` should be set to 1 if squaring are faster than multiplication by `x`.

GEN `gen_factorback`(GEN `L`, GEN `e`, void `*D`, F2 `mul`, F2 `pow`, GEN (`*one`)(void \*)`D`) generic form of `factorback`. The pair  $[L, e]$  is of the form

- `[fa, NULL]`, `fa` a two-column factorization matrix: expand it.
- `[v, NULL]`, `v` a vector of objects: return their product.
- or `[v, e]`, `v` a vector of objects, `e` a vector of integral exponents (a `ZV` or `zv`): return the product of the  $v[i]^{e[i]}$ .

`mul`(`D`, `x`, `y`) and `pow`(`D`, `x`, `n`) return  $xy$  and  $x^n$  respectively.

`one`(`D`) returns the neutral element. If `one` is `NULL`, `gen_1` is used instead.

## 9.14 Matrix and polynomial norms.

This section concerns only standard norms of **R** and **C** vector spaces, not algebraic norms given by the determinant of some multiplication operator. We have already seen type-specific functions like `ZM_supnorm` or `RgM_fpnorml2` and limit ourselves to generic functions assuming nothing about their `GEN` argument; these functions allow the following scalar types: `t_INT`, `t_FRAC`, `t_REAL`, `t_COMPLEX`, `t_QUAD` and are defined recursively (in terms of norms of their components) for the following “container” types: `t_POL`, `t_VEC`, `t_COL` and `t_MAT`. They raise an error if some other type appears in the argument.

**GEN gnorml2(GEN x)** The norm of a scalar is the square of its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the *square* of the usual  $L^2$  norm. In most applications, the missing square root computation can be skipped.

**GEN gnorml1(GEN x, long prec)** The norm of a scalar is its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the usual  $L^1$  norm. One must include a real precision `prec` in case the inputs include `t_COMPLEX` or `t_QUAD` with exact rational components: a square root must be computed and we must choose an accuracy.

**GEN gnorml1\_fake(GEN x)** as `gnorml1`, except that the norm of a `t_QUAD`  $x + wy$  or `t_COMPLEX`  $x + Iy$  is defined as  $|x| + |y|$ , where we use the ordinary real absolute value. This is still a norm of **R** vector spaces, which is easier to compute than `gnorml1` and can often be used in its place.

**GEN gsupnorm(GEN x, long prec)** The norm of a scalar is its complex modulus, the norm of a recursive type is the max of the norms of its components. A precision `prec` must be included for the same reason as in `gnorml1`.

**void gsupnorm\_aux(GEN x, GEN \*m, GEN \*m2, long prec)** is the low-level function underlying `gsupnorm`, used as follows:

```
GEN m = NULL, m2 = NULL;
gsupnorm_aux(x, &m, &m2);
```

After the call, the sup norm of  $x$  is the min of `m` and the square root of `m2`; one or both of `m`, `m2` may be `NULL`, in which case it must be omitted. You may initially set `m` and `m2` to non-`NULL` values, in which case, the above procedure yields the max of (the initial) `m`, the square root of (the initial) `m2`, and the sup norm of  $x$ .

The strange interface is due to the fact that  $|z|^2$  is easier to compute than  $|z|$  for a `t_QUAD` or `t_COMPLEX`  $z$ : `m2` is the max of those  $|z|^2$ , and `m` is the max of the other  $|z|$ .



## 9.15 Substitution and evaluation.

GEN gsubst(GEN x, long v, GEN y) substitutes the object y into x for the variable number v.

GEN poleval(GEN q, GEN x) evaluates the t\_POL or t\_RFRAC  $q$  at  $x$ . For convenience, a t\_VEC or t\_COL is also recognized as the t\_POL gtovectrev(q).

GEN RgX\_cxeval(GEN T, GEN x, GEN xi) evaluate the t\_POL  $T$  at  $x$  via Horner's scheme. If  $xi$  is not NULL it must be equal to  $1/x$  and we evaluate  $x^{\deg T}T(1/x)$  instead. This is useful when  $|x| > 1$  is a t\_REAL or an inexact t\_COMPLEX and  $T$  has “balanced” coefficients, since the evaluation becomes numerically stable.

GEN RgXY\_cxevalx(GEN T, GEN x, GEN xi) Apply RgX\_cxeval to all the polynomials coefficients of  $T$ .

GEN RgX\_RgM\_eval(GEN q, GEN x) evaluates the t\_POL  $q$  at the square matrix  $x$ .

GEN RgX\_RgMV\_eval(GEN f, GEN V) returns the evaluation  $f(x)$ , assuming that  $V$  was computed by FpXQ\_powers( $x, n$ ) for some  $n > 1$ .

GEN qfeval(GEN q, GEN x) evaluates the quadratic form  $q$  (symmetric matrix) at  $x$  (column vector of compatible dimensions).

GEN qfevalb(GEN q, GEN x, GEN y) evaluates the polar bilinear form attached to the quadratic form  $q$  (symmetric matrix) at  $x, y$  (column vectors of compatible dimensions).

GEN hqfeval(GEN q, GEN x) evaluates the Hermitian form  $q$  (a Hermitian complex matrix) at  $x$ .

GEN qf\_apply\_RgM(GEN q, GEN M)  $q$  is a symmetric  $n \times n$  matrix,  $M$  an  $n \times k$  matrix, return  $M'qM$ .

GEN qf\_apply\_ZM(GEN q, GEN M) as above assuming that both  $q$  and  $M$  have integer entries.



## Chapter 10:

### Miscellaneous mathematical functions

#### 10.1 Fractions.

**GEN** `absfrac(GEN x)` returns the absolute value of the `t_FRAC`  $x$ .

**GEN** `absfrac_shallow(GEN x)`  $x$  being a `t_FRAC`, returns a shallow copy of  $|x|$ , in particular returns  $x$  itself when  $x \geq 0$ , and `gneg(x)` otherwise.

**GEN** `sqrfrac(GEN x)` returns the square of the `t_FRAC`  $x$ .

#### 10.2 Binomials.

**GEN** `binomial(GEN x, long k)`

**GEN** `binomialuu(ulong n, ulong k)`

**GEN** `vecbinomial(long n)`, which returns a vector  $v$  with  $n + 1$  `t_INT` components such that  $v[k + 1] = \text{binomial}(n, k)$  for  $k$  from 0 up to  $n$ .

#### 10.3 Real numbers.

**GEN** `R_abs(GEN x)`  $x$  being a `t_INT`, a `t_REAL` or a `t_FRAC`, returns  $|x|$ .

**GEN** `R_abs_shallow(GEN x)`  $x$  being a `t_INT`, a `t_REAL` or a `t_FRAC`, returns a shallow copy of  $|x|$ , in particular returns  $x$  itself when  $x \geq 0$ , and `gneg(x)` otherwise.

**GEN** `modRr_safe(GEN x, GEN y)` let  $x$  be a `t_INT`, a `t_REAL` or `t_FRAC` and let  $y$  be a `t_REAL`. Return  $x \% y$  unless the input accuracy is insufficient to compute the floor or  $x/y$  in which case we return `NULL`.

## 10.4 Complex numbers.

GEN `gimag`(GEN `x`) returns a copy of the imaginary part of `x`.

GEN `greal`(GEN `x`) returns a copy of the real part of `x`. If `x` is a `t_QUAD`, returns the coefficient of 1 in the “canonical” integral basis  $(1, \omega)$ .

GEN `gconj`(GEN `x`) returns `greal(x) - 2gimag(x)`, which is the ordinary complex conjugate except for a real `t_QUAD`.

GEN `imag_i`(GEN `x`), shallow variant of `gimag`.

GEN `real_i`(GEN `x`), shallow variant of `greal`.

GEN `conj_i`(GEN `x`), shallow variant of `gconj`.

GEN `mulreal`(GEN `x`, GEN `y`) returns the real part of  $xy$ ;  $x, y$  have type `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`. See also `RgM_mulreal`.

GEN `cxnorm`(GEN `x`) norm of the `t_COMPLEX`  $x$  (modulus squared).

GEN `cxexpm1`(GEN `x`) returns  $\exp(x) - 1$ , for a `t_COMPLEX`  $x$ .

int `cx_approx_equal`(GEN `a`, GEN `b`) test whether (`t_INT`, `t_FRAC`, `t_REAL`, or `t_COMPLEX` of those)  $a$  and  $b$  are approximately equal. This returns 1 if and only if the division by  $a - b$  would produce a division by 0 (which is a less stringent test than testing whether  $a - b$  evaluates to 0).

int `cx_approx0`(GEN `a`, GEN `b`) test whether (`t_INT`, `t_FRAC`, `t_REAL`, or `t_COMPLEX` of those)  $a$  is approximately 0, where  $b$  is a reference point. A non-0 `t_REAL` component  $x$  is approximately 0 if

$$\text{exponent}(b) - \text{exponent}(x) > \text{bit\_prec}(x).$$

## 10.5 Quadratic numbers and binary quadratic forms.

GEN `quad_disc`(GEN `x`) returns the discriminant of the `t_QUAD`  $x$ . Not stack-clean but suitable for `gerepileupto`.

GEN `quadnorm`(GEN `x`) norm of the `t_QUAD`  $x$ .

GEN `qfb_disc`(GEN `x`) returns the discriminant of the `t_QFB`  $x$ .

GEN `qfb_disc3`(GEN `x`, GEN `y`, GEN `z`) returns  $y^2 - 4xz$  assuming all inputs are `t_INTs`. Not stack-clean.

GEN `qfb_apply_ZM`(GEN `q`, GEN `g`) returns  $q \circ g$ .

GEN `qfbforms`(GEN `D`) given a discriminant  $D < 0$ , return the list of reduced forms of discriminant  $D$  as `t_VECSMALL` with 3 components. The primitive forms in the list enumerate the class group of the quadratic order of discriminant  $D$ ; if  $D$  is fundamental, all returned forms are automatically primitive.

## 10.6 Polynomials.

GEN `truecoef`(GEN `x`, long `n`) returns `polcoef`(`x`, `n`, `-1`), i.e. the coefficient of the term of degree `n` in the main variable. This is a safe but expensive function that must *copy* its return value so that it be *gerepile*-safe. Use `polcoef_i` for a fast internal variant.

GEN `polcoef_i`(GEN `x`, long `n`, long `v`) internal shallow function. Rewrite  $x$  as a Laurent polynomial in the variable  $v$  and returns its coefficient of degree  $n$  (`gen_0` if this falls outside the coefficient array). Allow `t_POL`, `t_SER`, `t_RFRAC` and scalars.

long `degree`(GEN `x`) returns `poldegree`(`x`, `-1`), the degree of  $x$  with respect to its main variable, with the usual meaning if the leading coefficient of  $x$  is nonzero. If the sign of  $x$  is 0, this function always returns `-1`. Otherwise, we return the index of the leading coefficient of  $x$ , i.e. the coefficient of largest index stored in  $x$ . For instance the “degrees” of

```
0. E-38 * x^4 + 0.E-19 * x + 1
Mod(0,2) * x^0    \\ sign is 0 !
```

are 4 and `-1` respectively.

long `degpol`(GEN `x`) is a simple macro returning `lg`(`x`) `- 3`. This is the degree of the `t_POL`  $x$  with respect to its main variable, *if* its leading coefficient is nonzero (a rational 0 is impossible, but an inexact 0 is allowed, as well as an exact modular 0, e.g. `Mod(0,2)`). If  $x$  has no coefficients (rational 0 polynomial), its length is 2 and we return the expected `-1`.

GEN `characteristic`(GEN `x`) returns the characteristic of the base ring over which the polynomial is defined (as defined by `t_INTMOD` and `t_FFELT` components). The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components. Shallow function.

GEN `residual_characteristic`(GEN `x`) returns a kind of “residual characteristic” of the base ring over which the polynomial is defined. This is defined as the gcd of all moduli `t_INTMOD`s occurring in the structure, as well as primes  $p$  arising from `t_PADIC`s or `t_FFELT`s. The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components. Shallow function.

GEN `resultant`(GEN `x`, GEN `y`) resultant of  $x$  and  $y$ , with respect to the main variable of highest priority. Uses either the subresultant algorithm (generic case), a modular algorithm (inputs in  $\mathbf{Q}[X]$ ) or Sylvester’s matrix (inexact inputs).

GEN `resultant2`(GEN `x`, GEN `y`) resultant of  $x$  and  $y$ , with respect to the main variable of highest priority. Computes the determinant of Sylvester’s matrix.

GEN `cleanroots`(GEN `x`, long `prec`) returns the complex roots of the complex polynomial  $x$  (with coefficients `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX` of the above). The roots are returned as `t_REAL` or `t_COMPLEX` of `t_REALS` of precision `prec` (guaranteeing a nonzero imaginary part). See `QX_complex_roots`.

double `fujiwara_bound`(GEN `x`) return a quick upper bound for the logarithm in base 2 of the modulus of the largest complex roots of the polynomial  $x$  (complex coefficients).

double `fujiwara_bound_real`(GEN `x`, long `sign`) return a quick upper bound for the logarithm in base 2 of the absolute value of the largest real root of sign *sign* (1 or `-1`), for the polynomial  $x$  (real coefficients).

GEN `polmod_to_embed`(GEN `x`, long `prec`) return the vector of complex embeddings of the `t_POLMOD`  $x$  (with complex coefficients). Shallow function, simple complex variant of `conjvec`.

GEN `pollegendre_reduced`(long `n`, long `v`) let  $P_n(t) \in \mathbf{Q}[t]$  be the  $n$ -th Legendre polynomial in variable  $v$ . Return  $p \in \mathbf{Z}[t]$  such that  $2^n P_n(t) = p(t^2)$  ( $n$  even) or  $tp(t^2)$  ( $n$  odd).

## 10.7 Power series.

GEN `sertoser`(GEN `x`, long `prec`) return the `t_SER`  $x$  truncated or extended (with zeros) to `prec` terms. Shallow function, assume that `prec`  $\geq 0$ .

GEN `derivser`(GEN `x`) returns the derivative of the `t_SER`  $x$  with respect to its main variable.

GEN `integser`(GEN `x`) returns the primitive of the `t_SER`  $x$  with respect to its main variable.

GEN `truecoef`(GEN `x`, long `n`) returns `polcoef`(`x`, `n`, -1), i.e. the coefficient of the term of degree `n` in the main variable. This is a safe but expensive function that must *copy* its return value so that it be `gerepile`-safe. Use `polcoef_i` for a fast internal variant.

GEN `ser_unscale`(GEN `P`, GEN `h`) return  $P(hx)$ , not memory clean.

GEN `ser_normalize`(GEN `x`) divide  $x$  by its “leading term” so that the series is either 0 or equal to  $t^v(1 + O(t))$ . Shallow function if the “leading term” is 1.

int `ser_isexactzero`(GEN `x`) return 1 if  $x$  is a zero series, all of whose known coefficients are exact zeroes; this implies that `sign`( $x$ ) = 0 and `lg`( $x$ )  $\leq 3$ .

GEN `ser_inv`(GEN `x`) return the inverse of the `t_SER`  $x$  using Newton iteration.

GEN `psi1series`(long `n`, long `v`, long `prec`) creates the `t_SER`  $\psi(1 + x + O(x^n))$  in variable  $v$ .

## 10.8 Functions to handle `t_FFELT`.

These functions define the public interface of the `t_FFELT` type to use in generic functions. However, in specific functions, it is better to use the functions class `FpXQ` and/or `F1xq` as appropriate.

GEN `FF_p`(GEN `a`) returns the characteristic of the definition field of the `t_FFELT` element `a`.

long `FF_f`(GEN `a`) returns the dimension of the definition field over its prime field; the cardinality of the dimension field is thus  $p^f$ .

GEN `FF_p_i`(GEN `a`) shallow version of `FF_p`.

GEN `FF_q`(GEN `a`) returns the cardinality of the definition field of the `t_FFELT` element `a`.

GEN `FF_mod`(GEN `a`) returns the polynomial (with reduced `t_INT` coefficients) defining the finite field, in the variable used to display  $a$ .

long `FF_var`(GEN `a`) returns the variable used to display  $a$ .

GEN `FF_gen`(GEN `a`) returns the standard generator of the definition field of the `t_FFELT` element `a`, see `ffgen`, that is  $x \pmod{T}$  where  $T$  is the polynomial over the prime field that define the finite field.

GEN `FF_to_FpXQ`(GEN `a`) converts the `t_FFELT` `a` to a polynomial  $P$  with reduced `t_INT` coefficients such that  $a = P(g)$  where  $g$  is the generator of the finite field returned by `ffgen`, in the variable used to display  $g$ .

GEN `FF_to_FpXQ_i`(GEN `a`) shallow version of `FF_to_FpXQ`.

GEN `FF_to_F2xq`(GEN `a`) converts the `t_FFELT` `a` to a `F2x`  $P$  such that  $a = P(g)$  where  $g$  is the generator of the finite field returned by `ffgen`, in the variable used to display  $g$ . This only work if the characteristic is 2.

GEN FF\_to\_F2xq\_i(GEN a) shallow version of FF\_to\_F2xq.

GEN FF\_to\_Flxq(GEN a) converts the  $t\_FFELT$   $a$  to a  $Flx$   $P$  such that  $a = P(g)$  where  $g$  is the generator of the finite field returned by `ffgen`, in the variable used to display  $g$ . This only work if the characteristic is small enough.

GEN FF\_to\_Flxq\_i(GEN a) shallow version of FF\_to\_Flxq.

GEN p\_to\_FF(GEN p, long v) returns a  $t\_FFELT$  equal to 1 in the finite field  $\mathbf{Z}/p\mathbf{Z}$ . Useful for generic code that wants to handle (inefficiently)  $\mathbf{Z}/p\mathbf{Z}$  as if it were not a prime field.

GEN Tp\_to\_FF(GEN T, GEN p) returns a  $t\_FFELT$  equal to 1 in the finite field  $\mathbf{F}_p/(T)$ , where  $T$  is a  $ZX$ , assumed to be irreducible modulo  $p$ , or NULL in which case the routine acts as `p_to_FF(p,0)`. No checks.

GEN Fq\_to\_FF(GEN x, GEN ff) returns a  $t\_FFELT$  equal to  $x$  in the finite field defined by the  $t\_FFELT$   $ff$ , where  $x$  is an  $Fq$  (either a  $t\_INT$  or a  $ZX$ : a  $t\_POL$  with  $t\_INT$  coefficients). No checks.

GEN FqX\_to\_FFX(GEN x, GEN ff) given an  $FqX$   $x$ , return the polynomial with  $t\_FFELT$  coefficients obtained by applying `Fq_to_FF` coefficientwise. No checks, and no normalization if the leading coefficient maps to 0.

GEN FF\_1(GEN a) returns the unity in the definition field of the  $t\_FFELT$  element  $a$ .

GEN FF\_zero(GEN a) returns the zero element of the definition field of the  $t\_FFELT$  element  $a$ .

int FF\_equal0(GEN a) returns 1 if the  $t\_FFELT$   $a$  is equal to 0 else returns 0.

int FF\_equal1(GEN a) returns 1 if the  $t\_FFELT$   $a$  is equal to 1 else returns 0.

int FF\_equalm1(GEN a) returns  $-1$  if the  $t\_FFELT$   $a$  is equal to 1 else returns 0.

int FF\_equal(GEN a, GEN b) return 1 if the  $t\_FFELT$   $a$  and  $b$  have the same definition field and are equal, else 0.

int FF\_samefield(GEN a, GEN b) return 1 if the  $t\_FFELT$   $a$  and  $b$  have the same definition field, else 0.

int Rg\_is\_FF(GEN c, GEN \*ff) to be called successively on many objects, setting  $*ff = \text{NULL}$  (unset) initially. Returns 1 as long as  $c$  is a  $t\_FFELT$  defined over the same field as  $*ff$  (setting  $*ff = c$  if unset), and 0 otherwise.

int RgC\_is\_FFC(GEN x, GEN \*ff) apply `Rg_is_FF` successively to all components of the  $t\_VEC$  or  $t\_COL$   $x$ . Return 0 if one call fails, and 1 otherwise.

int RgM\_is\_FFM(GEN x, GEN \*ff) apply `Rg_is_FF` to all components of the  $t\_MAT$ . Return 0 if one call fails, and 1 otherwise.

GEN FF\_add(GEN a, GEN b) returns  $a + b$  where  $a$  and  $b$  are  $t\_FFELT$  having the same definition field.

GEN FF\_Z\_add(GEN a, GEN x) returns  $a + x$ , where  $a$  is a  $t\_FFELT$ , and  $x$  is a  $t\_INT$ , the computation being performed in the definition field of  $a$ .

GEN FF\_Q\_add(GEN a, GEN x) returns  $a + x$ , where  $a$  is a  $t\_FFELT$ , and  $x$  is a  $t\_RFRAC$ , the computation being performed in the definition field of  $a$ .

GEN FF\_sub(GEN a, GEN b) returns  $a - b$  where  $a$  and  $b$  are  $t\_FFELT$  having the same definition field.

GEN `FF_mul`(GEN `a`, GEN `b`) returns  $ab$  where `a` and `b` are `t_FFELT` having the same definition field.

GEN `FF_Z_mul`(GEN `a`, GEN `b`) returns  $ab$ , where `a` is a `t_FFELT`, and `b` is a `t_INT`, the computation being performed in the definition field of `a`.

GEN `FF_div`(GEN `a`, GEN `b`) returns  $a/b$  where `a` and `b` are `t_FFELT` having the same definition field.

GEN `FF_neg`(GEN `a`) returns  $-a$  where `a` is a `t_FFELT`.

GEN `FF_neg_i`(GEN `a`) shallow function returning  $-a$  where `a` is a `t_FFELT`.

GEN `FF_inv`(GEN `a`) returns  $a^{-1}$  where `a` is a `t_FFELT`.

GEN `FF_sqr`(GEN `a`) returns  $a^2$  where `a` is a `t_FFELT`.

GEN `FF_mul2n`(GEN `a`, long `n`) returns  $a^{2^n}$  where `a` is a `t_FFELT`.

GEN `FF_pow`(GEN `a`, GEN `n`) returns  $a^n$  where `a` is a `t_FFELT` and `n` is a `t_INT`.

GEN `FF_Frobenius`(GEN `a`, GEN `n`) returns  $x^{p^n}$  where `x` is the standard generator of the definition field of the `t_FFELT` element `a`, `t_FFELT`, `n` is a `t_INT`, and  $p$  is the characteristic of the definition field of `a`.

GEN `FF_Z_Z_muldiv`(GEN `a`, GEN `x`, GEN `y`) returns  $ay/z$ , where `a` is a `t_FFELT`, and `x` and `y` are `t_INT`, the computation being performed in the definition field of `a`.

GEN `Z_FF_div`(GEN `x`, GEN `a`) return  $x/a$  where `a` is a `t_FFELT`, and `x` is a `t_INT`, the computation being performed in the definition field of `a`.

GEN `FF_norm`(GEN `a`) returns the norm of the `t_FFELT` `a` with respect to its definition field.

GEN `FF_trace`(GEN `a`) returns the trace of the `t_FFELT` `a` with respect to its definition field.

GEN `FF_conjvec`(GEN `a`) returns the vector of conjugates  $[a, a^p, a^{p^2}, \dots, a^{p^{n-1}}]$  where the `t_FFELT` `a` belong to a field with  $p^n$  elements.

GEN `FF_charpoly`(GEN `a`) returns the characteristic polynomial) of the `t_FFELT` `a` with respect to its definition field.

GEN `FF_minpoly`(GEN `a`) returns the minimal polynomial of the `t_FFELT` `a`.

GEN `FF_sqrt`(GEN `a`) returns an `t_FFELT` `b` such that  $a = b^2$  if it exist, where `a` is a `t_FFELT`.

long `FF_issquareall`(GEN `x`, GEN `*pt`) returns 1 if `x` is a square, and 0 otherwise. If `x` is indeed a square, set `pt` to its square root.

long `FF_issquare`(GEN `x`) returns 1 if `x` is a square and 0 otherwise.

long `FF_ispower`(GEN `x`, GEN `K`, GEN `*pt`) Given  $K$  a positive integer, returns 1 if `x` is a  $K$ -th power, and 0 otherwise. If `x` is indeed a  $K$ -th power, set `pt` to its  $K$ -th root.

GEN `FF_sqrtn`(GEN `a`, GEN `n`, GEN `*zn`) returns an  $n$ -th root of `a` if it exist. If `zn` is non-NULL set it to a primitive  $n$ -th root of the unity.

GEN `FF_log`(GEN `a`, GEN `g`, GEN `o`) the `t_FFELT` `g` being a generator for the definition field of the `t_FFELT` `a`, returns a `t_INT` `e` such that  $a^e = g$ . If `e` does not exists, the result is currently undefined. If `o` is not NULL it is assumed to be a factorization of the multiplicative order of `g` (as set by `FF_primroot`)



GEN FF\_order(GEN a, GEN o) returns the order of the  $t\_FFELT$  a. If o is non-NULL, it is assumed that o is a multiple of the order of a.

GEN FF\_primroot(GEN a, GEN \*o) returns a generator of the multiplicative group of the definition field of the  $t\_FFELT$  a. If o is not NULL, set it to the factorization of the order of the primitive root (to speed up FF\_log).

GEN FF\_map(GEN m, GEN a) returns  $A(m)$  where  $A=a.pol$  assuming  $a$  and  $m$  belongs to fields having the same characteristic.

### 10.8.1 FFX.

The functions in this sections take polynomial arguments and a  $t\_FFELT$  a. The coefficients of the polynomials must be of type  $t\_INT$ ,  $t\_INTMOD$  or  $t\_FFELT$  and compatible with a.

GEN FFX\_add(GEN P, GEN Q, GEN a) returns the sum of the polynomials P and Q defined over the definition field of the  $t\_FFELT$  a.

GEN FFX\_mul(GEN P, GEN Q, GEN a) returns the product of the polynomials P and Q defined over the definition field of the  $t\_FFELT$  a.

GEN FFX\_sqr(GEN P, GEN a) returns the square of the polynomial P defined over the definition field of the  $t\_FFELT$  a.

GEN FFX\_rem(GEN P, GEN Q, GEN a) returns the remainder of the polynomial P modulo the polynomial Q, where P and Q are defined over the definition field of the  $t\_FFELT$  a.

GEN FFX\_gcd(GEN P, GEN Q, GEN a) returns the GCD of the polynomials P and Q defined over the definition field of the  $t\_FFELT$  a.

GEN FFX\_extgcd(GEN P, GEN Q, GEN a, GEN \*U, GEN \*V) returns the GCD of the polynomials P and Q defined over the definition field of the  $t\_FFELT$  a and sets \*U, \*V to the Bezout coefficients such that  $*UP + *VQ = d$ . If \*U is set to NULL, it is not computed which is a bit faster.

GEN FFX\_halfgcd(GEN x, GEN y, GEN a) returns a two-by-two matrix  $M$  with determinant  $\pm 1$  such that the image  $(a, b)$  of  $(x, y)$  by  $M$  has the property that  $\deg a \geq \frac{\deg x}{2} > \deg b$ .

GEN FFX\_resultant(GEN P, GEN Q, GEN a) returns the resultant of the polynomials P and Q where P and Q are defined over the definition field of the  $t\_FFELT$  a.

GEN FFX\_disc(GEN P, GEN a) returns the discriminant of the polynomial P where P is defined over the definition field of the  $t\_FFELT$  a.

GEN FFX\_ishpower(GEN P, ulong k, GEN a, GEN \*py) return 1 if the FFX P is a  $k$ -th power, 0 otherwise, where P is defined over the definition field of the  $t\_FFELT$  a. If py is not NULL, set it to  $g$  such that  $g^k = f$ .

GEN FFX\_factor(GEN f, GEN a) returns the factorization of the univariate polynomial f over the definition field of the  $t\_FFELT$  a. The coefficients of f must be of type  $t\_INT$ ,  $t\_INTMOD$  or  $t\_FFELT$  and compatible with a.

GEN FFX\_factor\_squarefree(GEN f, GEN a) returns the squarefree factorization of the univariate polynomial f over the definition field of the  $t\_FFELT$  a. This is a vector  $[u_1, \dots, u_k]$  of pairwise coprime FFX such that  $u_k \neq 1$  and  $f = \prod u_i$ .

GEN FFX\_ddf(GEN f, GEN a) assuming that f is squarefree, returns the distinct degree factorization of f modulo p. The returned value v is a  $t\_VEC$  with two components:  $F=v[1]$  is a vector of (FFX)

factors, and  $E=v[2]$  is a `t_VECSMALL`, such that  $f$  is equal to the product of the  $F[i]$  and each  $F[i]$  is a product of irreducible factors of degree  $E[i]$ .

`GEN FFX_degfact(GEN f, GEN a)`, as `FFX_factor`, but the degrees of the irreducible factors are returned instead of the factors themselves (as a `t_VECSMALL`).

`GEN FFX_roots(GEN f, GEN a)` returns the roots (`t_FFELT`) of the univariate polynomial  $f$  over the definition field of the `t_FFELT`  $a$ . The coefficients of  $f$  must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with  $a$ .

`GEN FFX_preimagerel(GEN F, GEN x, GEN a)` returns  $P\%F$  where  $P=x.\text{pol}$  assuming  $a$  and  $x$  belongs to fields having the same characteristic, and that the coefficients of  $F$  belong to the definition field of  $a$ .

`GEN FFX_preimage(GEN F, GEN x, GEN a)` as `FFX_preimagerel` but return `NULL` if the remainder is of degree greater or equal to 1, the constant coefficient otherwise.

### 10.8.2 FFM.

`GEN FFM_FFC_gauss(GEN M, GEN C, GEN ff)` given a matrix  $M$  (`t_MAT`) and a column vector  $C$  (`t_COL`) over the finite field given by  $ff$  (`t_FFELT`) such that  $M$  is invertible, return the unique column vector  $X$  such that  $MX = C$ .

`GEN FFM_FFC_invimage(GEN M, GEN C, GEN ff)` given a matrix  $M$  (`t_MAT`) and a column vector  $C$  (`t_COL`) over the finite field given by  $ff$  (`t_FFELT`), return a column vector  $X$  such that  $MX = C$ , or `NULL` if no such vector exists.

`GEN FFM_FFC_mul(GEN M, GEN C, GEN ff)` returns the product of the matrix  $M$  (`t_MAT`) and the column vector  $C$  (`t_COL`) over the finite field given by  $ff$  (`t_FFELT`).

`GEN FFM_deplin(GEN M, GEN ff)` returns a nonzero vector (`t_COL`) in the kernel of the matrix  $M$  over the finite field given by  $ff$ , or `NULL` if no such vector exists.

`GEN FFM_det(GEN M, GEN ff)` returns the determinant of the matrix  $M$  over the finite field given by  $ff$ .

`GEN FFM_gauss(GEN M, GEN N, GEN ff)` given two matrices  $M$  and  $N$  (`t_MAT`) over the finite field given by  $ff$  (`t_FFELT`) such that  $M$  is invertible, return the unique matrix  $X$  such that  $MX = N$ .

`GEN FFM_image(GEN M, GEN ff)` returns a matrix whose columns span the image of the matrix  $M$  over the finite field given by  $ff$ .

`GEN FFM_indexrank(GEN M, GEN ff)` given a matrix  $M$  of rank  $r$  over the finite field given by  $ff$ , returns a vector with two `t_VECSMALL` components  $y$  and  $z$  containing  $r$  row and column indices, respectively, such that the  $r \times r$ -matrix formed by the  $M[i,j]$  for  $i$  in  $y$  and  $j$  in  $z$  is invertible.

`GEN FFM_inv(GEN M, GEN ff)` returns the inverse of the square matrix  $M$  over the finite field given by  $ff$ , or `NULL` if  $M$  is not invertible.

`GEN FFM_invimage(GEN M, GEN N, GEN ff)` given two matrices  $M$  and  $N$  (`t_MAT`) over the finite field given by  $ff$  (`t_FFELT`), return a matrix  $X$  such that  $MX = N$ , or `NULL` if no such matrix exists.

`GEN FFM_ker(GEN M, GEN ff)` returns the kernel of the `t_MAT`  $M$  over the finite field given by the `t_FFELT`  $ff$ .

`GEN FFM_mul(GEN M, GEN N, GEN ff)` returns the product of the matrices  $M$  and  $N$  (`t_MAT`) over the finite field given by  $ff$  (`t_FFELT`).

`long FFM_rank(GEN M, GEN ff)` returns the rank of the matrix `M` over the finite field given by `ff`.

`GEN FFM_suppl(GEN M, GEN ff)` given a matrix `M` over the finite field given by `ff` whose columns are linearly independent, returns a square invertible matrix whose first columns are those of `M`.

### 10.8.3 FFXQ.

`GEN FFXQ_mul(GEN P, GEN Q, GEN T, GEN a)` returns the product of the polynomials `P` and `Q` modulo the polynomial `T`, where `P`, `Q` and `T` are defined over the definition field of the `t_FFELT a`.

`GEN FFXQ_sqr(GEN P, GEN T, GEN a)` returns the square of the polynomial `P` modulo the polynomial `T`, where `P` and `T` are defined over the definition field of the `t_FFELT a`.

`GEN FFXQ_inv(GEN P, GEN Q, GEN a)` returns the inverse of the polynomial `P` modulo the polynomial `Q`, where `P` and `Q` are defined over the definition field of the `t_FFELT a`.

`GEN FFXQ_minpoly(GEN Pf, GEN Qf, GEN a)` returns the minimal polynomial of the polynomial `P` modulo the polynomial `Q`, where `P` and `Q` are defined over the definition field of the `t_FFELT a`.

## 10.9 Transcendental functions.

The following two functions are only useful when interacting with `gp`, to manipulate its internal default precision (expressed as a number of decimal digits, not in words as used everywhere else):

`long getrealprecision(void)` returns `realprecision`.

`long setrealprecision(long n, long *prec)` sets the new `realprecision` to `n`, which is returned. As a side effect, set `prec` to the corresponding number of words `ndec2prec(n)`.

### 10.9.1 Transcendental functions with `t_REAL` arguments.

In the following routines,  $x$  is assumed to be a `t_REAL` and the result is a `t_REAL` (sometimes a `t_COMPLEX` with `t_REAL` components), with the largest accuracy which can be deduced from the input. The naming scheme is inconsistent here, since we sometimes use the prefix `mp` even though `t_INT` inputs are forbidden:

`GEN sqrtr(GEN x)` returns the square root of  $x$ .

`GEN cbrtr(GEN x)` returns the real cube root of  $x$ .

`GEN sqrtnr(GEN x, long n)` returns the  $n$ -th root of  $x$ , assuming  $n \geq 1$  and  $x \geq 0$ .

`GEN sqrtnr_abs(GEN x, long n)` returns the  $n$ -th root of  $|x|$ , assuming  $n \geq 1$  and  $x \neq 0$ .

`GEN mpcos[z](GEN x[, GEN z])` returns  $\cos(x)$ .

`GEN mpsin[z](GEN x[, GEN z])` returns  $\sin(x)$ .

`GEN mplog[z](GEN x[, GEN z])` returns  $\log(x)$ . We must have  $x > 0$  since the result must be a `t_REAL`. Use `glog` for the general case, where you want such computations as  $\log(-1) = I$ .

`GEN mpexp[z](GEN x[, GEN z])` returns  $\exp(x)$ .

`GEN mpexpm1(GEN x)` returns  $\exp(x) - 1$ , but is more accurate than `subrs(mpexp(x), 1)`, which suffers from catastrophic cancellation if  $|x|$  is very small.

`void mpsincosm1(GEN x, GEN *s, GEN *c)` sets  $s$  and  $c$  to  $\sin(x)$  and  $\cos(x) - 1$  respectively, where  $x$  is a `t_REAL`; the latter is more accurate than `subrs(mpcos(y), 1)`, which suffers from catastrophic cancellation if  $|x|$  is very small.

`GEN mpveceint1(GEN C, GEN eC, long n)` as `veceint1`; assumes that  $C > 0$  is a `t_REAL` and that `eC` is `NULL` or `mpexp(C)`.

`GEN mpeint1(GEN x, GEN expx)` returns `eint1(x)`, for a `t_REAL`  $x \neq 0$ , assuming that `expx` is `mpexp(x)`.

A few variants on the Lambert function: they actually work when `gtofp` can map all `GEN` arguments to a `t_REAL`.

`GEN mplambertW(GEN y)` solution  $x = W_0(y)$  of the implicit equation  $x \exp(x) = y$ , for  $y > -1/e$  a `t_REAL`.

`GEN mplambertx_logx(GEN a, GEN b, long bit)` solve  $x - a \log(x) = b$  with  $a > 0$  and  $b \geq a(1 - \log(a))$ .

`GEN mplambertX(GEN y, long bit)` as `mplambertx_logx` in the special case  $a = 1$ ,  $b = \log(y)$ . In other words, solve  $e^x/x = y$  with  $y \geq e$ .

`GEN mplambertxlogx_x(GEN a, GEN b, long bit)` solve  $x \log(x) - ax = b$ ; if  $b < 0$ , assume  $a \geq 1 + \log|b|$ .

Useful low-level functions which *disregard* the sign of  $x$ :

`GEN sqrtr_abs(GEN x)` returns  $\sqrt{|x|}$  assuming  $x \neq 0$ .

`GEN cbrtr_abs(GEN x)` returns  $|x|^{1/3}$  assuming  $x \neq 0$ .

`GEN exp1r_abs(GEN x)` returns  $\exp(|x|) - 1$ , assuming  $x \neq 0$ .

`GEN logr_abs(GEN x)` returns  $\log(|x|)$ , assuming  $x \neq 0$ .

### 10.9.2 Other complex transcendental functions.

`GEN atanhuu(ulong u, ulong v, long prec)` computes  $\operatorname{atanh}(u/v)$  using binary splitting, assuming  $0 < u < v$ . Not memory clean but suitable for `gerepileupto`.

`GEN atanhui(ulong u, GEN v, long prec)` computes  $\operatorname{atanh}(u/v)$  using binary splitting, assuming  $0 < u < v$ . Not memory clean but suitable for `gerepileupto`.

`GEN szeta(long s, long prec)` returns the value of Riemann's zeta function at the (possibly negative) integer  $s \neq 1$ , in relative accuracy `prec`.

`GEN veczeta(GEN a, GEN b, long N, long prec)` returns in a vector all the  $\zeta(aj + b)$ , where  $j = 0, 1, \dots, N - 1$ , where  $a$  and  $b$  are real numbers (of arbitrary type, although `t_INT` is treated more efficiently) and  $b > 1$ . Assumes that  $N \geq 1$ .

`GEN ggamma1m1(GEN x, long prec)` return  $\Gamma(1 + x) - 1$  assuming  $|x| < 1$ . Guard against cancellation when  $x$  is small.

A few variants on  $\sin$  and  $\cos$ :

`void mpsincos(GEN x, GEN *s, GEN *c)` sets  $s$  and  $c$  to  $\sin(x)$  and  $\cos(x)$  respectively, where  $x$  is a `t_REAL`

`void mpsinhcosh(GEN x, GEN *s, GEN *c)` sets  $s$  and  $c$  to  $\sinh(x)$  and  $\cosh(x)$  respectively, where  $x$  is a `t_REAL`

GEN expIr(GEN x) returns  $\exp(ix)$ , where  $x$  is a `t_REAL`. The return type is `t_COMPLEX` unless the imaginary part is equal to 0 to the current accuracy (its sign is 0).

GEN expIPiR(GEN x, long prec) return  $\exp(i\pi x)$ , where  $x$  is a real number (`t_INT`, `t_FRAC` or `t_REAL`).

GEN expIPiC(GEN z, long prec) return  $\exp(i\pi x)$ , where  $x$  is a complex number (`t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`).

GEN expIxy(GEN x, GEN y, long prec) returns  $\exp(ixy)$ . Efficient when  $x$  is real and  $y$  pure imaginary.

GEN pow2Pis(GEN s, long prec) returns  $(2\pi)^s$ . The intent of this function and the next ones is to be accurate even if  $s$  has a huge imaginary part:  $\pi$  is computed at an accuracy taking into account the cancellation induced by argument reduction when computing the sine or cosine of  $\Im s \log 2\pi$ .

GEN powPis(GEN s, long prec) returns  $\pi^s$ , as `pow2Pis`.

long powcx\_prec(long e, GEN s, long prec) if  $e \approx \log_2 |x|$  return the precision at which  $\log(x)$  must be computed to evaluate  $x^s$  reliably (taking into account argument reduction).

GEN powcx(GEN x, GEN logx, GEN s, long prec) assuming  $s$  is a `t_COMPLEX` and `logx` is  $\log(x)$  computed to accuracy `powcx_prec`, return  $x^s$ .

void gsincos(GEN x, GEN \*s, GEN \*c, long prec) general case.

GEN rootsof1\_cx(GEN d, long prec) return  $e(1/d)$  at precision `prec`,  $e(x) = \exp(2i\pi x)$ .

GEN rootsof1u\_cx(ulong d, long prec) return  $e(1/d)$  at precision `prec`.

GEN rootsof1q\_cx(long a, long b, long prec) return  $e(a/b)$  at precision `prec`.

GEN rootsof1powinit(long a, long b, long prec) precompute  $b$ -th roots of 1 for `rootsof1pow`, i.e. to later compute  $e(ac/b)$  for varying  $c$ .

GEN rootsof1pow(GEN T, long c) given  $T = \text{rootsof1powinit}(a, b, \text{prec})$ , return  $e(ac/b)$ .

A generalization of `affrr_fixlg`

GEN affc\_fixlg(GEN x, GEN res) assume `res` was allocated using `cgetc`, and that  $x$  is either a `t_REAL` or a `t_COMPLEX` with `t_REAL` components. Assign  $x$  to `res`, first shortening the components of `res` if needed (in a `gerepile`-safe way). Further convert `res` to a `t_REAL` if  $x$  is a `t_REAL`.

GEN trans\_eval(const char \*fun, GEN (\*f)(GEN, long), GEN x, long prec) evaluate the transcendental function  $f$  (named "fun" at the argument  $x$  and precision `prec`. This is a quick way to implement a transcendental function to be made available under GP, starting from a  $C$  function handling only `t_REAL` and `t_COMPLEX` arguments. This routine first converts  $x$  to a suitable type:

- `t_INT/t_FRAC` to `t_REAL` of precision `prec`, `t_QUAD` to `t_REAL` or `t_COMPLEX` of precision `prec`.
- `t_POLMOD` to a `t_COL` of complex embeddings (as in `conjvec`)

Then evaluates the function at `t_VEC`, `t_COL`, `t_MAT` arguments coefficientwise.

GEN trans\_evalgen(const char \*fun, void \*E, GEN (\*f)(void\*, GEN, long), GEN x, long prec), general variant evaluating  $f(E, x, \text{prec})$ , where the function prototype allows to wrap an arbitrary context given by the argument  $E$ .

### 10.9.3 Modular functions.

GEN `cxredsl2`(GEN `z`, GEN `*g`) given  $t$  a `t_COMPLEX` belonging to the upper half-plane, find  $\gamma \in \mathrm{SL}_2(\mathbf{Z})$  such that  $\gamma \cdot z$  belongs to the standard fundamental domain and set `*g` to  $\gamma$ .

GEN `cxredsl2_i`(GEN `z`, GEN `*g`, GEN `*czd`) as `cxredsl2`; also sets `*czd` to  $cz+d$ , if  $\gamma = [a, b; c, d]$ .

GEN `cxEk`(GEN `tau`, long `k`, long `prec`) returns  $E_k(\tau)$  by direct evaluation of  $1 + 2/\zeta(1-k) \sum_n n^{k-1} q^n / (1 - q^n)$ ,  $q = e(\tau)$ . Assume that  $\Im \tau > 0$  and  $k$  even. Very slow unless  $\tau$  is already reduced modulo  $\mathrm{SL}_2(\mathbf{Z})$ . Not `gerepile-clean` but suitable for `gerepileupto`.

### 10.9.4 Transcendental functions with `t_PADIC` arguments.

The argument  $x$  is assumed to be a `t_PADIC`.

GEN `Qp_exp`(GEN `x`) shortcut for `gexp(x, /*ignored*/prec)`

long `Qp_exp_prec`(GEN `x`) number of terms to sum in the  $\exp(x)$  series to reach the same  $p$ -adic accuracy as  $x \neq 0$ . If  $n = p - 1$ ,  $e = v_p(x)$  and  $b = \text{prec}_p(x)$ , this is the ceiling of  $nb/(ne - 1)$ . Return  $-1$  if the series does not converge ( $ne \leq 1$ ).

GEN `Qp_gamma`(GEN `x`) shortcut for `ggamma(x, /*ignored*/prec)`

GEN `Qp_zeta`(GEN `x`) shortcut for `gzeta(x, /*ignored*/prec)`; assume that  $x \neq 1$ .

GEN `Qp_log`(GEN `x`) shortcut for `glog(x, /*ignored*/prec)`

GEN `Qp_sqrt`(GEN `x`) shortcut for `gsqrt(x, /*ignored*/prec)` Return NULL if  $x$  is not a square.

GEN `Qp_sqrtn`(GEN `x`, GEN `n`, GEN `*z`) shortcut for `gsqrtn(x, n, z, /*ignored*/prec)`. Return NULL if  $x$  is not an  $n$ -th power.

GEN `Qp_agm2_sequence`(GEN `a1`, GEN `b1`) assume  $a_1/b_1 = 1 \bmod p$  if  $p$  odd and  $\bmod 2^4$  if  $p = 2$ . Let  $A_1 = a_1/p^v$  and  $B_1 = b_1/p^v$  with  $v = v_p(a_1) = v_p(b_1)$ ; let further  $A_{n+1} = (A_n + B_n + 2B_{n+1})/4$ ,  $B_{n+1} = B_n \sqrt{A_n/B_n}$  (the square root of  $A_n B_n$  congruent to  $B_n \bmod p$ ) and  $R_n = p^v(A_n - B_n)$ . We stop when  $R_n$  is 0 at the given  $p$ -adic accuracy. This function returns in a triplet `t_VEC` the three sequences  $(A_n)$ ,  $(B_n)$  and  $(R_n)$ , corresponding to a sequence of 2-isogenies on the Tate curve  $y^2 = x(x - a_1)(x + a_1 - b_1)$ . The common limit of  $A_n$  and  $B_n$  is the  $M_2(a_1, b_1)$ , the square of the  $p$ -adic AGM of  $\sqrt{a_1}$  and  $\sqrt{b_1}$ . This is given by `ellQp_Ei` and is used by corresponding ascending and descending  $p$ -adic Landen transforms:

void `Qp_ascending_Landen`(GEN `ABR`, GEN `*ptx`, GEN `*pty`)

void `Qp_descending_Landen`(GEN `ABR`, GEN `*ptx`, GEN `*pty`)

### 10.9.5 Cached constants.

The cached constant is returned at its current precision, which may be larger than `prec`. One should always use the `mpxxx` variant: `mppi`, `mpeuler`, or `mplog2`.

GEN `consteuler`(long `prec`) precomputes Euler-Mascheroni's constant at precision `prec`.

GEN `constcatalan`(long `prec`) precomputes Catalan's constant at precision `prec`.

GEN `constpi`(long `prec`) precomputes  $\pi$  at precision `prec`.

GEN `constlog2`(long `prec`) precomputes  $\log(2)$  at precision `prec`.

`void constbern(long n)` precomputes the  $n$  even Bernoulli numbers  $B_2, \dots, B_{2n}$  as `t_FRAC`. No more than  $n$  Bernoulli numbers will ever be stored (by `bernfrac` or `bernreal`), unless a subsequent call to `constbern` increases the cache.

`GEN constzeta(long n, long prec)` ensures that the  $n$  values  $\gamma, \zeta(2), \dots, \zeta(n)$  are cached at accuracy bigger than or equal to `prec` and return a vector containing at least those value. Note that  $\gamma = \lim_1 \zeta(s) - 1/(s-1)$ . If the accuracy of cached data is too low or  $n$  is greater than the cache length, the cache is recomputed at the given parameters.

The following functions use cached data if `prec` is smaller than the precision of the cached value; otherwise the newly computed data replaces the old cache.

`GEN mppi(long prec)` returns  $\pi$  at precision `prec`.

`GEN Pi2n(long n, long prec)` returns  $2^n\pi$  at precision `prec`.

`GEN PiI2(long n, long prec)` returns the complex number  $2\pi i$  at precision `prec`.

`GEN PiI2n(long n, long prec)` returns the complex number  $2^n\pi i$  at precision `prec`.

`GEN mpeuler(long prec)` returns Euler-Mascheroni's constant at precision `prec`.

`GEN mpeuler(long prec)` returns Catalan's number at precision `prec`.

`GEN mplog2(long prec)` returns  $\log 2$  at precision `prec`.

The following functions use the Bernoulli numbers cache initialized by `constbern`:

`GEN bernreal(long i, long prec)` returns the Bernoulli number  $B_i$  as a `t_REAL` at precision `prec`. If `constbern(n)` was called previously with  $n \geq i$ , then the cached value is (converted to a `t_REAL` of accuracy `prec` then) returned. Otherwise, the missing value is computed; the cache is not updated.

`GEN bernfrac(long i)` returns the Bernoulli number  $B_i$  as a rational number (`t_FRAC` or `t_INT`). If the `constbern` cache includes  $B_i$ , the latter is returned. Otherwise, the missing value is computed; the cache is not updated.

### 10.9.6 Obsolete functions.

`void mpbern(long n, long prec)`

## 10.10 Permutations .

Permutations are represented in two different ways

- (perm) a `t_VECSMALL`  $p$  representing the bijection  $i \mapsto p[i]$ ; unless mentioned otherwise, this is the form used in the functions below for both input and output,

- (cyc) a `t_VEC` of `t_VECSMALL`s representing a product of disjoint cycles.

`GEN identity_perm(long n)` return the identity permutation on  $n$  symbols.

`GEN cyclic_perm(long n, long d)` return the cyclic permutation mapping  $i$  to  $i + d \pmod{n}$  in  $S_n$ . Assume that  $d \leq n$ .

`GEN perm_mul(GEN s, GEN t)` multiply  $s$  and  $t$  (composition  $s \circ t$ )

`GEN perm_sqr(GEN s)` multiply  $s$  by itself (composition  $s \circ s$ )

`GEN perm_conj(GEN s, GEN t)` return  $sts^{-1}$ .  
`int perm_commute(GEN p, GEN q)` return 1 if  $p$  and  $q$  commute, 0 otherwise.  
`GEN perm_inv(GEN p)` returns the inverse of  $p$ .  
`GEN perm_pow(GEN p, GEN n)` returns  $p^n$ .  
`GEN perm_powu(GEN p, ulong n)` returns  $p^n$ .  
`GEN cyc_pow_perm(GEN p, long n)` the permutation  $p$  is given as a product of disjoint cycles (`cyc`); return  $p^n$  (as a `perm`).  
`GEN cyc_pow(GEN p, long n)` the permutation  $p$  is given as a product of disjoint cycles (`cyc`); return  $p^n$  (as a `cyc`).  
`GEN perm_cycles(GEN p)` return the cyclic decomposition of  $p$ .  
`GEN perm_order(GEN p)` returns the order of the permutation  $p$  (as the lcm of its cycle lengths).  
`ulong perm_orderu(GEN p)` returns the order of the permutation  $p$  (as the lcm of its cycle lengths) assuming it fits in a `ulong`.  
`long perm_sign(GEN p)` returns the sign of the permutation  $p$ .  
`GEN vecperm_orbits(GEN gen, long n)` return the orbits of  $\{1, 2, \dots, n\}$  under the action of the subgroup of  $S_n$  generated by  $gen$ .  
`GEN Z_to_perm(long n, GEN x)` as `numtoperm`, returning a `t_VECSMALL`.  
`GEN perm_to_Z(GEN v)` as `permtonum` for a `t_VECSMALL` input.  
`GEN perm_to_GAP(GEN p)` return a `t_STR` which is a representation of  $p$  compatible with the GAP computer algebra system.

## 10.11 Small groups.

The small (finite) groups facility is meant to deal with subgroups of Galois groups obtained by `galoisinit` and thus is currently limited to weakly super-solvable groups.

A group  $grp$  of order  $n$  is represented by its regular representation (for an arbitrary ordering of its element) in  $S_n$ . A subgroup of such group is represented by the restriction of the representation to the subgroup. A *small group* can be either a group or a subgroup. Thus it is embedded in some  $S_n$ , where  $n$  is the multiple of the order. Such an  $n$  is called the *domain* of the small group. The domain of a trivial subgroup cannot be derived from the subgroup data, so some functions require the subgroup domain as argument.

The small group  $grp$  is represented by a `t_VEC` with two components:

$grp[1]$  is a generating subset  $[s_1, \dots, s_g]$  of  $grp$  expressed as a vector of permutations of length  $n$ .

$grp[2]$  contains the relative orders  $[o_1, \dots, o_g]$  of the generators  $grp[1]$ .

See `galoisinit` for the technical details.

`GEN checkgroup(GEN gal, GEN *elts)` check whether  $gal$  is a small group or a Galois group. Returns the underlying small group and set  $elts$  to the list of elements or to `NULL` if it is not known.



`GEN checkgroupelts(GEN gal)` check whether *gal* is a small group or a Galois group, or a vector of permutations listing the group elements. Returns the list of group elements as permutations.

`GEN galois_group(GEN gal)` return the underlying small group of the Galois group *gal*.

`GEN cyclicgroup(GEN g, long s)` return the cyclic group with generator *g* of order *s*.

`GEN trivialgroup(void)` return the trivial group.

`GEN dicyclicgroup(GEN g1, GEN g2, long s1, long s2)` returns the group with generators *g1*, *g2* with respecting relative orders *s1*, *s2*.

`GEN abelian_group(GEN v)` let *v* be a `t_VECSMALL` seen as the SNF of a small abelian group, return its regular representation.

`long group_domain(GEN grp)` returns the domain of the *nontrivial* small group *grp*. Return an error if *grp* is trivial.

`GEN group_elts(GEN grp, long n)` returns the list of elements of the small group *grp* of domain *n* as permutations.

`GEN groupelts_to_group(GEN elts)`, where *elts* is the list of elements of a group, returns the corresponding small group, if it exists, otherwise return `NULL`.

`GEN group_set(GEN grp, long n)` returns a  $F2v$  *b* such that *b*[*i*] is set if and only if the small group *grp* of domain *n* contains a permutation sending 1 to *i*.

`GEN groupelts_set(GEN elts, long n)`, where *elts* is the list of elements of a small group of domain *n*, returns a  $F2v$  *b* such that *b*[*i*] is set if and only if the small group contains a permutation sending 1 to *i*.

`GEN groupelts_conj_set(GEN elts, GEN p)`, where *elts* is the list of elements of a small group of domain *n*, returns a  $F2v$  *b* such that *b*[*i*] is set if and only if the small group contains a permutation sending  $p^{-1}[1]$  to  $p^{-1}[i]$ .

`int group_subgroup_is_faithful(GEN G, GEN H)` return 1 if the action of *G* on  $G/H$  by translation is faithful, 0 otherwise.

`GEN groupelts_conjclasses(GEN elts, long *pn)`, where *elts* is the list of elements of a small group (sorted with respect to `vecsmall_lexcmp`), return a `t_VECSMALL` *conj* of the same length such that *conj*[*i*] is the index in  $\{1, \dots, n\}$  of the conjugacy class of *elts*[*i*] for some unspecified but deterministic ordering of the classes, where *n* is the number of conjugacy classes. If *pn* is non `NULL`, *\*pn* is set to *n*.

`GEN conjclasses_repr(GEN conj, long nb)`, where *conj* and *nb* are as returned by the call `groupelts_conjclasses(elts)`, return `t_VECSMALL` of length *nb* which gives the indices in *elts* of a representative of each conjugacy class.

`GEN group_to_cc(GEN G)`, where *G* is a small group or a Galois group, returns a *cc* (conjugacy classes) structure [*elts*,*conj*,*rep*,*flag*], as obtained by `alggroupcenter`, where *conj* is `groupelts_conjclasses(elts)` and *rep* is the attached `conjclasses_repr`. *flag* is 1 if the permutation representation is transitive (in which case an element *g* of *G* is characterized by *g*[1]), and 0 otherwise. Shallow function.

`long group_order(GEN grp)` returns the order of the small group *grp* (which is the product of the relative orders).

`long group_isabelian(GEN grp)` returns 1 if the small group *grp* is Abelian, else 0.

`GEN group_abelianHNF(GEN grp, GEN elts)` if *grp* is not Abelian, returns NULL, else returns the HNF matrix of *grp* with respect to the generating family *grp*[1]. If *elts* is no NULL, it must be the list of elements of *grp*.

`GEN group_abelianSNF(GEN grp, GEN elts)` if *grp* is not Abelian, returns NULL, else returns its cyclic decomposition. If *elts* is no NULL, it must be the list of elements of *grp*.

`long group_subgroup_isnormal(GEN G, GEN H)`, *H* being a subgroup of the small group *G*, returns 1 if *H* is normal in *G*, else 0.

`long group_isA4S4(GEN grp)` returns 1 if the small group *grp* is isomorphic to  $A_4$ , 2 if it is isomorphic to  $S_4$ , 3 if it is isomorphic to  $(3 \times 3) : 4$  and 0 else. This is mainly to deal with the idiosyncrasy of the format.

`GEN group_leftcoset(GEN G, GEN g)` where *G* is a small group and *g* a permutation of the same domain, the left coset  $gG$  as a vector of permutations.

`GEN group_rightcoset(GEN G, GEN g)` where *G* is a small group and *g* a permutation of the same domain, the right coset  $Gg$  as a vector of permutations.

`long group_perm_normalize(GEN G, GEN g)` where *G* is a small group and *g* a permutation of the same domain, return 1 if  $gGg^{-1} = G$ , else 0.

`GEN group_quotient(GEN G, GEN H)`, where *G* is a small group and *H* is a subgroup of *G*, returns the quotient map  $G \rightarrow G/H$  as an abstract data structure.

`GEN groupeelts_quotient(GEN elts, GEN H)`, where *elts* is the list of elements of a small group *G*, *H* is a subgroup of *G*, returns the quotient map  $G \rightarrow G/H$  as an abstract data structure.

`GEN quotient_perm(GEN C, GEN g)` where *C* is the quotient map  $G \rightarrow G/H$  for some subgroup *H* of *G* and *g* an element of *G*, return the image of *g* by *C* (i.e. the coset  $gH$ ).

`GEN quotient_group(GEN C, GEN G)` where *C* is the quotient map  $G \rightarrow G/H$  for some *normal* subgroup *H* of *G*, return the quotient group  $G/H$  as a small group.

`GEN quotient_groupeelts(GEN C)` where *C* is the quotient map  $G \rightarrow G/H$  for some group *G* and some *normal* subgroup *H* of *G*, return the list of elements of the quotient group  $G/H$  (as permutations over corresponding to the regular representation).

`GEN quotient_subgroup_lift(GEN C, GEN H, GEN S)` where *C* is the quotient map  $G \rightarrow G/H$  for some group *G* normalizing *H* and *S* is a subgroup of  $G/H$ , return the inverse image of *S* by *C*.

`GEN group_subgroups(GEN grp)` returns the list of subgroups of the small group *grp* as a `t_VEC`.

`GEN groupeelts_solvablesubgroups(GEN elts)` where *elts* is the list of elements of a finite group, returns the list of its solvable subgroups, each as a list of its elements.

`GEN subgroups_tableset(GEN S, long n)` where *S* is a vector of subgroups of domain *n*, returns a table which matches the set of elements of the subgroups against the index of the subgroups.

`long tableset_find_index(GEN tbl, GEN set)` searches the set *set* in the table *tbl* and returns its attached index, or 0 if not found.

`GEN groupeelts_abelian_group(GEN elts)` where *elts* is the list of elements of an *Abelian* small group, returns the corresponding small group.

`long groupeelts_exponent(GEN elts)` where *elts* is the list of elements of a small group, returns the exponent the group (the LCM of the order of the elements of the group).

`GEN groupelts_center(GEN elts)` where *elts* is the list of elements of a small group, returns the list of elements of the center of the group.

`GEN group_export(GEN grp, long format)` convert a small group to another format, as a `t_STR` describing the group for the given syntax, see `galoisexport`.

`GEN group_export_GAP(GEN G)` export a small group to GAP format.

`GEN group_export_MAGMA(GEN G)` export a small group to MAGMA format.

`long group_ident(GEN grp, GEN elts)` returns the index of the small group *grp* in the GAP4 Small Group library, see `galoisidentify`. If *elts* is not `NULL`, it must be the list of elements of *grp*.

`long group_ident_trans(GEN grp, GEN elts)` returns the index of the regular representation of the small group *grp* in the GAP4 Transitive Group library, see `polgalois`. If *elts* is no `NULL`, it must be the list of elements of *grp*.



## Chapter 11:

### Standard data structures

#### 11.1 Character strings.

##### 11.1.1 Functions returning a char \*.

`char* pari_strdup(const char *s)` returns a malloc'ed copy of *s* (uses `pari_malloc`).

`char* pari_strndup(const char *s, long n)` returns a malloc'ed copy of at most *n* chars from *s* (uses `pari_malloc`). If *s* is longer than *n*, only *n* characters are copied and a terminal null byte is added.

`char* stack_strdup(const char *s)` returns a copy of *s*, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_strcat(const char *s, const char *t)` returns the concatenation of *s* and *t*, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_sprintf(const char *fmt, ...)` runs `pari_sprintf` on the given arguments, returning a string allocated on the PARI stack.

`char* uordinal(ulong x)` return the ordinal number attached to *x* (i.e. 1st, 2nd, etc.) as a `stack_malloc`'ed string.

`char* itostr(GEN x)` writes the `t_INT` *x* to a `stack_malloc`'ed string.

`char* GENTostr(GEN x)`, using the current default output format (`GP_DATA->fmt`, which contains the output style and the number of significant digits to print), converts *x* to a malloc'ed string. Simple variant of `pari_sprintf`.

`char* GENTostr_raw(GEN x)` as `GENTostr` with the following differences: 1) the output format is `f_RAW`; 2) the result is allocated on the stack and *must not* be freed.

`char* GENTostr_unquoted(GEN x)` as `GENTostr_raw` with the following additional difference: a `t_STR` *x* is printed without enclosing quotes (to be used by `print`).

`char* GENToTeXstr(GEN x)`, as `GENTostr`, except that `f_TEX` overrides the output format from `GP_DATA->fmt`.

`char* RgV_to_str(GEN g, long flag)` *g* being a vector of GENs, returns a malloc'ed string, the concatenation of the `GENTostr` applied to its elements, except that `t_STR` are printed without enclosing quotes. `flag` determines the output format: `f_RAW`, `f_PRETTYMAT` or `f_TEX`.

### 11.1.2 Functions returning a `t_STR`.

`GEN strtogenstr(const char *s)` returns a `t_STR` with content `s`.

`GEN strntogenstr(const char *s, long n)` returns a `t_STR` containing the first `n` characters of `s`.

`GEN chartogenstr(char c)` returns a `t_STR` containing the character `c`.

`GEN GENTogenstr(GEN x)` returns a `t_STR` containing the printed form of `x` (in `raw` format). This is often easier to use than `GENTostr` (which returns a malloc'ed `char*`) since there is no need to free the string after use.

`GEN GENTogenstr_nospace(GEN x)` as `GENTogenstr`, removing all spaces from the output.

`GEN Str(GEN g)` as `RgV_to_str` with output format `f_RAW`, but returns a `t_STR`, not a malloc'ed string.

`GEN strtex(GEN g)` as `RgV_to_str` with output format `f_TEX`, but returns a `t_STR`, not a malloc'ed string.

`GEN strexpend(GEN g)` as `RgV_to_str` with output format `f_RAW`, performing tilde and environment expansion on the result. Returns a `t_STR`, not a malloc'ed string.

`GEN gsprintf(const char *fmt, ...)` equivalent to `pari_sprintf(fmt, ...)`, followed by `strtoGENstr`. Returns a `t_STR`, not a malloc'ed string.

`GEN gvsprintf(const char *fmt, va_list ap)` variadic version of `gsprintf`

### 11.1.3 Dynamic strings.

A `pari_str` is a dynamic string which grows dynamically as needed. This structure contains private data and two public members `char *string`, which is the string itself and `use_stack` which tells whether the string lives

- on the PARI stack (value 1), meaning that it will be destroyed by any manipulation of the stack, e.g. a `gerepile` call or resetting `avma`;
- in malloc'ed memory (value 0), in which case it is impervious to stack manipulation but will need to be explicitly freed by the user after use, via `pari_free(s.string)`.

`void str_init(pari_str *S, int use_stack)` initializes a dynamic string; if `use_stack` is 0, then the string is malloc'ed, else it lives on the PARI stack.

`void str_printf(pari_str *S, const char *fmt, ...)` write to the end of `S` the remaining arguments according to PARI format `fmt`.

`void str_putc(pari_str *S, char c)` write the character `c` to the end of `S`.

`void str_puts(pari_str *S, const char *s)` write the string `s` to the end of `S`.

## 11.2 Output.

### 11.2.1 Output contexts.

An output context, of type `PariOUT`, is a `struct` that models a stream and contains the following function pointers:

```
void (*putch)(char);          /* fputc()-alike */
void (*puts)(const char*);    /* fputs()-alike */
void (*flush)(void);          /* fflush()-alike */
```

The methods `putch` and `puts` are used to print a character or a string respectively. The method `flush` is called to finalize a messages.

The generic functions `pari_putc`, `pari_puts`, `pari_flush` and `pari_printf` print according to a *default output context*, which should be sufficient for most purposes. Lower level functions are available, which take an explicit output context as first argument:

`void out_putc(PariOUT *out, char c)` essentially equivalent to `out->putc(c)`. In addition, registers whether the last character printed was a `\n`.

`void out_puts(PariOUT *out, const char *s)` essentially equivalent to `out->puts(s)`. In addition, registers whether the last character printed was a `\n`.

`void out_printf(PariOUT *out, const char *fmt, ...)`

`void out_vprintf(PariOUT *out, const char *fmt, va_list ap)`

N.B. The function `out_flush` does not exist since it would be identical to `out->flush()`

`int pari_last_was_newline(void)` returns a nonzero value if the last character printed via `out_putc` or `out_puts` was `\n`, and 0 otherwise.

`void pari_set_last_newline(int last)` sets the boolean value to be returned by the function `pari_last_was_newline` to *last*.

**11.2.2 Default output context.** They are defined by the global variables `pariOut` and `pariErr` for normal outputs and warnings/errors, and you probably do not want to change them. If you *do* change them, diverting output in nontrivial ways, this probably means that you are rewriting `gp`. For completeness, we document in this section what the default output contexts do.

**pariOut.** writes output to the `FILE*` `pari_outfile`, initialized to `stdout`. The low-level methods are actually the standard `putc` / `fputs`, plus some magic to handle a log file if one is open.

**pariErr.** prints to the `FILE*` `pari_errfile`, initialized to `stderr`. The low-level methods are as above.

You can stick with the default `pariOut` output context and change PARI's standard output, redirecting `pari_outfile` to another file, using

`void switchout(const char *name)` where `name` is a character string giving the name of the file you want to write to; the output is *appended* at the end of the file. To close the file and revert to outputting to `stdout`, call `switchout(NULL)`.

**11.2.3 PARI colors.** In this section we describe the low-level functions used to implement GP's color scheme, attached to the `colors` default. The following symbolic names are attached to gp's output strings:

- `c_ERR` an error message
- `c_HIST` a history number (as in `%1 = ...`)
- `c_PROMPT` a prompt
- `c_INPUT` an input line (minus the prompt part)
- `c_OUTPUT` an output
- `c_HELP` a help message
- `c_TIME` a timer
- `c_NONE` everything else

If the `colors` default is set to a nonempty value, before gp outputs a string, it first outputs an ANSI colors escape sequence — understood by most terminals —, according to the `colors` specifications. As long as this is in effect, the following strings are rendered in color, possibly in bold or underlined.

`void term_color(long c)` prints (as if using `pari_puts`) the ANSI color escape sequence attached to output object `c`. If `c` is `c_NONE`, revert to default printing style.

`void out_term_color(PariOUT *out, long c)` as `term_color`, using output context `out`.

`char* term_get_color(char *s, long c)` returns as a character string the ANSI color escape sequence attached to output object `c`. If `c` is `c_NONE`, the value used to revert to default printing style is returned. The argument `s` is either `NULL` (string allocated on the PARI stack), or preallocated storage (in which case, it must be able to hold at least 16 chars, including the final `\0`).

#### 11.2.4 Obsolete output functions.

These variants of `void output(GEN x)`, which prints `x`, followed by a newline and a buffer flush are complicated to use and less flexible than what we saw above, or than the `pari_printf` variants. They are provided for backward compatibility and are scheduled to disappear.

`void brute(GEN x, char format, long dec)`

`void matbrute(GEN x, char format, long dec)`

`void texe(GEN x, char format, long dec)`



## 11.3 Files.

The following routines are trivial wrappers around system functions (possibly around one of several functions depending on availability). They are usually integrated within PARI's diagnostics system, printing messages if the debug level for "files" is high enough.

`int pari_is_dir(const char *name)` returns 1 if `name` points to a directory, 0 otherwise.

`int pari_is_file(const char *name)` returns 1 if `name` points to a file, 0 otherwise.

`int file_is_binary(FILE *f)` returns 1 if the file `f` is a binary file (in the `writebin` sense), 0 otherwise.

`void pari_unlink(const char *s)` deletes the file named `s`. Warn if the operation fails.

`void pari_fread_chars(void *b, size_t n, FILE *f)` read `n` chars from stream `f`, storing the result in pre-allocated buffer `b` (assumed to be large enough).

`char* path_expand(const char *s)` perform tilde and environment expansion on `s`. Returns a malloc'ed buffer.

`void strftime_expand(const char *s, char *buf, long max)` perform time expansion on `s`, storing the result (at most `max` chars) in buffer `buf`. Trivial wrapper around

```
time_t t = time(NULL);
strftime(buf, max, s, localtime(&t));
```

`char* pari_get_homedir(const char *user)` expands `~user` constructs, returning the home directory of user `user`, or NULL if it could not be determined (in particular if the operating system has no such concept). The return value may point to static area and may be overwritten by subsequent system calls: use immediately or `strdup` it.

`int pari_stdin_isatty(void)` returns 1 if our standard input `stdin` is attached to a terminal. Trivial wrapper around `isatty`.

### 11.3.1 pariFILE.

PARI maintains a linked list of open files, to reclaim resources (file descriptors) on error or interrupts. The corresponding data structure is a `pariFILE`, which is a wrapper around a standard `FILE*`, containing further the file name, its type (regular file, pipe, input or output file, etc.). The following functions create and manipulate this structure; they are integrated within PARI's diagnostics system, printing messages if the debug level for "files" is high enough.

`pariFILE* pari_fopen(const char *s, const char *mode)` wrapper around `fopen(s, mode)`, return NULL on failure.

`pariFILE* pari_fopen_or_fail(const char *s, const char *mode)` simple wrapper around `fopen(s, mode)`; error on failure.

`pariFILE* pari_fopengz(const char *s)` opens the file whose name is `s`, and associates a (read-only) `pariFILE` with it. If `s` is a compressed file (`.gz` suffix), it is uncompressed on the fly. If `s` cannot be opened, also try to open `s.gz`. Returns NULL on failure.

`void pari_fclose(pariFILE *f)` closes the underlying file descriptor and deletes the `pariFILE` struct.

`pariFILE* pari_safeopen(const char *s, const char *mode)` creates a *new* file `s` (a priori for writing) with 600 permissions. Error if the file already exists. To avoid symlink attacks, a symbolic link exists, regardless of where it points to.

### 11.3.2 Temporary files.

PARI has its own idea of the system temp directory derived from an environment variable (\$GPTMPDIR, else \$TMPDIR), or the first writable directory among /tmp, /var/tmp and ..

`char* pari_unique_dir(const char *s)` creates a “unique directory” and return its name built from the string *s*, the user id and process pid (on Unix systems). This directory is itself located in the temp directory mentioned above. The name returned is malloc’ed.

`char* pari_unique_filename(const char *s)` creates a *new* empty file in the temp directory, whose name contains the id-string *s* (truncated to its first 8 chars), followed by a system-dependent suffix (incorporating the ids of both the user and the running process, for instance). The function returns the tempfile name and creates an empty file with that name. The name returned is malloc’ed.

`char* pari_unique_filename_suffix(const char *s, const char *suf)` analogous to above `pari_unique_filename`, creating a (previously nonexistent) tempfile whose name ends with suffix *suf*.

## 11.4 Errors.

This section documents the various error classes, and the corresponding arguments to `pari_err`. The general syntax is

```
void pari_err(numerr, ...)
```

In the sequel, we mostly use sequences of arguments of the form

```
const char *s
const char *fmt, ...
```

where *fmt* is a PARI format, producing a string *s* from the remaining arguments. Since providing the correct arguments to `pari_err` is quite error-prone, we also provide specialized routines `pari_err_ERRORCLASS(...)` instead of `pari_err(e_ERRORCLASS, ...)` so that the C compiler can check their arguments.

We now inspect the list of valid keywords (error classes) for `numerr`, and the corresponding required arguments.

### 11.4.1 Internal errors, “system” errors.

**11.4.1.1 e\_ARCH.** A requested feature *s* is not available on this architecture or operating system.

```
pari_err(e_ARCH)
```

prints the error message: `sorry, 's' not available on this system.`

**11.4.1.2 e\_BUG.** A bug in the PARI library, in function *s*.

```
pari_err(e_BUG, const char *s)
pari_err_BUG(const char *s)
```

prints the error message: `Bug in s, please report.`

**11.4.1.3 e\_FILE.** Error while trying to open a file.

```
pari_err(e_FILE, const char *what, const char *name)
pari_err_FILE(const char *what, const char *name)
```

prints the error message: error opening *what*: '*name*'.

**11.4.1.4 e\_FILEDESC.** Error while handling a file descriptor.

```
pari_err(e_FILEDESC, const char *where, long n)
pari_err_FILEDESC(const char *where, long n)
```

prints the error message: invalid file descriptor in *where*: '*name*'.

**11.4.1.5 e\_IMPL.** A requested feature *s* is not implemented.

```
pari_err(e_IMPL, const char *s)
pari_err_IMPL(const char *s)
```

prints the error message: sorry, *s* is not yet implemented.

**11.4.1.6 e\_PACKAGE.** Missing optional package *s*.

```
pari_err(e_PACKAGE, const char *s)
pari_err_PACKAGE(const char *s)
```

prints the error message: package *s* is required, please install it

**11.4.2 Syntax errors, type errors.**

**11.4.2.1 e\_DIM.** arguments submitted to function *s* have inconsistent dimensions. E.g., when solving a linear system, or trying to compute the determinant of a nonsquare matrix.

```
pari_err(e_DIM, const char *s)
pari_err_DIM(const char *s)
```

prints the error message: inconsistent dimensions in *s*.

**11.4.2.2 e\_FLAG.** A flag argument is out of bounds in function *s*.

```
pari_err(e_FLAG, const char *s)
pari_err_FLAG(const char *s)
```

prints the error message: invalid flag in *s*.

**11.4.2.3 e\_NOTFUNC.** Generated by the PARI evaluator; tried to use a GEN which is not a t\_CLOSURE in a function call syntax (as in `f = 1; f(2);`).

```
pari_err(e_NOTFUNC, GEN fun)
```

prints the error message: not a function in a function call.

**11.4.2.4 e\_OP.** Impossible operation between two objects than cannot be typecast to a sensible common domain for deeper reasons than a type mismatch, usually for arithmetic reasons. As in  $0(2) + 0(3)$ : it is valid to add two t\_PADICs, provided the underlying prime is the same; so the addition is not forbidden a priori for type reasons, it only becomes so when inspecting the objects and trying to perform the operation.

```
pari_err(e_OP, const char *op, GEN x, GEN y)
pari_err_OP(const char *op, GEN x, GEN y)
```

As e.TYPE2, replacing forbidden by inconsistent.

**11.4.2.5 e\_PRIORITY.** object  $o$  in function  $s$  contains variables whose priority is incompatible with the expected operation. E.g. `Pol([x,1], 'y')`: this raises an error because it's not possible to create a polynomial whose coefficients involve variables with higher priority than the main variable.

```
pari_err(e_PRIORITY, const char *s, GEN o, const char *op, long v)
pari_err_PRIORITY(const char *s, GEN o, const char *op, long v)
```

prints the error message: `incorrect priority in s, variable  $v_o$  op  $v$ , were  $v_o$  is gvar(o).`

**11.4.2.6 e\_SYNTAX.** Syntax error, generated by the PARI parser.

```
pari_err(e_SYNTAX, const char *msg, const char *e, const char *entry)
```

where `msg` is a complete error message, and `e` and `entry` point into the *same* character string, which is the input that was incorrectly parsed: `e` points to the character where the parser failed, and `entry`  $\leq$  `e` points somewhat before.

Prints the error message: `msg`, followed by a colon, then a part of the input character string (in general `entry` itself, but an initial segment may be truncated if `e - entry` is large); a caret points at `e`, indicating where the error took place.

**11.4.2.7 e\_TYPE.** An argument  $x$  of function  $s$  had an unexpected type. (As in `factor("blah").`)

```
pari_err(e_TYPE, const char *s, GEN x)
pari_err_TYPE(const char *s, GEN x)
```

prints the error message: `incorrect type in s ( $t_x$ ), where  $t_x$  is the type of  $x$ .`

**11.4.2.8 e\_TYPE2.** Forbidden operation between two objects than cannot be typecast to a sensible common domain, because their types do not match up. (As in `Mod(1,2) + Pi.`)

```
pari_err(e_TYPE2, const char *op, GEN x, GEN y)
pari_err_TYPE2(const char *op, GEN x, GEN y)
```

prints the error message: `forbidden  $s\ t_x\ op\ t_y$ , where  $t_z$  denotes the type of  $z$ . Here,  $s$  denotes the spelled out name of the operator  $op \in \{+, *, /, \%, =\}$ , e.g. addition for "+" or assignment for "=". If  $op$  is not in the above operator, list, it is taken to be the already spelled out name of a function, e.g. "gcd", and the error message becomes forbidden  $op\ t_x, t_y$ .`

**11.4.2.9 e\_VAR.** polynomials  $x$  and  $y$  submitted to function  $s$  have inconsistent variables. E.g., considering the algebraic number `Mod(t,t^2+1)` in `nfini(x^2+1)`.

```
pari_err(e_VAR, const char *s, GEN x, GEN y)
pari_err_VAR(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent variables in s  $X \neq Y$ , where  $X$  and  $Y$  are the names of the variables of  $x$  and  $y$ , respectively.`

### 11.4.3 Overflows.

**11.4.3.1 e\_COMPONENT.** Trying to access an inexistent component of a vector/matrix/list: the index is less than 1 or greater than the allowed length.

```
pari_err(e_COMPONENT, const char *f, const char *op, GEN lim, GEN x)
pari_err_COMPONENT(const char *f, const char *op, GEN lim, GEN x)
```

prints the error message: `nonexistent component in f: index op lim`. Special case: if  $f$  is the empty string (no meaningful public function name can be used), we ignore it and print the message: `nonexistent component: index op lim`.

**11.4.3.2 e\_DOMAIN.** An argument  $x$  is not in the function's domain (as in `moebius(0)` or `zeta(1)`).

```
pari_err(e_DOMAIN, char *f, char *v, char *op, GEN lim, GEN x)
pari_err_DOMAIN(char *f, char *v, char *op, GEN lim, GEN x)
```

prints the error message: `domain error in f: v op lim`. Special case: if `op` is the empty string, we ignore `lim` and print the error message: `domain error in f: v out of range`.

**11.4.3.3 e\_MAXPRIME.** A function using the precomputed list of prime numbers ran out of primes.

```
pari_err(e_MAXPRIME, ulong c)
pari_err_MAXPRIME(ulong c)
```

prints the error message: `not enough precomputed primes, need primelimit ~c if c is nonzero`. And simply `not enough precomputed primes` otherwise.

**11.4.3.4 e\_MEM.** A call to `pari_malloc` or `pari_realloc` failed.

```
pari_err(e_MEM)
```

prints the error message: `not enough memory`.

**11.4.3.5 e\_OVERFLOW.** An object in function  $s$  becomes too large to be represented within PARI's hardcoded limits. (As in `2^2^2^10` or `exp(1e100)`, which overflow in `lg` and `expo`.)

```
pari_err(e_OVERFLOW, const char *s)
pari_err_OVERFLOW(const char *s)
```

prints the error message: `overflow in s`.

**11.4.3.6 e\_PREC.** Function  $s$  fails because input accuracy is too low. (As in `floor(1e100)` at default accuracy.)

```
pari_err(e_PREC, const char *s)
pari_err_PREC(const char *s)
```

prints the error message: `precision too low in s`.

**11.4.3.7 e\_STACK.** The PARI stack overflows.

```
pari_err(e_STACK)
```

prints the error message: `the PARI stack overflows !` as well as some statistics concerning stack usage.

#### 11.4.4 Errors triggered intentionally.

**11.4.4.1 e\_ALARM.** A timeout, generated by the `alarm` function.

```
pari_err(e_ALARM, const char *fmt, ...)
```

prints the error message: `s`.

**11.4.4.2 e\_USER.** A user error, as triggered by `error( $g_1, \dots, g_n$ )` in GP.

```
pari_err(e_USER, GEN g)
```

prints the error message: `user error:`, then the entries of the vector  $g$ .

### 11.4.5 Mathematical errors.

**11.4.5.1 e\_CONSTPOL.** An argument of function  $s$  is a constant polynomial, which does not make sense. (As in `galoisinit(Pol(1))`.)

```
pari_err(e_CONSTPOL, const char *s)
pari_err_CONSTPOL(const char *s)
```

prints the error message: `constant polynomial in s`.

**11.4.5.2 e\_COPRIME.** Function  $s$  expected two coprime arguments, and did receive  $x, y$  which were not.

```
pari_err(e_COPRIME, const char *s, GEN x, GEN y)
pari_err_COPRIME(const char *s, GEN x, GEN y)
```

prints the error message: `elements not coprime in s: x,y`.

**11.4.5.3 e\_INV.** Tried to invert a noninvertible object  $x$ .

```
pari_err(e_INV, const char *s, GEN x)
pari_err_INV(const char *s, GEN x)
```

prints the error message: `impossible inverse in s: x`. If  $x = \text{Mod}(a, b)$  is a `t_INTMOD` and  $a$  is not 0 mod  $b$ , this allows to factor the modulus, as  $\text{gcd}(a, b)$  is a nontrivial divisor of  $b$ .

**11.4.5.4 e\_IRREDPOL.** Function  $s$  expected an irreducible polynomial, and did not receive one. (As in `nfinit(x^2-1)`.)

```
pari_err(e_IRREDPOL, const char *s, GEN x)
pari_err_IRREDPOL(const char *s, GEN x)
```

prints the error message: `not an irreducible polynomial in s: x`.

**11.4.5.5 e\_MISC.** Generic uncategorized error.

```
pari_err(e_MISC, const char *fmt, ...)
```

prints the error message: `s`.

**11.4.5.6 e\_MODULUS.** moduli  $x$  and  $y$  submitted to function  $s$  are inconsistent. E.g., considering the algebraic number  $\text{Mod}(t, t^2+1)$  in `nfinit(t^3-2)`.

```
pari_err(e_MODULUS, const char *s, GEN x, GEN y)
pari_err_MODULUS(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent moduli in s, then the moduli`.

**11.4.5.7 e\_PRIME.** Function  $s$  expected a prime number, and did receive  $p$ , which was not. (As in `idealprimedec(nf, 4)`.)

```
pari_err(e_PRIME, const char *s, GEN x)
pari_err_PRIME(const char *s, GEN x)
```

prints the error message: `not a prime in s: x`.

**11.4.5.8 e\_ROOTS0.** An argument of function  $s$  is a zero polynomial, and we need to consider its roots. (As in `polroots(0)`.)

```
pari_err(e_ROOTS0, const char *s)
pari_err_ROOTS0(const char *s)
```

prints the error message: `zero polynomial in s`.

**11.4.5.9 e\_SQRTN.** Tried to compute an  $n$ -th root of  $x$ , which does not exist, in function  $s$ . (As in `sqrt(Mod(-1,3))`.)

```
pari_err(e_SQRTN, GEN x)
pari_err_SQRTN(GEN x)
```

prints the error message: `not an n-th power residue in s: x`.

## 11.4.6 Miscellaneous functions.

`long name_numerr(const char *s)` return the error number corresponding to an error name. E.g. `name_numerr("e_DIM")` returns `e_DIM`.

`const char* numerr_name(long errnum)` returns the error name corresponding to an error number. E.g. `name_numerr(e_DIM)` returns `"e_DIM"`.

`char* pari_err2str(GEN err)` returns the error message that would be printed on `t_ERROR err`. The name is allocated on the PARI stack and must not be freed.

## 11.5 Hashtables.

A **hashtable**, or associative array, is a set of pairs  $(k, v)$  of keys and values. PARI implements general extensible hashtables for fast data retrieval: when creating a table, we may either choose to use the PARI stack, or `malloc` so as to be stack-independent. A hashtable is implemented as a table of linked lists, each list containing all entries sharing the same hash value. The table length is a prime number, which roughly doubles as the table overflows by gaining new entries; both the current number of entries and the threshold before the table grows are stored in the table. Finally the table remembers the functions used to hash the entries's keys and to test for equality two entries hashed to the same value.

An entry, or **hashentry**, contains

- a key/value pair  $(k, v)$ , both of type `void*` for maximal flexibility,
- the hash value of the key, for the table hash function. This hash is mapped to a table index (by reduction modulo the table length), but it contains more information, and is used to bypass costly general equality tests if possible,
- a link pointer to the next entry sharing the same table cell.

```
typedef struct {
    void *key, *val;
    ulong hash; /* hash(key) */
    struct hashentry *next;
} hashentry;

typedef struct {
    ulong len; /* table length */
```

```

    hashentry **table; /* the table */
    ulong nb, maxnb; /* number of entries stored and max nb before enlarging */
    ulong pindex; /* prime index */
    ulong (*hash)(void *k); /* hash function */
    int (*eq)(void *k1, void *k2); /* equality test */
    int use_stack; /* use the PARI stack, resp. malloc */
} hashtable;

```

```

hashtable* hash_create(size, hash, eq, use_stack)
    ulong size;
    ulong (*hash)(void*);
    int (*eq)(void*,void*);
    int use_stack;

```

creates a hashtable with enough room to contain `size` entries. The functions `hash` and `eq` compute the hash value of keys and test keys for equality, respectively. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`hashtable* hash_create_ulong(ulong size, long stack)` special case when the keys are `ulong`s with ordinary equality test.

`hashtable* hash_create_str(ulong size, long stack)` special case when the keys are character strings with string equality test (and `hash_str` hash function).

`void hash_init(hashtable *h, ulong size, ulong (*hash)(void*), int (*eq)(void*, void*), use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `void*`. The functions `eq` test keys for equality. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`void hash_init_GEN(hashtable *h, ulong size, int (*eq)(GEN, GEN), use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `GEN`. The functions `eq` test keys for equality. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`. The hash used is `hash_GEN`.

`void hash_init_ulong(hashtable *h, ulong size, use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `ulong`. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`void hash_insert(hashtable *h, void *k, void *v)` inserts  $(k, v)$  in hashtable `h`. No copy is made: `k` and `v` themselves are stored. The implementation does not prevent one to insert two entries with equal keys `k`, but which of the two is affected by later commands is undefined.

`void hash_insert2(hashtable *h, void *k, void *v, ulong hash)` as `hash_insert`, assuming `h->hash(k)` is `hash`.

`void hash_insert_long(hashtable *h, void *k, long v)` as `hash_insert` but `v` is a `long`.

`hashentry* hash_search(hashtable *h, void *k)` look for an entry with key `k` in `h`. Return it if it one exists, and `NULL` if not.

`hashentry* hash_search2(hashtable *h, void *k, ulong hash)` as `hash_search` assuming `h->hash(k)` is `hash`.

`GEN hash_haskey_GEN(hashtable *h, void *k)` returns the associate value if the key `k` belongs to the hash, otherwise returns `NULL`.



`int hash_haskey_long(hashtable *h, void *k, long *v)` returns 1 if the key  $k$  belongs to the hash and set  $v$  to its value, otherwise returns 0.

`hashentry * hash_select(hashtable *h, void *k, void *E, int (*select)(void *, hashentry *))` variant of `hash_search`, useful when entries with identical keys are inserted: among the entries attached to key  $k$ , return one satisfying the selection criterion (such that `select(E,e)` is nonzero), or NULL if none exist.

`hashentry* hash_remove(hashtable *h, void *k)` deletes an entry  $(k,v)$  with key  $k$  from  $h$  and return it. (Return NULL if none was found.) Only the linking structures are freed, memory attached to  $k$  and  $v$  is not reclaimed.

`hashentry* hash_remove_select(hashtable *h, void *k, void *E, int(*select)(void*, hashentry *))` a variant of `hash_remove`, useful when entries with identical keys are inserted: among the entries attached to key  $k$ , return one satisfying the selection criterion (such that `select(E,e)` is nonzero) and delete it, or NULL if none exist. Only the linking structures are freed, memory attached to  $k$  and  $v$  is not reclaimed.

`GEN hash_keys(hashtable *h)` return in a `t_VECSMALL` the keys stored in hashtable  $h$ .

`GEN hash_values(hashtable *h)` return in a `t_VECSMALL` the values stored in hashtable  $h$ .

`void hash_destroy(hashtable *h)` deletes the hashtable, by removing all entries.

`void hash_dbg(hashtable *h)` print statistics for hashtable  $h$ , allows to evaluate the attached hash function performance on actual data.

Some interesting hash functions are available:

`ulong hash_str(const char *s)`

`ulong hash_str_len(const char *s, long len)` hash the prefix string containing the first `len` characters (assume `strlen(s) ≥ len`).

`ulong hash_GEN(GEN x)` generic hash function.

`ulong hash_zv(GEN x)` hash a `t_VECSMALL`.

## 11.6 Dynamic arrays.

A **dynamic array** is a generic way to manage stacks of data that need to grow dynamically. It allocates memory using `pari_malloc`, and is independent of the PARI stack; it even works before the `pari_init` call.

### 11.6.1 Initialization.

To create a stack of objects of type `foo`, we proceed as follows:

```
foo *t_foo;
pari_stack s_foo;
pari_stack_init(&s_foo, sizeof(*t_foo), (void**)&t_foo);
```

Think of `s_foo` as the controlling interface, and `t_foo` as the (dynamic) array tied to it. The value of `t_foo` may be changed as you add more elements.

**11.6.2 Adding elements.** The following function pushes an element on the stack.

```
/* access globals t_foo and s_foo */
void push_foo(foo x)
{
    long n = pari_stack_new(&s_foo);
    t_foo[n] = x;
}
```

**11.6.3 Accessing elements.**

Elements are accessed naturally through the `t_foo` pointer. For example this function swaps two elements:

```
void swapfoo(long a, long b)
{
    foo x;
    if (a > s_foo.n || b > s_foo.n) pari_err_BUG("swapfoo");
    x = t_foo[a];
    t_foo[a] = t_foo[b];
    t_foo[b] = x;
}
```

**11.6.4 Stack of stacks.** Changing the address of `t_foo` is not supported in general. In particular `realloc()`'ed array of stacks and stack of stacks are not supported.

**11.6.5 Public interface.** Let `s` be a `pari_stack` and `data` the data linked to it. The following public fields are defined:

- `s.alloc` is the number of elements allocated for `data`.
- `s.n` is the number of elements in the stack and `data[s.n-1]` is the topmost element of the stack. `s.n` can be changed as long as  $0 \leq s.n \leq s.alloc$  holds.

`void pari_stack_init(pari_stack *s, size_t size, void **data)` links `*s` to the data pointer `*data`, where `size` is the size of data element. The pointer `*data` is set to `NULL`, `s->n` and `s->alloc` are set to 0: the array is empty.

`void pari_stack_alloc(pari_stack *s, long nb)` makes room for `nb` more elements, i.e. makes sure that  $s.alloc \geq s.n + nb$ , possibly reallocating `data`.

`long pari_stack_new(pari_stack *s)` increases `s.n` by one unit, possibly reallocating `data`, and returns `s.n - 1`.

**Caveat.** The following construction is incorrect because `stack_new` can change the value of `t_foo`:

```
t_foo[ pari_stack_new(&s_foo) ] = x;
```

`void pari_stack_delete(pari_stack *s)` frees `data` and resets the stack to the state immediately following `stack_init` (`s->n` and `s->alloc` are set to 0).

`void * pari_stack_pushp(pari_stack *s, void *u)` This function assumes that `*data` is of pointer type. Pushes the element `u` on the stack `s`.

`void ** pari_stack_base(pari_stack *s)` returns the address of `data`, typecast to a `void **`.

## 11.7 Vectors and Matrices.

**11.7.1 Access and extract.** See Section 9.3.1 and Section 9.3.2 for various useful constructors. Coefficients are accessed and set using `gel`, `gcoeff`, see Section 5.2.7. There are many internal functions to extract or manipulate subvectors or submatrices but, like the accessors above, none of them are suitable for `gerepileupto`. Worse, there are no type verification, nor bound checking, so use at your own risk.

`GEN shallowcopy(GEN x)` returns a `GEN` whose components are the components of  $x$  (no copy is made). The result may now be used to compute in place without destroying  $x$ . This is essentially equivalent to

```
GEN y = cgetg(lg(x), typ(x));
for (i = 1; i < lg(x); i++) y[i] = x[i];
return y;
```

except that `t_MAT` is treated specially since shallow copies of all columns are made. The function also works for nonrecursive types, but is useless in that case since it makes a deep copy. If  $x$  is known to be a `t_MAT`, you may call `RgM_shallowcopy` directly; if  $x$  is known not to be a `t_MAT`, you may call `leafcopy` directly.

`GEN RgM_shallowcopy(GEN x)` returns `shallowcopy(x)`, where  $x$  is a `t_MAT`.

`GEN shallowtrans(GEN x)` returns the transpose of  $x$ , *without* copying its components, i. e., it returns a `GEN` whose components are (physically) the components of  $x$ . This is the internal function underlying `gtrans`.

`GEN shallowconcat(GEN x, GEN y)` concatenate  $x$  and  $y$ , *without* copying components, i. e., it returns a `GEN` whose components are (physically) the components of  $x$  and  $y$ .

`GEN shallowconcat1(GEN x)`  $x$  must be `t_VEC`, `t_COL` or `t_LIST`, concatenate its elements from left to right. Shallow version of `gconcat1`.

`GEN shallowmatconcat(GEN v)` shallow version of `matconcat`.

`GEN shallowextract(GEN x, GEN y)` extract components of the vector or matrix  $x$  according to the selection parameter  $y$ . This is the shallow analog of `extract0(x, y, NULL)`, see `vecextract`.

`GEN shallowmatextract(GEN M, GEN l1, GEN l2)` extract components of the matrix  $M$  according to the `t_VECSMALL`  $l1$  (list of lines indices) and  $l2$  (list of columns indices). This is the shallow analog of `extract0(x, l1, l2)`, see `vecextract`.

`GEN RgM_minor(GEN A, long i, long j)` given a square `t_MAT`  $A$ , return the matrix with  $i$ -th row and  $j$ -th column removed.

`GEN vconcat(GEN A, GEN B)` concatenate vertically the two `t_MAT`  $A$  and  $B$  of compatible dimensions. A `NULL` pointer is accepted for an empty matrix. See `shallowconcat`.

`GEN matslice(GEN A, long a, long b, long c, long d)` returns the submatrix  $A[a..b, c..d]$ . Assume  $a \leq b$  and  $c \leq d$ .

`GEN row(GEN A, long i)` return  $A[i, ]$ , the  $i$ -th row of the `t_MAT`  $A$ .

`GEN row_i(GEN A, long i, long j1, long j2)` return part of the  $i$ -th row of `t_MAT`  $A$ :  $A[i, j_1], A[i, j_1 + 1] \dots, A[i, j_2]$ . Assume  $j_1 \leq j_2$ .

GEN rowcopy(GEN A, long i) return the row  $A[i,]$  of the  $\mathbf{t\_MAT}$   $A$ . This function is memory clean and suitable for `gerepileupto`. See `row` for the shallow equivalent.

GEN rowslice(GEN A, long i1, long i2) return the  $\mathbf{t\_MAT}$  formed by the  $i_1$ -th through  $i_2$ -th rows of  $\mathbf{t\_MAT}$   $A$ . Assume  $i_1 \leq i_2$ .

GEN rowsplice(GEN A, long i) return the  $\mathbf{t\_MAT}$  formed from the coefficients of  $\mathbf{t\_MAT}$   $A$  with  $j$ -th row removed.

GEN rowpermute(GEN A, GEN p),  $p$  being a  $\mathbf{t\_VECSMALL}$  representing a list  $[p_1, \dots, p_n]$  of rows of  $\mathbf{t\_MAT}$   $A$ , returns the matrix whose rows are  $A[p_1,], \dots, A[p_n,]$ .

GEN rowslicepermute(GEN A, GEN p, long x1, long x2), short for

`rowslice(rowpermute(A,p), x1, x2)`

(more efficient).

GEN vecslice(GEN A, long j1, long j2), return  $A[j_1], \dots, A[j_2]$ . If  $A$  is a  $\mathbf{t\_MAT}$ , these correspond to *columns* of  $A$ . The object returned has the same type as  $A$  ( $\mathbf{t\_VECSMALL}$ ,  $\mathbf{t\_VEC}$ ,  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$ ). Assume  $j_1 \leq j_2$  or  $j_2 = j_1 - 1$  (return empty vector/matrix).

GEN vecsplice(GEN A, long j) return  $A$  with  $j$ -th entry removed ( $\mathbf{t\_VEC}$ ,  $\mathbf{t\_COL}$ ) or  $j$ -th column removed ( $\mathbf{t\_MAT}$ ).

GEN vecreverse(GEN A). Returns a GEN which has the same type as  $A$  ( $\mathbf{t\_VEC}$ ,  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$ ), and whose components are the  $A[n], \dots, A[1]$ . If  $A$  is a  $\mathbf{t\_MAT}$ , these are the *columns* of  $A$ .

void vecreverse\_inplace(GEN A) as `vecreverse`, but reverse  $A$  in place.

GEN vecpermute(GEN A, GEN p)  $p$  is a  $\mathbf{t\_VECSMALL}$  representing a list  $[p_1, \dots, p_n]$  of indices. Returns a GEN which has the same type as  $A$  ( $\mathbf{t\_VEC}$ ,  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$ ), and whose components are  $A[p_1], \dots, A[p_n]$ . If  $A$  is a  $\mathbf{t\_MAT}$ , these are the *columns* of  $A$ .

GEN vecsmallpermute(GEN A, GEN p) as `vecpermute` when  $A$  is a  $\mathbf{t\_VECSMALL}$ .

GEN vecslicepermute(GEN A, GEN p, long y1, long y2) short for

`vecslice(vecpermute(A,p), y1, y2)`

(more efficient).

### 11.7.2 Componentwise operations.

The following convenience routines automate trivial loops of the form

`for (i = 1; i < lg(a); i++) gel(v,i) = f(gel(a,i), gel(b,i))`

for suitable  $f$ :

GEN vecinv(GEN a). Given a vector  $a$ , returns the vector whose  $i$ -th component is `ginv(a[i])`.

GEN vecmul(GEN a, GEN b). Given  $a$  and  $b$  two vectors of the same length, returns the vector whose  $i$ -th component is `gmul(a[i], b[i])`.

GEN vecdiv(GEN a, GEN b). Given  $a$  and  $b$  two vectors of the same length, returns the vector whose  $i$ -th component is `gdiv(a[i], b[i])`.

GEN vecpow(GEN a, GEN n). Given  $n$  a  $\mathbf{t\_INT}$ , returns the vector whose  $i$ -th component is  $a[i]^n$ .

`GEN vecmodii(GEN a, GEN b)`. Assuming  $a$  and  $b$  are two ZV of the same length, returns the vector whose  $i$ -th component is `modii(a[i], b[i])`.

`GEN vecmoduu(GEN a, GEN b)`. Assuming  $a$  and  $b$  are two `t_VECSMALL` of the same length, returns the vector whose  $i$ -th component is  $a[i] \% b[i]$ .

Note that `vecadd` or `vecsub` do not exist since `gadd` and `gsub` have the expected behavior. On the other hand, `ginv` does not accept vector types, hence `vecinv`.

### 11.7.3 Low-level vectors and columns functions.

These functions handle `t_VEC` as an abstract container type of `GEN`s. No specific meaning is attached to the content. They accept both `t_VEC` and `t_COL` as input, but `col` functions always return `t_COL` and `vec` functions always return `t_VEC`.

**Note.** All the functions below are shallow.

`GEN const_col(long n, GEN x)` returns a `t_COL` of  $n$  components equal to  $x$ .

`GEN const_vec(long n, GEN x)` returns a `t_VEC` of  $n$  components equal to  $x$ .

`int vec_isconst(GEN v)` Returns 1 if all the components of  $v$  are equal, else returns 0.

`void vec_setconst(GEN v, GEN x)`  $v$  a pre-existing vector. Set all its components to  $x$ .

`int vec_is1to1(GEN v)` Returns 1 if the components of  $v$  are pair-wise distinct, i.e. if  $i \mapsto v[i]$  is a 1-to-1 mapping, else returns 0.

`GEN vec_append(GEN V, GEN s)` append  $s$  to the vector  $V$ .

`GEN vec_prepend(GEN V, GEN s)` prepend  $s$  to the vector  $V$ .

`GEN vec_shorten(GEN v, long n)` shortens the vector  $v$  to  $n$  components.

`GEN vec_lengthen(GEN v, long n)` lengthens the vector  $v$  to  $n$  components. The extra components are not initialized.

`GEN vec_insert(GEN v, long n, GEN x)` inserts  $x$  at position  $n$  in the vector  $v$ .

`GEN vec_equiv(GEN O)` given a vector of objects  $O$ , return a vector with  $n$  components where  $n$  is the number of distinct objects in  $O$ . The  $i$ -th component is a `t_VECSMALL` containing the indices of the elements in  $O$  having the same value. Applied to the image of a function evaluated on some finite set, it computes the fibers of the function.

`GEN vec_reduce(GEN O, GEN *pE)` given a vector of objects  $O$ , return the vector  $v$  (of the same type as  $O$ ) of *distinct* elements of  $O$  and set a `t_VECSMALL`  $E$  with the same length as  $v$ , such that  $E[i]$  is the multiplicity of object  $v[i]$  in the original  $O$ . Shallow function.

## 11.8 Vectors of small integers.

### 11.8.1 t\_VECSMALL.

These functions handle `t_VECSMALL` as an abstract container type of small signed integers. No specific meaning is attached to the content.

`GEN const_vecsmall(long n, long c)` returns a `t_VECSMALL` of `n` components equal to `c`.

`GEN vec_to_vecsmall(GEN z)` identical to `ZV_to_zv(z)`.

`GEN vecsmall_to_vec(GEN z)` identical to `zv_to_ZV(z)`.

`GEN vecsmall_to_col(GEN z)` identical to `zv_to_ZC(z)`.

`GEN vecsmall_to_vec_inplace(GEN z)` apply `stoi` to all entries of `z` and set its type to `t_VEC`.

`GEN vecsmall_copy(GEN x)` makes a copy of `x` on the stack.

`GEN vecsmall_shorten(GEN v, long n)` shortens the `t_VECSMALL` `v` to `n` components.

`GEN vecsmall_lengthen(GEN v, long n)` lengthens the `t_VECSMALL` `v` to `n` components. The extra components are not initialized.

`GEN vecsmall_indexsort(GEN x)` performs an indirect sort of the components of the `t_VECSMALL` `x` and return a permutation stored in a `t_VECSMALL`.

`void vecsmall_sort(GEN v)` sorts the `t_VECSMALL` `v` in place.

`GEN vecsmall_reverse(GEN v)` as `vecreverse` for a `t_VECSMALL` `v`.

`long vecsmall_max(GEN v)` returns the maximum of the elements of `t_VECSMALL` `v`, assumed nonempty.

`long vecsmall_indexmax(GEN v)` returns the index of the largest element of `t_VECSMALL` `v`, assumed nonempty.

`long vecsmall_min(GEN v)` returns the minimum of the elements of `t_VECSMALL` `v`, assumed nonempty.

`long vecsmall_indexmin(GEN v)` returns the index of the smallest element of `t_VECSMALL` `v`, assumed nonempty.

`int vecsmall_isconst(GEN v)` Returns 1 if all the components of `v` are equal, else returns 0.

`int vecsmall_is1to1(GEN v)` Returns 1 if the components of `v` are pair-wise distinct, i.e. if  $i \mapsto v[i]$  is a 1-to-1 mapping, else returns 0.

`long vecsmall_isin(GEN v, long x)` returns the first index  $i$  such that  $v[i]$  is equal to  $x$ . Naive search in linear time, does not assume that `v` is sorted.

`GEN vecsmall_uniq(GEN v)` given a `t_VECSMALL` `v`, return the vector of unique occurrences.

`GEN vecsmall_uniq_sorted(GEN v)` same as `vecsmall_uniq`, but assumes `v` sorted.

`long vecsmall_duplicate(GEN v)` given a `t_VECSMALL` `v`, return 0 if there is no duplicates, or the index of the first duplicate (`vecsmall_duplicate([1,1])` returns 2).

`long vecsmall_duplicate_sorted(GEN v)` same as `vecsmall_duplicate`, but assume `v` sorted.

`int vecsmall_lexcmp(GEN x, GEN y)` compares two `t_VECSMALL` lexically.

`int vecsmall_prefixcmp(GEN x, GEN y)` truncate the longest `t_VECSMALL` to the length of the shortest and compares them lexicographically.

`GEN vecsmall_prepend(GEN V, long s)` prepend `s` to the `t_VECSMALL` `V`.

`GEN vecsmall_append(GEN V, long s)` append `s` to the `t_VECSMALL` `V`.

`GEN vecsmall_concat(GEN u, GEN v)` concat the `t_VECSMALL` `u` and `v`.

`long vecsmall_coincidence(GEN u, GEN v)` returns the numbers of indices where `u` and `v` agree.

`long vecsmall_pack(GEN v, long base, long mod)` handles the `t_VECSMALL` `v` as the digit of a number in base `base` and return this number modulo `mod`. This can be used as an hash function.

`GEN vecsmall_prod(GEN v)` given a `t_VECSMALL` `v`, return the product of its entries.

**11.8.2 Vectors of `t_VECSMALL`.** These functions manipulate vectors of `t_VECSMALL` (`vecvecsmall`).

`GEN vecvecsmall_sort(GEN x)` sorts lexicographically the components of the vector `x`.

`GEN vecvecsmall_sort_shallow(GEN x)`, shallow variant of `vecvecsmall_sort`.

`void vecvecsmall_sort_inplace(GEN x, GEN *perm)` sort lexicographically `x` in place, without copying its components. If `perm` is not `NULL`, it is set to the permutation that would sort the original `x`.

`GEN vecvecsmall_sort_uniq(GEN x)` sorts lexicographically the components of the vector `x`, removing duplicates entries.

`GEN vecvecsmall_indexsort(GEN x)` performs an indirect lexicographic sorting of the components of the vector `x` and return a permutation stored in a `t_VECSMALL`.

`long vecvecsmall_search(GEN x, GEN y)` `x` being a sorted `vecvecsmall` and `y` a `t_VECSMALL`, search `y` inside `x`.

`GEN vecvecsmall_max(GEN x)` returns the largest entry in all  $x[i]$ , assumed nonempty. Shallow function.





## Chapter 12:

### Functions related to the GP interpreter

#### 12.1 Handling closures.

##### 12.1.1 Functions to evaluate `t_CLOSURE`.

`void closure_disassemble(GEN C)` print the `t_CLOSURE` `C` in GP assembly format.

`GEN closure_callgenall(GEN C, long n, ...)` evaluate the `t_CLOSURE` `C` with the `n` arguments (of type `GEN`) following `n` in the function call. Assumes `C` has arity  $\geq n$ .

`GEN closure_callgenvec(GEN C, GEN args)` evaluate the `t_CLOSURE` `C` with the arguments supplied in the vector `args`. Assumes `C` has arity  $\geq \lg(\text{args}) - 1$ .

`GEN closure_callgenvecprec(GEN C, GEN args, long prec)` as `closure_callgenvec` but set the precision locally to `prec`.

`GEN closure_callgenvecdef(GEN C, GEN args, GEN def)` evaluate the `t_CLOSURE` `C` with the arguments supplied in the vector `args`, where the `t_VECSMALL` `def` indicates which arguments are actually present. Assumes `C` has arity  $\geq \lg(\text{args}) - 1$ .

`GEN closure_callgenvecdefprec(GEN C, GEN args, GEN def, long prec)` as `closure_callgenvecdef` but set the precision locally to `prec`.

`GEN closure_callgen0prec(GEN C, long prec)` evaluate the `t_CLOSURE` `C` without arguments, but set the precision locally to `prec`.

`GEN closure_callgen1(GEN C, GEN x)` evaluate the `t_CLOSURE` `C` with argument `x`. Assumes `C` has arity  $\geq 1$ .

`GEN closure_callgen1prec(GEN C, GEN x, long prec)` as `closure_callgen1`, but set the precision locally to `prec`.

`GEN closure_callgen2(GEN C, GEN x, GEN y)` evaluate the `t_CLOSURE` `C` with argument `x`, `y`. Assumes `C` has arity  $\geq 2$ .

`void closure_callvoid1(GEN C, GEN x)` evaluate the `t_CLOSURE` `C` with argument `x` and discard the result. Assumes `C` has arity  $\geq 1$ .

The following technical functions are used to evaluate *inline* closures and closures of arity 0.

The control flow statements (`break`, `next` and `return`) will cause the evaluation of the closure to be interrupted; this is called below a *flow change*. When that occurs, the functions below generally return `NULL`. The caller can then adopt three positions:

- raises an exception (`closure_evalnobrk`).
- passes through (by returning `NULL` itself).
- handles the flow change.

`GEN closure_evalgen(GEN code)` evaluates a closure and returns the result, or `NULL` if a flow change occurred.

`GEN closure_evalnobrk(GEN code)` as `closure_evalgen` but raise an exception if a flow change occurs. Meant for iterators where interrupting the closure is meaningless, e.g. `intnum` or `sumnum`.

`void closure_evalvoid(GEN code)` evaluates a closure whose return value is ignored. The caller has to deal with eventual flow changes by calling `loop_break`.

The remaining functions below are for exceptional situations:

`GEN closure_evalres(GEN code)` evaluates a closure and returns the result. The difference with `closure_evalgen` being that, if the flow end by a `return` statement, the result will be the returned value instead of `NULL`. Used by the main GP loop.

`GEN closure_evalbrk(GEN code, long *status)` as `closure_evalres` but set `status` to a nonzero value if a flow change occurred. This variant is not stack clean. Used by the break loop.

`GEN closure_trapgen(long numerr, GEN code)` evaluates closure, while trapping error `numerr`. Return `(GEN)1L` if error trapped, and the result otherwise, or `NULL` if a flow change occurred. Used by trap.

### 12.1.2 Functions to handle control flow changes.

`long loop_break(void)` processes an eventual flow changes inside an iterator. If this function return 1, the iterator should stop.

### 12.1.3 Functions to deal with lexical local variables.

Function using the prototype code 'V' need to manually create and delete a lexical variable for each code 'V', which will be given a number  $-1, -2, \dots$

`void push_lex(GEN a, GEN code)` creates a new lexical variable whose initial value is  $a$  on the top of the stack. This variable get the number  $-1$ , and the number of the other variables is decreased by one unit. When the first variable of a closure is created, the argument `code` must be the closure that references this lexical variable. The argument `code` must be `NULL` for all subsequent variables (if any). (The closure contains the debugging data for the variable).

`void pop_lex(long n)` deletes the  $n$  topmost lexical variables, increasing the number of other variables by  $n$ . The argument  $n$  must match the number of variables allocated through `push_lex`.

`GEN get_lex(long vn)` get the value of the variable with number  $vn$ .

`void set_lex(long vn, GEN x)` set the value of the variable with number  $vn$ .

#### 12.1.4 Functions returning new closures.

GEN `compile_str(const char *s)` returns the closure corresponding to the GP expression `s`.

GEN `closure_deriv(GEN code)` returns a closure corresponding to the numerical derivative of the closure `code`.

GEN `closure_derivn(GEN code, long n)` returns a closure corresponding to the numerical derivative of order  $n > 0$  of the closure `code`.

GEN `snm_closure(entree *ep, GEN data)` Let `data` be a vector of length  $m$ , `ep` be an `entree` pointing to a C function  $f$  of arity  $n + m$ , returns a `t_CLOSURE` object  $g$  of arity  $n$  such that  $g(x_1, \dots, x_n) = f(x_1, \dots, x_n, \text{gel}(\text{data}, 1), \dots, \text{gel}(\text{data}, m))$ . If `data` is `NULL`, then  $m = 0$  is assumed. Shallow function.

GEN `strtofunction(char *str)` returns a closure corresponding to the built-in or install'ed function named `str`.

GEN `strtoclosure(char *str, long n, ...)` returns a closure corresponding to the built-in or install'ed function named `str` with the  $n$  last parameters set to the  $n$  GENs following `n`. This is analogous to `snm_closure(isentry(str), mkvecn(...))` but the latter has lower overhead since it does not copy arguments, nor does it validate inputs.

In the example code below, `agm1` is set to the function `x->agm(x,1)` and `res` is set to `agm(2,1)`.

```
GEN agm1 = strtoclosure("agm",1, gen_1);
GEN res = closure_callgen1(agm1, gen_2);
```

**12.1.5 Functions used by the gp debugger (break loop).** long `closure_context(long s)` restores the compilation context starting at frame `s+1`, and returns the index of the topmost frame. This allow to compile expressions in the topmost lexical scope.

void `closure_err(long level)` prints a backtrace of the last 20 stack frames, starting at frame `level`, the numbering starting at 0.

**12.1.6 Standard wrappers for iterators.** Two families of standard wrappers are provided to interface iterators like `intnum` or `sumnum` with GP.

**12.1.6.1 Standard wrappers for inline closures.** These wrappers are used to implement GP functions taking inline closures as input. The object (GEN)E must be an inline closure which is evaluated with the lexical variable number  $-1$  set to  $x$ .

GEN `gp_eval(void *E, GEN x)` is used for the prototype code 'E'.

GEN `gp_evalprec(void *E, GEN x, long prec)` as `gp_eval`, but set the precision locally to `prec`.

long `gp_evalvoid(void *E, GEN x)` is used for the prototype code 'I'. The resulting value is discarded. Return a nonzero value if a control-flow instruction request the iterator to terminate immediately.

long `gp_evalbool(void *E, GEN x)` returns the boolean `gp_eval(E, x)` evaluates to (i.e. true iff the value is nonzero).

GEN `gp_evalupto(void *E, GEN x)` memory-safe version of `gp_eval`, `gcopy`-ing the result, when the evaluator returns components of previously allocated objects (e.g. member functions).

**12.1.6.2 Standard wrappers for true closures.** These wrappers are used to implement GP functions taking true closures as input.

`GEN gp_call(void *E, GEN x)` evaluates the closure (GEN)E on  $x$ .

`GEN gp_callprec(void *E, GEN x, long prec)` as `gp_call`, but set the precision locally to `prec`.

`GEN gp_call2(void *E, GEN x, GEN y)` evaluates the closure (GEN)E on  $(x, y)$ .

`long gp_callbool(void *E, GEN x)` evaluates the closure (GEN)E on  $x$ , returns 1 if its result is nonzero, and 0 otherwise.

`long gp_callvoid(void *E, GEN x)` evaluates the closure (GEN)E on  $x$ , discarding the result. Return a nonzero value if a control-flow instruction request the iterator to terminate immediately.

## 12.2 Defaults.

`entree* pari_is_default(const char *s)` return the `entree` structure attached to  $s$  if it is the name of a default, NULL otherwise.

`GEN setdefault(const char *s, const char *v, long flag)` is the low-level function underlying `default0`. If  $s$  is NULL, call all default setting functions with string argument NULL and flag `d_ACKNOWLEDGE`. Otherwise, check whether  $s$  corresponds to a default and call the corresponding default setting function with arguments  $v$  and *flag*.

We shall describe these functions below: if  $v$  is NULL, we only look at the default value (and possibly print or return it, depending on *flag*); otherwise the value of the default to  $v$ , possibly after some translation work. The flag is one of

- `d_INITRC` called while reading the `gprc`: print and return `gnil`, possibly defer until `gp` actually starts.
- `d_RETURN` return the current value, as a `t_INT` if possible, as a `t_STR` otherwise.
- `d_ACKNOWLEDGE` print the current value, return `gnil`.
- `d_SILENT` print nothing, return `gnil`.

Low-level functions called by `setdefault`:

`GEN sd_Texstyle(const char *v, long flag)`

`GEN sd_breakloop(const char *v, long flag)`

`GEN sd_colors(const char *v, long flag)`

`GEN sd_compatible(const char *v, long flag)`

`GEN sd_datadir(const char *v, long flag)`

`GEN sd_debug(const char *v, long flag)`

`GEN sd_debugfiles(const char *v, long flag)`

`GEN sd_debugmem(const char *v, long flag)`

`GEN sd_echo(const char *v, long flag)`

`GEN sd_factor_add_primes(const char *v, long flag)`

```

GEN sd_factor_proven(const char *v, long flag)
GEN sd_format(const char *v, long flag)
GEN sd_graphcolormap(const char *v, long flag)
GEN sd_graphcolors(const char *v, long flag)
GEN sd_help(const char *v, long flag)
GEN sd_histfile(const char *v, long flag)
GEN sd_histsize(const char *v, long flag)
GEN sd_lines(const char *v, long flag)
GEN sd_linewrap(const char *v, long flag)
GEN sd_log(const char *v, long flag)
GEN sd_logfile(const char *v, long flag)
GEN sd_nbthreads(const char *v, long flag)
GEN sd_new_galois_format(const char *v, long flag)
GEN sd_output(const char *v, long flag)
GEN sd_parisize(const char *v, long flag)
GEN sd_parisizemax(const char *v, long flag)
GEN sd_path(const char *v, long flag)
GEN sd_plothsizes(const char *v, long flag)
GEN sd_prettyprinter(const char *v, long flag)
GEN sd_primelimit(const char *v, long flag)
GEN sd_prompt(const char *v, long flag)
GEN sd_prompt_cont(const char *v, long flag)
GEN sd_psfile(const char *v, long flag) The psfile default is obsolete, don't use this func-
tion.
GEN sd_readline(const char *v, long flag)
GEN sd_realbitprecision(const char *v, long flag)
GEN sd_realprecision(const char *v, long flag)
GEN sd_recover(const char *v, long flag)
GEN sd_secure(const char *v, long flag)
GEN sd_seriesprecision(const char *v, long flag)
GEN sd_simplify(const char *v, long flag)
GEN sd_sopath(const char *v, int flag)
GEN sd_strictargs(const char *v, long flag)

```

GEN sd\_strictmatch(const char \*v, long flag)

GEN sd\_timer(const char \*v, long flag)

GEN sd\_threadsize(const char \*v, long flag)

GEN sd\_threadsizemax(const char \*v, long flag)

Generic functions used to implement defaults: most of the above routines are implemented in terms of the following generic ones. In all routines below

- **v** and **flag** are the arguments passed to **default**: **v** is a new value (or the empty string: no change), and **flag** is one of **d\_INITRC**, **d\_RETURN**, etc.

- **s** is the name of the default being changed, used to display error messages or acknowledgements.

GEN sd\_toggle(const char \*v, long flag, const char \*s, int \*ptn)

- if **v** is neither "0" nor "1", an error is raised using **pari\_err**.
- **ptn** points to the current numerical value of the toggle (1 or 0), and is set to the new value (when **v** is nonempty).

For instance, here is how the timer default is implemented internally:

```
GEN
sd_timer(const char *v, long flag)
{ return sd_toggle(v,flag,"timer", &(GP_DATA->chrono)); }
```

The exact behavior and return value depends on **flag**:

- **d\_RETURN**: returns the new toggle value, as a **GEN**.
- **d\_ACKNOWLEDGE**: prints a message indicating the new toggle value and return **gnil**.
- other cases: print nothing and return **gnil**.

GEN sd\_ulong(const char \*v, long flag, const char \*s, ulong \*ptn, ulong Min, ulong Max, const char \*\*msg)

- **ptn** points to the current numerical value of the toggle, and is set to the new value (when **v** is nonempty).

- **Min** and **Max** point to the minimum and maximum values allowed for the default.

- **v** must translate to an integer in the allowed ranger, a suffix among **k/K** ( $\times 10^3$ ), **m/M** ( $\times 10^6$ ), or **g/G** ( $\times 10^9$ ) is allowed, but no arithmetic expression.

- **msg** is a [NULL]-terminated array of messages or NULL (ignored). If **msg** is not NULL, **msg[i]** contains a message attached to the value *i* of the default. The last entry in the **msg** array is used as a message attached to all subsequent ones.

The exact behavior and return value depends on **flag**:

- **d\_RETURN**: returns the new value, as a **GEN**.
- **d\_ACKNOWLEDGE**: prints a message indicating the new value, possibly a message attached to it via the **msg** argument, and return **gnil**.
- other cases: print nothing and return **gnil**.

GEN sd\_intarray(const char \*v, long flag, const char \*s, GEN \*pz)

- records a `t_VECSMALL` array of nonnegative integers.
- `pz` points to the current `t_VECSMALL` value, and is set to the new value (when `v` is nonempty).

The exact return value depends on `flag`:

- `d_RETURN`: returns the new value, as a `t_VEC` (converted via `zv_to_ZV`)
- `d_ACKNOWLEDGE`: prints a message indicating the new value, (as a `t_VEC`) and return `gnil`.
- other cases: print nothing and return `gnil`.

GEN sd\_string(const char \*v, long flag, const char \*s, char \*\*pstr) • `v` is subject to environment expansion, then time expansion.

- `pstr` points to the current string value, and is set to the new value (when `v` is nonempty).

## 12.3 Records and Lazy vectors.

The functions in this section are used to implement `ell` structures and analogous objects, which are vectors some of whose components are initialized to dummy values, later computed on demand. We start by initializing the structure:

GEN obj\_init(long d, long n) returns an *obj* `S`, a `t_VEC` with  $d$  regular components, accessed as `gel(S,1), ..., gel(S,d)`; together with a record of  $n$  members, all initialized to 0. The arguments  $d$  and  $n$  must be nonnegative.

After `S = obj_init(d, n)`, the prototype of our other functions are of the form

GEN obj\_do(GEN S, long tag, ...)

The first argument `S` holds the structure to be managed. The second argument *tag* is the index of the struct member (from 1 to  $n$ ) we operate on. We recommend to define an `enum` and use descriptive names instead of hardcoded numbers. For instance, if  $n = 3$ , after defining

```
enum { TAG_p = 1, TAG_list, TAG_data };
```

one may use `TAG_list` or 2 indifferently as a tag. The former being preferred, of course.

**Technical note.** In the current implementation,  $S$  is a `t_VEC` with  $d + 1$  entries. The first  $d$  components are ordinary `t_GEN` entries, which you can read or assign to in the customary way. But the last component `gel(S, d + 1)`, a `t_VEC` of length  $n$  initialized to `zerovec(n)`, must be handled in a special way: you should never access or modify its components directly, only through the API we are about to describe. Indeed, its entries are meant to contain dynamic data, which will be stored, retrieved and replaced (for instance by a value computed to a higher accuracy), while interacting safely with intermediate `gerepile` calls. This mechanism allows to simulate C `structs`, in a simpler way than with general hashtables, while remaining compatible with the GP language, which knows neither `structs` nor hashtables. It also serialize the structure in an ordinary `GEN`, which facilitates copies and garbage collection (use `gcopy` or `gerepile`), rather than having to deal with individual components of actual C `structs`.

`GEN obj_reinit(GEN S)` make a shallow copy of  $S$ , re-initializing all dynamic components. This allows “forking” a lazy vector while avoiding both a memory leak, and storing pointers to the same data in different objects (with risks of a double free later).

`GEN obj_check(GEN S, long tag)` if the *tag*-component in  $S$  is non empty, return it. Otherwise return `NULL`. The `t_INT 0` (initial value) is used as a sentinel to indicated an empty component.

`GEN obj_insert(GEN S, long tag, GEN O)` insert (a clone of)  $O$  as *tag*-component of  $S$ . Any previous value is deleted, and data pointing to it become invalid.

`GEN obj_insert_shallow(GEN S, long K, GEN O)` as `obj_insert`, inserting  $O$  as-is, not via a clone.

`GEN obj_checkbuild(GEN S, long tag, GEN (*build)(GEN))` if the *tag*-component of  $S$  is non empty, return it. Otherwise insert (a clone of) `build(S)` as *tag*-component in  $S$ , and return it.

`GEN obj_checkbuild_padicprec(GEN S, long tag, GEN (*build)(GEN, long), long prec)` if the *tag*-component of  $S$  is non empty *and* has relative  $p$ -adic precision  $\geq$  `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in  $S$ , and return it.

`GEN obj_checkbuild_realprec(GEN S, long tag, GEN (*build)(GEN, long), long prec)` if the *tag*-component of  $S$  is non empty *and* satisfies `gprecision`  $\geq$  `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in  $S$ , and return it.

`GEN obj_checkbuild_prec(GEN S, long tag, GEN (*build)(GEN, long), GEN (*gpr)(GEN), long prec)` if the *tag*-component of  $S$  is non empty *and* has precision `gpr(x)`  $\geq$  `prec`, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in  $S$ , and return it.

`void obj_free(GEN S)` destroys all clones stored in the  $n$  tagged components, and replace them by the initial value 0. The regular entries of  $S$  are unaffected, and  $S$  remains a valid object. This is used to avoid memory leaks.



# Chapter 13:

## Algebraic Number Theory

### 13.1 General Number Fields.

#### 13.1.1 Number field types.

None of the following routines thoroughly check their input: they distinguish between *bona fide* structures as output by PARI routines, but designing perverse data will easily fool them. To give an example, a square matrix will be interpreted as an ideal even though the  $\mathbf{Z}$ -module generated by its columns may not be an  $\mathbf{Z}_K$ -module (i.e. the expensive `nfisideal` routine will *not* be called).

`long nftyp(GEN x)`. Returns the type of number field structure stored in `x`, `typ_NF`, `typ_BNF`, or `typ_BNR`. Other answers are possible, meaning `x` is not a number field structure.

`GEN get_nf(GEN x, long *t)`. Extract an *nf* structure from `x` if possible and return it, otherwise return NULL. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_bnf(GEN x, long *t)`. Extract a *bnf* structure from `x` if possible and return it, otherwise return NULL. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_nfpol(GEN x, GEN *nf)` try to extract an *nf* structure from `x`, and sets `*nf` to NULL (failure) or to the *nf*. Returns the (monic, integral) polynomial defining the field.

`GEN get_bnfpol(GEN x, GEN *bnf, GEN *nf)` try to extract a *bnf* and an *nf* structure from `x`, and sets `*bnf` and `*nf` to NULL (failure) or to the corresponding structure. Returns the (monic, integral) polynomial defining the field.

`GEN checknf(GEN x)` if an *nf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_nf` is often more flexible.

`GEN checkbnf(GEN x)` if an *bnf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_bnf` is often more flexible.

`GEN checkbnf_i(GEN bnf)` same as `checkbnf` but return NULL instead of raising an exception.

`void checkbnr(GEN bnr)` Raise an exception if the argument is not a *bnr* structure.

`GEN checkbnr_i(GEN bnr)` same as `checkbnr` but returns the *bnr* or NULL instead of raising an exception.

`GEN checknf_i(GEN nf)` same as `checknf` but return NULL instead of raising an exception.

`void checkrnf(GEN rnf)` Raise an exception if the argument is not an *rnf* structure.

`int checkrnf_i(GEN rnf)` same as `checkrnf` but return 0 on failure and 1 on success.

`void checkbid(GEN bid)` Raise an exception if the argument is not a *bid* structure.

`GEN checkbid_i(GEN bid)` same as `checkbid` but return NULL instead of raising an exception and return `bid` on success.

`GEN checkznstar_i(GEN G)` return  $G$  if it is a *znstar*; else return `NULL` on failure.

`GEN checkgal(GEN x)` if a *galoisinit* structure can be extracted from  $x$ , return it; otherwise raise an exception.

`void checksqmat(GEN x, long N)` check whether  $x$  is a square matrix of dimension  $N$ . May be used to check for ideals if  $N$  is the field degree.

`void checkprid(GEN pr)` Raise an exception if the argument is not a prime ideal structure.

`int checkprid_i(GEN pr)` same as `checkprid` but return 0 instead of raising an exception and return 1 on success.

`int is_nf_factor(GEN F)` return 1 if  $F$  is an ideal factorization and 0 otherwise.

`int is_nf_extfactor(GEN F)` return 1 if  $F$  is an extended ideal factorization (allowing 0 or negative exponents) and 0 otherwise.

`int RgV_is_prV(GEN v)` returns 1 if the vector  $v$  contains only prime ideals and 0 otherwise.

`GEN get_prid(GEN ideal)` return the underlying prime ideal structure if one can be extracted from *ideal* (ideal or extended ideal), and return `NULL` otherwise.

`void checkabgrp(GEN v)` Raise an exception if the argument is not an abelian group structure, i.e. a `t_VEC` with either 2 or 3 entries:  $[N, cyc]$  or  $[N, cyc, gen]$ .

`GEN abgrp_get_no(GEN x)` extract the cardinality  $N$  from an abelian group structure.

`GEN abgrp_get_cyc(GEN x)` extract the elementary divisors *cyc* from an abelian group structure.

`GEN abgrp_get_gen(GEN x)` extract the generators *gen* from an abelian group structure.

`GEN cyc_get_expo(GEN cyc)` return the exponent of the group with structure *cyc*; 0 for an infinite group.

`void checkmodpr(GEN modpr)` Raise an exception if the argument is not a `modpr` structure (from `nfmodprinit`).

`GEN get_modpr(GEN x)` return  $x$  if it is a `modpr` structure and `NULL` otherwise.

`GEN checknfelt_mod(GEN nf, GEN x, const char *s)` given an *nf* structure *nf* and a `t_POLMOD`  $x$ , return the attached polynomial representative (shallow) if  $x$  and *nf* are compatible. Raise an exception otherwise. Set *s* to the name of the caller for a meaningful error message.

`int check_ZKmodule_i(GEN x)` return 1 if  $x$  looks like a projective  $\mathbf{Z}_K$ -module, i.e., a pair  $[A, I]$  where  $A$  is a matrix and  $I$  is a list of ideals and  $A$  has as many columns as  $I$  has elements. Or possibly a longer list  $[A, I, \dots]$  such as the output of `rnfpsudobasis`. Otherwise return 0.

`void check_ZKmodule(GEN x, const char *s)` raise an exception unless  $x$  is recognized as a projective  $\mathbf{Z}_K$ -module. Set *s* to the name of the caller for a meaningful error message.

`long idealtyp(GEN *ideal, GEN *fa)` The input is *ideal*, a pointer to an ideal or extended ideal; returns the type of the underlying ideal among `id_PRINCIPAL` (a number field element), `id_PRIME` (a prime ideal) `id_MAT` (an ideal in matrix form).

As a first side effect, *\*ideal* is set to the underlying ideal, possibly simplified (for instance the zero ideal represented by an empty matrix is replaced by `gen_0`).

If *fa* is not `NULL`, then *\*fa* is set to the extended part in the input: either `NULL` (regular ideal) or the extended part of an extended ideal.

### 13.1.2 Extracting info from a nf structure.

These functions expect a true *nf* argument attached to a number field  $K = \mathbf{Q}[x]/(T)$ , e.g. a *bnf* will not work. Let  $n = [K : \mathbf{Q}]$  be the field degree.

`GEN nf_get_pol(GEN nf)` returns the polynomial  $T$  (monic, in  $\mathbf{Z}[x]$ ).

`long nf_get_varn(GEN nf)` returns the variable number of the number field defining polynomial.

`long nf_get_r1(GEN nf)` returns the number of real places  $r_1$ .

`long nf_get_r2(GEN nf)` returns the number of complex places  $r_2$ .

`void nf_get_sign(GEN nf, long *r1, long *r2)` sets  $r_1$  and  $r_2$  to the number of real and complex places respectively. Note that  $r_1 + 2r_2$  is the field degree.

`long nf_get_degree(GEN nf)` returns the number field degree,  $n = r_1 + 2r_2$ .

`GEN nf_get_disc(GEN nf)` returns the field discriminant.

`GEN nf_get_index(GEN nf)` returns the index of  $T$ , i.e. the index of the order generated by the power basis  $(1, x, \dots, x^{n-1})$  in the maximal order of  $K$ .

`GEN nf_get_zk(GEN nf)` returns a basis  $(w_1, w_2, \dots, w_n)$  for the maximal order of  $K$ . Those are polynomials in  $\mathbf{Q}[x]$  of degree  $< n$ ; it is guaranteed that  $w_1 = 1$ .

`GEN nf_get_zkden(GEN nf)` returns the denominator of `nf_get_zk`, as a positive `t_INT`.

`GEN nf_get_zkprimpart(GEN nf)` returns `nf_get_zk` times its denominator.

`GEN nf_get_invzk(GEN nf)` returns the matrix  $(m_{i,j}) \in M_n(\mathbf{Z})$  giving the power basis  $(x^i)$  in terms of the  $(w_j)$ , i.e. such that  $x^{j-1} = \sum_{i=1}^n m_{i,j} w_i$  for all  $1 \leq j \leq n$ ; since  $w_1 = 1 = x^0$ , we have  $m_{i,1} = \delta_{i,1}$  for all  $i$ . The conversion functions in the `algtobasis` family essentially amount to a left multiplication by this matrix.

`GEN nf_get_roots(GEN nf)` returns the  $r_1$  real roots of the polynomial defining the number fields: first the  $r_1$  real roots (as `t_REALs`), then the  $r_2$  representatives of the pairs of complex conjugates.

`GEN nf_get_allroots(GEN nf)` returns all the complex roots of  $T$ : first the  $r_1$  real roots (as `t_REALs`), then the  $r_2$  pairs of complex conjugates.

`GEN nf_get_M(GEN nf)` returns the  $(r_1 + r_2) \times n$  matrix  $M$  giving the embeddings of  $K$ :  $M[i, j]$  contains  $w_j(\alpha_i)$ , where  $\alpha_i$  is the  $i$ -th element of `nf_get_roots(nf)`. In particular, if  $v$  is an  $n$ -th dimensional `t_COL` representing the element  $\sum_{i=1}^n v[i] w_i$  of  $K$ , then `RgM_RgC_mul(M, v)` represents the embeddings of  $v$ .

`GEN nf_get_G(GEN nf)` returns a  $n \times n$  real matrix  $G$  such that  $Gv \cdot Gv = T_2(v)$ , where  $v$  is an  $n$ -th dimensional `t_COL` representing the element  $\sum_{i=1}^n v[i] w_i$  of  $K$  and  $T_2$  is the standard Euclidean form on  $K \otimes \mathbf{R}$ , i.e.  $T_2(v) = \sum_{\sigma} |\sigma(v)|^2$ , where  $\sigma$  runs through all  $n$  complex embeddings of  $K$ .

`GEN nf_get_roundG(GEN nf)` returns a rescaled version of  $G$ , rounded to nearest integers, specifically `RM_round_maxrank(G)`.

`GEN nf_get_ramified_primes(GEN nf)` returns the vector of ramified primes.

`GEN nf_get_Tr(GEN nf)` returns the matrix of the Trace quadratic form on the basis  $(w_1, \dots, w_n)$ : its  $(i, j)$  entry is  $\text{Tr} w_i w_j$ .

`GEN nf_get_diff(GEN nf)` returns the primitive part of the inverse of the above Trace matrix.

`long nf_get_prec(GEN nf)` returns the precision (in words) to which the *nf* was computed.

### 13.1.3 Extracting info from a bnf structure.

These functions expect a true *bnf* argument, e.g. a *bnr* will not work.

`GEN bnf_get_nf(GEN bnf)` returns the underlying *nf*.

`GEN bnf_get_clgp(GEN bnf)` returns the class group in *bnf*, which is a 3-component vector  $[h, cyc, gen]$ .

`GEN bnf_get_cyc(GEN bnf)` returns the elementary divisors of the class group (cyclic components)  $[d_1, \dots, d_k]$ , where  $d_k \mid \dots \mid d_1$ .

`GEN bnf_get_gen(GEN bnf)` returns the generators  $[g_1, \dots, g_k]$  of the class group. Each  $g_i$  has order  $d_i$ , and the full module of relations between the  $g_i$  is generated by the  $d_i g_i = 0$ .

`GEN bnf_get_no(GEN bnf)` returns the class number.

`GEN bnf_get_reg(GEN bnf)` returns the regulator.

`GEN bnf_get_logfu(GEN bnf)` returns (complex floating point approximations to) the logarithms of the complex embeddings of our system of fundamental units.

`GEN bnf_get_fu(GEN bnf)` returns the fundamental units. Raise an error if the *bnf* does not contain units in algebraic form.

`GEN bnf_get_fu_nocheck(GEN bnf)` as `bnf_get_fu` without checking whether units are present. Do not use this unless you initialize the *bnf* yourself!

`GEN bnf_get_tuU(GEN bnf)` returns a generator of the torsion part of  $\mathbf{Z}_K^*$ .

`long bnf_get_tuN(GEN bnf)` returns the order of the torsion part of  $\mathbf{Z}_K^*$ , i.e. the number of roots of unity in  $K$ .

`GEN bnf_get_sunits(GEN bnf)` allows access to the algebraic data stored by `bnfinit(,1)`. The function returns NULL unless the *bnf* was initialized by `bnfinit(,1)`, else a vector  $[X, U, E, \text{lim}]$  where

- $X$  is a vector of rational primes and algebraic integers all of whose prime divisors have norm less than `lim`,
- $U$  is a matrix of exponents whose columns yield the fundamental units `bnf.fu`. More precisely,

$$\text{bnf.fu}[j] = \prod_i X[i]^{U[i,j]}.$$

- $G$  is a matrix of exponents whose columns yield the generators of principal ideals attached to the HNF of the *bnf* relation matrix between the maximal ideals of norm less `lim` (that generate the class group under GRH). More precisely, `bnf[5]` contains the prime factor base  $P$  (its first  $r$  elements being independant class group generators), `bnf[1]` contains a matrix  $W$  in HNF in  $M_r(\mathbf{Z})$  and `bnf[2]`, contains a matrix  $B$  in  $M_{r \times c}(\mathbf{Z})$ . We define algebraic numbers  $e_j$  for  $j \leq r + c$  such that

$$\prod_{i \leq r} p_i^{w[i,j]} = (e_j), \quad j \leq r$$

$$P_j \prod_{i \leq r} p_i^{b[i,j]} = (e_j), \quad j > r$$

Then  $e_j = \prod_i X[i]^{E[i,j]}$ .

`GEN bnf_has_fu(GEN bnf)` return fundamental units in expanded form if `bnf` contains them. Else return `NULL`.

`GEN bnf_compactfu(GEN bnf)` return fundamental units as a vector of algebraic numbers in compact form if `bnf` contains them. Else return `NULL`.

`GEN bnf_compactfu_mat(GEN bnf)` as a pair  $(X, U)$ , where  $X$  is a vector of  $S$ -units and  $U$  is a matrix with integer entries (without 0 rows), see `bnf_get_sunits`, if `bnf` contains them. Else return `NULL`.

#### 13.1.4 Extracting info from a `bnr` structure.

These functions expect a true *bnr* argument.

`GEN bnr_get_bnf(GEN bnr)` returns the underlying *bnf*.

`GEN bnr_get_nf(GEN bnr)` returns the underlying *nf*.

`GEN bnr_get_clgp(GEN bnr)` returns the ray class group.

`GEN bnr_get_no(GEN bnr)` returns the ray class number.

`GEN bnr_get_cyc(GEN bnr)` returns the elementary divisors of the ray class group (cyclic components)  $[d_1, \dots, d_k]$ , where  $d_k \mid \dots \mid d_1$ .

`GEN bnr_get_gen(GEN bnr)` returns the generators  $[g_1, \dots, g_k]$  of the ray class group. Each  $g_i$  has order  $d_i$ , and the full module of relations between the  $g_i$  is generated by the  $d_i g_i = 0$ . Raise a generic error if the *bnr* does not contain the ray class group generators.

`GEN bnr_get_gen_nocheck(GEN bnr)` as `bnr_get_gen` without checking whether generators are present. Do not use this unless you initialize the *bnr* yourself!

`GEN bnr_get_bid(GEN bnr)` returns the *bid* attached to the *bnr* modulus.

`GEN bnr_get_mod(GEN bnr)` returns the modulus attached to the *bnr*.

#### 13.1.5 Extracting info from an *rnf* structure.

These functions expect a true *rnf* argument, attached to an extension  $L/K$ ,  $K = \mathbf{Q}[y]/(T)$ ,  $L = K[x]/(P)$ .

`long rnf_get_degree(GEN rnf)` returns the *relative* degree  $[L : K]$ .

`long rnf_get_absdegree(GEN rnf)` returns the absolute degree  $[L : \mathbf{Q}]$ .

`long rnf_get_nfdegree(GEN rnf)` returns the degree of the base field  $[K : \mathbf{Q}]$ .

`GEN rnf_get_nf(GEN rnf)` returns the base field  $K$ , an *nf* structure.

`GEN rnf_get_nfpol(GEN rnf)` returns the polynomial  $T$  defining the base field  $K$ .

`long rnf_get_nfvarn(GEN rnf)` returns the variable  $y$  attached to the base field  $K$ .

`GEN rnf_get_nfzk(GEN rnf)` returns the integer basis of the base field  $K$ .

`GEN rnf_get_pol(GEN rnf)` returns the relative polynomial defining  $L/K$ .

`long rnf_get_varn(GEN rnf)` returns the variable  $x$  attached to  $L$ .

GEN `rnf_get_zk`(GEN `nf`) returns the relative integer basis generating  $\mathbf{Z}_L$  as a  $\mathbf{Z}_K$ -module, as a pseudo-matrix  $(A, I)$  in HNF.

GEN `rnf_get_disc`(GEN `rnf`) is the output  $[\mathfrak{d}, s]$  of `rnfdisc`.

GEN `rnf_get_ramified_primes`(GEN `rnf`) returns the vector of rational primes below ramified primes in the relative extension, i.e. all prime numbers appearing in the factorization of

`idealnrm(rnf_get_nf(rnf), rnf_get_disc(rnf));`

GEN `rnf_get_idealdisc`(GEN `rnf`) is the ideal discriminant  $\mathfrak{d}$  from `rnfdisc`.

GEN `rnf_get_index`(GEN `rnf`) is the index ideal  $\mathfrak{f}$

GEN `rnf_get_polabs`(GEN `rnf`) returns an absolute polynomial defining  $L/\mathbf{Q}$ .

GEN `rnf_get_alpha`(GEN `rnf`) a root  $\alpha$  of the polynomial defining the base field, modulo `polabs` (cf. `rnfequation`)

GEN `rnf_get_k`(GEN `rnf`) a small integer  $k$  such that  $\theta = \beta + k\alpha$  is a root of `polabs`, where  $\beta$  is a root of `pol` and  $\alpha$  a root of the polynomial defining the base field, as in `rnf_get_alpha` (cf. also `rnfequation`).

GEN `rnf_get_invzk`(GEN `rnf`) contains  $A^{-1}$ , where  $(A, I)$  is the chosen pseudo-basis for  $\mathbf{Z}_L$  over  $\mathbf{Z}_K$ .

GEN `rnf_get_map`(GEN `rnf`) returns technical data attached to the map  $K \rightarrow L$ . Currently, this contains data from `rnfequation`, as well as the polynomials  $T$  and  $P$ .

### 13.1.6 Extracting info from a bid structure.

These functions expect a true *bid* argument, attached to a modulus  $I = I_0 I_\infty$  in a number field  $K$ .

GEN `bid_get_mod`(GEN `bid`) returns the modulus attached to the *bid*.

GEN `bid_get_grp`(GEN `bid`) returns the abelian group attached to  $(\mathbf{Z}_K/I)^*$ .

GEN `bid_get_ideal`(GEN `bid`) return the finite part  $I_0$  of the *bid* modulus (an integer ideal).

GEN `bid_get_arch`(GEN `bid`) return the Archimedean part  $I_\infty$  of the *bid* modulus as a vector of real places in `vec01` format, see Section 13.1.20.

GEN `bid_get_archp`(GEN `bid`) return the Archimedean part  $I_\infty$  of the *bid* modulus, as a vector of real places in indices format see Section 13.1.20.

GEN `bid_get_fact`(GEN `bid`) returns the ideal factorization  $I_0 = \prod_i \mathfrak{p}_i^{e_i}$ .

GEN `bid_get_fact2`(GEN `bid`) as `bid_get_fact` with all factors  $\mathfrak{p}^1$  with  $\mathfrak{p}$  of norm 2 removed from the factorization. (They play no role in the structure of  $(\mathbf{Z}_K/I)^*$ , except that the generators must be made coprime to them.)

`bid_get_ideal(bid)`, via `idealfactor`.

GEN `bid_get_no`(GEN `bid`) returns the cardinality of the group  $(\mathbf{Z}_K/I)^*$ .

GEN `bid_get_cyc`(GEN `bid`) returns the elementary divisors of the group  $(\mathbf{Z}_K/I)^*$  (cyclic components)  $[d_1, \dots, d_k]$ , where  $d_k \mid \dots \mid d_1$ .

GEN `bid_get_gen`(GEN `bid`) returns the generators of  $(\mathbf{Z}_K/I)^*$  contained in *bid*. Raise a generic error if *bid* does not contain generators.

GEN `bid_get_gen_nocheck`(GEN `bid`) as `bid_get_gen` without checking whether generators are present. Do not use this unless you initialize the *bid* yourself!

GEN `bid_get_sprk`(GEN `bid`) return a list of structures attached to the  $(\mathbf{Z}_K/\mathfrak{p}^e)^*$  where  $\mathfrak{p}^e$  divides  $I_0$  exactly.

GEN `bid_get_sarch`(GEN `bid`) return the structure attached to  $(\mathbf{Z}_K/I_\infty)^*$ , by `nfarchstar`.

GEN `bid_get_U`(GEN `bid`) return the matrix with integral coefficients relating the local generators (from chinese remainders) to the global SNF generators (*bid.gen*).

### 13.1.7 Extracting info from a znstar structure.

These functions expect an argument  $G$  as returned by `znstar0`( $N$ , 1), attached to a positive  $N$  and the abelian group  $(\mathbf{Z}/N\mathbf{Z})^*$ . Let  $(g_i)$  be the SNF generators, where  $g_i$  has order  $d_i$ ; we call  $(g'_i)$  the (canonical) Conrey generators, where  $g'_i$  has order  $d'_i$ . Both sets of generators have the same cardinality.

GEN `znstar_get_N`(GEN `bid`) return  $N$ .

GEN `znstar_get_faN`(GEN  $G$ ) return the factorization `factor`( $N$ ),  $N = \prod_j p_j^{e_j}$ .

GEN `znstar_get_pe`(GEN  $G$ ) return the vector of primary factors  $(p_j^{e_j})$ .

GEN `znstar_get_no`(GEN  $G$ ) the cardinality  $\phi(N)$  of  $G$ .

GEN `znstar_get_cyc`(GEN  $G$ ) elementary divisors  $(d_i)$  of  $(\mathbf{Z}/N\mathbf{Z})^*$ .

GEN `znstar_get_gen`(GEN  $G$ ) SNF generators  $(g_i)$  of  $(\mathbf{Z}/N\mathbf{Z})^*$ .

GEN `znstar_get_conreycyc`(GEN  $G$ ) orders  $(d'_i)$  of Conrey generators.

GEN `znstar_get_conreygen`(GEN  $G$ ) Conrey generators  $(g'_i)$ .

GEN `znstar_get_U`(GEN  $G$ ) a square matrix  $U$  such that  $(g_i) = U(g'_i)$ .

GEN `znstar_get_Ui`(GEN  $G$ ) a square matrix  $U'$  such that  $U'(g_i) = (g'_i)$ . In general,  $UU'$  will not be the identity.

### 13.1.8 Inserting info in a number field structure.

If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

These functions expect a `bnf` argument:

GEN `bnf_build_cycgen`(GEN `bnf`) the *bnf* contains generators  $[g_1, \dots, g_k]$  of the class group, each with order  $d_i$ . Then  $g_i^{d_i} = (x_i)$  is a principal ideal. This function returns the  $x_i$  as a factorization matrix (`famat`) giving the element in factored form as a product of  $S$ -units.

GEN `bnf_build_matalpha`(GEN `bnf`) the class group was computed using a factorbase  $S$  of prime ideals  $\mathfrak{p}_i$ ,  $i \leq r$ . They satisfy relations of the form  $\prod_j \mathfrak{p}_i^{e_{i,j}} = (\alpha_j)$ , where the  $e_{i,j}$  are given by the matrices *bnf*[1] ( $W$ , singling out a minimal set of generators in  $S$ ) and *bnf*[2] ( $B$ , expressing the rest of  $S$  in terms of the singled out generators). This function returns the  $\alpha_j$  in factored form as a product of  $S$ -units.

GEN `bnf_build_units`(GEN `bnf`) returns a minimal set of generators for the unit group in expanded form. The first element is a torsion unit, the others have infinite order. This expands units

in compact form contained in a `bnf` from `bnfinit(,1)` and may be *very* expensive if the units are huge.

`GEN bnf_build_cheapfu(GEN bnf)` as `bnf_build_units` but only expand units in compact form if the computation is inexpensive (a few seconds). Return `NULL` otherwise.

These functions expect a `rnf` argument:

`GEN rnf_build_nfabs(GEN rnf, long prec)` given a *rnf* structure attached to  $L/K$ , (compute and) return an *nf* structure attached to  $L$  at precision `prec`.

`void rnfcomplete(GEN rnf)` as `rnf_build_nfabs` using the precision of  $K$  for `prec`.

`GEN rnf_zkabs(GEN rnf)` returns a  $\mathbf{Z}$ -basis in HNF for  $\mathbf{Z}_L$  as a pair  $[T, v]$ , where  $T$  is `rnf_get_polabs(rnf)` and  $v$  a vector of elements lifted from  $\mathbf{Q}[X]/(T)$ . Note that the function `rnf_build_nfabs` essentially applies `nfinit` to the output of this function.

### 13.1.9 Increasing accuracy.

`GEN nfnewprec(GEN x, long prec)`. Raise an exception if  $x$  is not a number field structure (*nf*, *bnf* or *bnr*). Otherwise, sets its accuracy to `prec` and return the new structure. This is mostly useful with `prec` larger than the accuracy to which  $x$  was computed, but it is also possible to decrease the accuracy of  $x$  (truncating relevant components, which may speed up later computations). This routine may modify the original  $x$  (see below).

This routine is straightforward for *nf* structures, but for the other ones, it requires all principal ideals corresponding to the *bnf* relations in algebraic form (they are originally only available via floating point approximations). This in turn requires many calls to `bnfisprincipal0`, which is often slow, and may fail if the initial accuracy was too low. In this case, the routine will not actually fail but recomputes a *bnf* from scratch!

Since this process may be very expensive, the corresponding data is cached (as a *clone*) in the *original*  $x$  so that later precision increases become very fast. In particular, the copy returned by `nfnewprec` also contains this additional data.

`GEN bnfnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *bnf* structure from  $x$  before increasing its accuracy, and returns only the latter.

`GEN bnrnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *bnr* structure from  $x$  before increasing its accuracy, and returns only the latter.

`GEN nfnewprec_shallow(GEN nf, long prec)`

`GEN bnfnewprec_shallow(GEN bnf, long prec)`

`GEN bnrnewprec_shallow(GEN bnr, long prec)` Shallow functions underlying the above, except that the first argument must now have the corresponding number field type. I.e. one cannot call `nfnewprec_shallow(nf, prec)` if `nf` is actually a *bnf*.



**13.1.10 Number field arithmetic.** The number field  $K = \mathbf{Q}[X]/(T)$  is represented by an `nf` (or `bnf` or `bnr` structure). An algebraic number belonging to  $K$  is given as

- a `t_INT`, `t_FRAC` or `t_POL` (implicitly modulo  $T$ ), or
- a `t_POLMOD` (modulo  $T$ ), or
- a `t_COL` `v` of dimension  $N = [K : \mathbf{Q}]$ , representing the element in terms of the computed integral basis  $(e_i)$ , as

```
sum(i = 1, N, v[i] * nf.zk[i])
```

The preferred forms are `t_INT` and `t_COL` of `t_INT`. Routines can handle denominators but it is much more efficient to remove denominators first (`Q_remove_denom`) and take them into account at the end.

**Safe routines.** The following routines do not assume that their `nf` argument is a true *nf* (it can be any number field type, e.g. a *bnf*), and accept number field elements in all the above forms. They return their result in `t_COL` form.

`GEN nfadd(GEN nf, GEN x, GEN y)` returns  $x + y$ .

`GEN nfsub(GEN nf, GEN x, GEN y)` returns  $x - y$ .

`GEN nfdiv(GEN nf, GEN x, GEN y)` returns  $x/y$ .

`GEN nfinv(GEN nf, GEN x)` returns  $x^{-1}$ .

`GEN nfmul(GEN nf, GEN x, GEN y)` returns  $xy$ .

`GEN nfpow(GEN nf, GEN x, GEN k)` returns  $x^k$ ,  $k$  is in  $\mathbf{Z}$ .

`GEN nfpow_u(GEN nf, GEN x, ulong k)` returns  $x^k$ ,  $k \geq 0$ ; the argument `nf` is a true *nf* structure.

`GEN nfsqr(GEN nf, GEN x)` returns  $x^2$ .

`long nfval(GEN nf, GEN x, GEN pr)` returns the valuation of  $x$  at the maximal ideal  $\mathfrak{p}$  attached to the *prid* `pr`. Returns `LONG_MAX` if  $x$  is 0.

`GEN nfnorm(GEN nf, GEN x)` absolute norm of  $x$ .

`GEN nftrace(GEN nf, GEN x)` absolute trace of  $x$ .

`GEN nfpoleval(GEN nf, GEN pol, GEN a)` evaluate the `t_POL` `pol` (with coefficients in `nf`) on the algebraic number  $a$  (also in *nf*).

`GEN FpX_FpC_nfpoleval(GEN nf, GEN pol, GEN a, GEN p)` evaluate the `FpX` `pol` on the algebraic number  $a$  (also in *nf*).

The following three functions implement trivial functionality akin to Euclidean division for which we currently have no real use. Of course, even if the number field is actually Euclidean, these do not in general implement a true Euclidean division.

`GEN nfdiveuc(GEN nf, GEN a, GEN b)` returns the algebraic integer closest to  $x/y$ . Functionally identical to `ground( nfdiv(nf,x,y) )`.

`GEN nfdivrem(GEN nf, GEN a, GEN b)` returns the vector  $[q, r]$ , where

```
q = nfdiveuc(nf, a, b);
r = nfsub(nf, a, nfmul(nf,q,b));    \\ or r = nfmod(nf,a,b);
```

GEN `nfmod`(GEN `nf`, GEN `a`, GEN `b`) returns  $r$  such that

```
q = nfdiveuc(nf, a, b);
r = nfsub(nf, a, nfmul(nf,q,b));
```

GEN `nf_to_scalar_or_basis`(GEN `nf`, GEN `x`) let  $x$  be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its basis representation (`nfaltobasis`). Shallow function.

GEN `nf_to_scalar_or_alg`(GEN `nf`, GEN `x`) let  $x$  be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its lifted `t_POLMOD` representation (lifted `nfbasistoalg`). Shallow function.

GEN `nfV_to_scalar_or_alg`(GEN `nf`, GEN `v`) apply `nf_to_scalar_or_alg` to all components of vector  $v$ .

GEN `RgX_to_nfX`(GEN `nf`, GEN `x`) let  $x$  be a `t_POL` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new polynomial. Shallow function.

GEN `RgM_to_nfM`(GEN `nf`, GEN `x`) let  $x$  be a `t_MAT` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new matrix. Shallow function.

GEN `RgC_to_nfC`(GEN `nf`, GEN `x`) let  $x$  be a `t_COL` or `t_VEC` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new `t_COL`. Shallow function.

GEN `nfX_to_monico`(GEN `nf`, GEN `T`, GEN `*pL`) given a nonzero `t_POL`  $T$  with coefficients in  $nf$ , return a monic polynomial  $f$  with integral coefficients such that  $f(x) = CT(x/L)$  for some integral  $L$  and some  $C$  in  $nf$ . The function allows coefficients in basis form; if  $L \neq 1$ , it will return them in algebraic form. If `pL` is not NULL, `*pL` is set to  $L$ . Shallow function.

**Unsafe routines.** The following routines assume that their `nf` argument is a true  $nf$  (e.g. a *bnf* is not allowed) and their argument are restricted in various ways, see the precise description below.

GEN `nfX_disc`(GEN `nf`, GEN `A`) given an  $nf$  structure attached to a number field  $K$  with main variable  $Y$  (`nf_get_varn(nf)`), a `t_POL`  $A \in K[X]$  given as a lift in  $\mathbf{Q}[X, Y]$  (implicitly modulo `nf_get_pol(nf)`), return the discriminant of  $A$  as a `t_POL` in  $\mathbf{Q}[Y]$  (representing an element of  $K$ ).

GEN `nfX_resultant`(GEN `nf`, GEN `A`, GEN `B`) analogous to `nfX_disc`,  $A, B \in \mathbf{Q}[X, Y]$ ; return the resultant of  $A$  and  $B$  with respect to  $X$  as a `t_POL` in  $\mathbf{Q}[Y]$  (representing an element of  $K$ ).

GEN `nfinvmideal`(GEN `nf`, GEN `x`, GEN `A`) given an algebraic integer  $x$  and a nonzero integral ideal  $A$  in HNF, returns a  $y$  such that  $xy \equiv 1$  modulo  $A$ .

GEN `nfpowmodideal`(GEN `nf`, GEN `x`, GEN `n`, GEN `ideal`) given an algebraic integer  $x$ , an integer  $n$ , and a nonzero integral ideal  $A$  in HNF, returns an algebraic integer congruent to  $x^n$  modulo  $A$ .

GEN `nfmuli`(GEN `nf`, GEN `x`, GEN `y`) returns  $x \times y$  assuming that both  $x$  and  $y$  are either `t_INTs` or ZVs of the correct dimension. The argument `nf` is a true  $nf$  structure.

GEN `nfsqri`(GEN `nf`, GEN `x`) returns  $x^2$  assuming that  $x$  is a `t_INT` or a ZV of the correct dimension. The argument `nf` is a true  $nf$  structure.

GEN `nfC_nf_mul`(GEN `nf`, GEN `v`, GEN `x`) given a `t_VEC` or `t_COL`  $v$  of elements of  $K$  in `t_INT`, `t_FRAC` or `t_COL` form, multiply it by the element  $x$  (arbitrary form). This is faster than multiplying

coordinatewise since pre-computations related to  $x$  (computing the multiplication table) are done only once. The components of the result are in most cases `t_COLs` but are allowed to be `t_INTs` or `t_FRACs`. Shallow function.

`GEN nfC_multable_mul(GEN v, GEN mx)` same as `nfC_nf_mul`, where the argument  $x$  is replaced by its multiplication table `mx`.

`GEN zkC_multable_mul(GEN v, GEN x)` same as `nfC_nf_mul`, where  $v$  is a vector of algebraic integers,  $x$  is an algebraic integer, and  $x$  is replaced by `zk_multable(x)`.

`GEN zk_multable(GEN nf, GEN x)` given a `ZC`  $x$  (implicitly representing an algebraic integer), returns the `ZM` giving the multiplication table by  $x$ . Shallow function (the first column of the result points to the same data as  $x$ ).

`GEN zk_inv(GEN nf, GEN x)` given a `ZC`  $x$  (implicitly representing an algebraic integer), returns the `QC` giving the inverse  $x^{-1}$ . Return `NULL` if  $x$  is 0. Not memory clean but safe for `gerepileupto`.

`GEN zkmultable_inv(GEN mx)` as `zk_inv`, where the argument given is `zk_multable(x)`.

`GEN zkmultable_capZ(GEN mx)` given a nonzero *zkmultable*  $mx$  attached to  $x \in \mathbf{Z}_K$ , return the positive generator of  $(x) \cap \mathbf{Z}$ .

`GEN zk_scalar_or_multable(GEN nf, GEN x)` given a `t_INT` or `ZC`  $x$ , returns a `t_INT` equal to  $x$  if the latter is a scalar (`t_INT` or `ZV_isscalar(x)` is 1) and `zk_multable(nf, x)` otherwise. Shallow function.

### 13.1.11 Number field arithmetic for linear algebra.

The following routines implement multiplication in a commutative  $R$ -algebra, generated by  $(e_1 = 1, \dots, e_n)$ , and given by a multiplication table  $M$ : elements in the algebra are  $n$ -dimensional `t_COLs`, and the matrix  $M$  is such that for all  $1 \leq i, j \leq n$ , its column with index  $(i-1)n + j$ , say  $(c_k)$ , gives  $e_i \cdot e_j = \sum c_k e_k$ . It is assumed that  $e_1$  is the neutral element for the multiplication (a convenient optimization, true in practice for all multiplications we needed to implement). If  $x$  has any other type than `t_COL` where an algebra element is expected, it is understood as  $x e_1$ .

`GEN multable(GEN M, GEN x)` given a column vector  $x$ , representing the quantity  $\sum_{i=1}^N x_i e_i$ , returns the multiplication table by  $x$ . Shallow function.

`GEN ei_multable(GEN M, long i)` returns the multiplication table by the  $i$ -th basis element  $e_i$ . Shallow function.

`GEN tablemul(GEN M, GEN x, GEN y)` returns  $x \cdot y$ .

`GEN tablesqr(GEN M, GEN x)` returns  $x^2$ .

`GEN tablemul_ei(GEN M, GEN x, long i)` returns  $x \cdot e_i$ .

`GEN tablemul_ei_ej(GEN M, long i, long j)` returns  $e_i \cdot e_j$ .

`GEN tablemulvec(GEN M, GEN x, GEN v)` given a vector  $v$  of elements in the algebra, returns the  $x \cdot v[i]$ .

The following routines implement naive linear algebra using the *black box field* mechanism:

`GEN nfM_det(GEN nf, GEN M)`

`GEN nfM_inv(GEN nf, GEN M)`

`GEN nfM_ker(GEN nf, GEN M)`

GEN nfM\_mul(GEN nf, GEN A, GEN B)

GEN nfM\_nfC\_mul(GEN nf, GEN A, GEN B)

### 13.1.12 Cyclotomic field arithmetic for linear algebra.

The following routines implement modular algorithms in cyclotomic fields. In the prototypes,  $P$  is the  $n$ -th cyclotomic polynomial  $\Phi_n$  and  $M$  is a `t_MAT` with `t_INT` or `ZX` coefficients, understood modulo  $P$ .

GEN ZabM\_ker(GEN M, GEN P, long n) returns an integral (primitive) basis of the kernel of  $M$ .

GEN ZabM\_indexrank(GEN M, GEN P, long n) return a vector with two `t_VEC` components giving the rank profile of  $M$ . Inefficient (but correct) when  $M$  does not have almost full column rank.

GEN ZabM\_inv(GEN M, GEN P, long n, GEN \*pden) assume that  $M$  is invertible; return  $N$  and sets the algebraic integer `*pden` (an integer or a `ZX`, implicitly modulo  $P$ ) such that  $MN = \text{den} \cdot \text{Id}$ .

GEN ZabM\_pseudoinv(GEN M, GEN P, long n, GEN \*pv, GEN \*pden) analog of `ZM_pseudoinv`. Not gerepile-safe.

GEN ZabM\_inv\_ratlift(GEN M, GEN P, long n, GEN \*pden) return a primitive matrix  $H$  such that  $MH$  is  $d$  times the identity and set `*pden` to  $d$ . Uses a multimodular algorithm, attempting rational reconstruction along the way. To be used when you expect that the denominator of  $M^{-1}$  is much smaller than  $\det M$  else use `ZabM_inv`.

### 13.1.13 Cyclotomic trace.

Given two positive integers  $m$  and  $n$  such that  $K_m = \mathbf{Q}(\zeta_m) \subset K_n = \mathbf{Q}(\zeta_n)$ , these functions implement relative trace computation from  $K_n$  to  $K_m$ . This is in particular useful for character values.

GEN Qab\_trace\_init(long n, long m, GEN Pn, GEN Pm) assume that `Pn` is `polcyclo(n)`, `Pm` is `polcyclo(m)` (both in the same variable), initialize a structure  $T$  used in the following routines. Shallow function.

GEN Qab\_tracerel(GEN T, long t, GEN z) assume  $T$  was created by `Qab_trace_init`,  $t$  is an integer such that  $0 \leq t < [K_n : K_m]$  and  $z$  belongs to the cyclotomic field  $\mathbf{Q}(\zeta_n) = \mathbf{Q}[X]/(\text{Pn})$ . Return the normalized relative trace  $[K_n : K_m]^{-1} \text{Tr}_{K_n/K_m}(\zeta_n^t z)$ . Shallow function.

GEN QabV\_tracerel(GEN T, long t, GEN v)  $v$  being a vector of entries belonging to  $K_n$ , apply `Qab_tracerel` to all entries. Shallow function.

GEN QabM\_tracerel(GEN T, long t, GEN m)  $m$  being a matrix of entries belonging to  $K_n$ , apply `Qab_tracerel` to all entries. Shallow function.

### 13.1.14 Elements in factored form.

Computational algebraic theory performs extensively linear algebra on  $\mathbf{Z}$ -modules with a natural multiplicative structure ( $K^*$ , fractional ideals in  $K$ ,  $\mathbf{Z}_K^*$ , ideal class group), thereby raising elements to horrendously large powers. A seemingly innocuous elementary linear algebra operation like  $C_i \leftarrow C_i - 10000C_1$  involves raising entries in  $C_1$  to the 10000-th power. Understandably, it is often more efficient to keep elements in factored form rather than expand every such expression. A *factorization matrix* (or *famat*) is a two column matrix, the first column containing *elements* (arbitrary objects which may be repeated in the column), and the second one contains *exponents* (`t_INTs`, allowed to be 0). By abuse of notation, the empty matrix `cgetg(1, t_MAT)` is recognized as the trivial factorization (no element, no exponent).

Even though we think of a *famat* with columns  $g$  and  $e$  as one meaningful object when fully expanded as  $\prod g[i]^{e[i]}$ , *famats* are basically about concatenating information to keep track of linear algebra: the objects stored in a *famat* need not be operation-compatible, they will not even be compared to each other (with one exception: `famat_reduce`). Multiplying two *famats* just concatenates their elements and exponents columns. In a context where a *famat* is expected, an object  $x$  which is not of type `t_MAT` will be treated as the factorization  $x^1$ . The following functions all return *famats*:

GEN `famat_mul`(GEN  $f$ , GEN  $g$ )  $f, g$  are *famat*, or objects whose type is *not* `t_MAT` (understood as  $f^1$  or  $g^1$ ). Returns  $fg$ . The empty factorization is the neutral element for *famat* multiplication.

GEN `famat_mul_shallow`(GEN  $f$ , GEN  $g$ ) shallow version of `famat_mul`.

GEN `famat_pow`(GEN  $f$ , GEN  $n$ )  $n$  is a `t_INT`. If  $f$  is a `t_MAT`, assume it is a *famat* and return  $f^n$  (multiplies the exponent column by  $n$ ). Otherwise, understand it as an element and returns the 1-line *famat*  $f^n$ .

GEN `famat_pow_shallow`(GEN  $f$ , GEN  $n$ ) shallow version of `famat_pow`.

GEN `famat_pows_shallow`(GEN  $f$ , long  $n$ ) shallow version of `famat_pow` where  $n$  is a small integer.

GEN `famat_mulpow_shallow`(GEN  $f$ , GEN  $g$ , GEN  $e$ ) *famat* corresponding to  $f \cdot g^e$ . Shallow function.

GEN `famat_mulpows_shallow`(GEN  $f$ , GEN  $g$ , long  $e$ ) *famat* shallow version of `famat_mulpow` where  $e$  is a small integer.

GEN `famat_sqr`(GEN  $f$ ) returns  $f^2$ .

GEN `famat_inv`(GEN  $f$ ) returns  $f^{-1}$ .

GEN `famat_div`(GEN  $f$ , GEN  $g$ ) return  $f/g$ .

GEN `famat_inv_shallow`(GEN  $f$ ) shallow version of `famat_inv`.

GEN `famat_div_shallow`(GEN  $f$ , GEN  $g$ ) return  $f/g$ ; shallow.

GEN `famat_Z_gcd`(GEN  $M$ , GEN  $n$ ) restrict the *famat*  $M$  to the prime power dividing  $n$ .

GEN `to_famat`(GEN  $x$ , GEN  $k$ ) given an element  $x$  and an exponent  $k$ , returns the *famat*  $x^k$ .

GEN `to_famat_shallow`(GEN  $x$ , GEN  $k$ ) same, as a shallow function.

GEN `famatV_factorback`(GEN  $v$ , GEN  $e$ ) given a vector of *famats*  $v$  and a ZV  $e$  return the *famat*  $\prod_i v[i]^{e[i]}$ . Shallow function.

`GEN famatV_zv_factorback(GEN v, GEN e)` given a vector of *famats*  $v$  and a *zv*  $e$  return the *famat*  $\prod_i v[i]^{e[i]}$ . Shallow function.

`GEN ZM_famat_limit(GEN f, GEN limit)` given a *famat*  $f$  with `t_INT` entries, returns a *famat*  $g$  with all factors larger than `limit` multiplied out as the last entry (with exponent 1). Shallow function.

Note that it is trivial to break up a *famat* into its two constituent columns: `gel(f,1)` and `gel(f,2)` are the elements and exponents respectively. Conversely, `mkmat2` builds a (shallow) *famat* from two `t_COLs` of the same length.

`GEN famat_reduce(GEN f)` given a *famat*  $f$ , returns a *famat*  $g$  without repeated elements or 0 exponents, such that the expanded forms of  $f$  and  $g$  would be equal. Shallow function.

`GEN famat_remove_trivial(GEN f)` given a *famat*  $f$ , returns a *famat*  $g$  without 0 exponents. Shallow function.

`GEN famatsmall_reduce(GEN f)` as `famat_reduce`, but for exponents given by a `t_VECSMALL`.

`GEN famat_to_nf(GEN nf, GEN f)` You normally never want to do this! This is a simplified form of `nf_factorback`, where we do not check the user input for consistency. The elements must be regular algebraic numbers (not *famats*) over the given number field.

Why should you *not* want to use this function? You should not need to: most of the functions useful in this context accept *famats* as inputs, for instance `nfsign`, `nfsign_arch`, `ideallog` and `bnfisunit`. Otherwise, we can hopefully make good use of a quotient operation (modulo a fixed conductor, modulo  $\ell$ -th powers); see the end of Section 13.1.26. If nothing else works, this function is available but is expected to be slow or even overflow the possibilities of the implementation.

`GEN famat_ideal_factor(GEN nf, GEN x)` This is a good alternative for `famat_to_nf`, returning the factorization of the ideal generated by  $x$ . Since the answer is still given in factorized form, there is no risk of coefficient explosion when the exponents are large. Of course, all components of  $x$  must be factored individually.

`GEN famat_nfvalrem(GEN nf, GEN x, GEN pr, GEN *py)` return the valuation  $v$  at  $\mathfrak{pr}$  of `famat_to_nf(x)`, without performing the expansion of course. Notice that the output is a `GEN` since it cannot be assumed to fit into a `long`. If `py` is not `NULL` it contains the *famat* obtained by applying `nfvalrem` to each entry of the first column and copying the second column, with 0 exponents removed. The expanded algebraic number is coprime to  $\mathfrak{pr}$  (in fact, all its components are coprime to  $\mathfrak{pr}$ ) and equal to  $x\tau^v$  where  $\tau$  is the fixed anti-uniformizer for  $\mathfrak{pr}$  (`pr_get_tau`).

**Caveat.** Receiving a *famat* input, `bnfisunit` assumes that it is an actual unit, since this is expensive to check, and normally easy to ensure from the user's side.

### 13.1.15 Ideal arithmetic.

## Conversion to HNF.

`GEN idealhnf(GEN nf, GEN x)` where the argument `nf` is a true *nf* structure. Returns the HNF of the ideal defined by  $x$ :  $x$  may be an algebraic number (defining a principal ideal), a maximal ideal (as given by `idealprimedec` or `idealfactor`), or a matrix whose columns give generators for the ideal. This last format is complicated, but useful to reduce general modules to the canonical form once in a while:

- if strictly less than  $N = [K : Q]$  generators are given,  $x$  is the  $\mathbf{Z}_K$ -module they generate,
- if  $N$  or more are given, it is assumed that they form a  $\mathbf{Z}$ -basis (that the matrix has maximal rank  $N$ ). This acts as `mathnf` since the  $\mathbf{Z}_K$ -module structure is (taken for granted hence) not taken into account in this case.

Extended ideals are also accepted, their principal part being discarded.

`GEN idealhnf0(GEN nf, GEN x, GEN y)` returns the HNF of the ideal generated by the two algebraic numbers  $x$  and  $y$ .

The following low-level functions underlie the above two: they all assume that `nf` is a true *nf* and perform no type checks:

`GEN idealhnf_principal(GEN nf, GEN x)` returns the ideal generated by the algebraic number  $x$ .

`GEN idealhnf_shallow(GEN nf, GEN x)` is `idealhnf` except that the result may not be suitable for `gerepile`: if  $x$  is already in HNF, we return  $x$ , not a copy!

`GEN idealhnf_two(GEN nf, GEN v)` assuming  $a = v[1]$  is a nonzero `t_INT` and  $b = v[2]$  is an algebraic integer, possibly given in regular representation by a `t_MAT` (the multiplication table by  $b$ , see `zk_multable`), returns the HNF of  $a\mathbf{Z}_K + b\mathbf{Z}_K$ .

## Operations.

The basic ideal routines accept all `nfs` (*nf*, *bnf*, *bnr*) and ideals in any form, including extended ideals, and return ideals in HNF, or an extended ideal when that makes sense:

`GEN idealadd(GEN nf, GEN x, GEN y)` returns  $x + y$ .

`GEN idealdiv(GEN nf, GEN x, GEN y)` returns  $x/y$ . Returns an extended ideal if  $x$  or  $y$  is an extended ideal.

`GEN idealmul(GEN nf, GEN x, GEN y)` returns  $xy$ . Returns an extended ideal if  $x$  or  $y$  is an extended ideal.

`GEN idealsqr(GEN nf, GEN x)` returns  $x^2$ . Returns an extended ideal if  $x$  is an extended ideal.

`GEN idealinv(GEN nf, GEN x)` returns  $x^{-1}$ . Returns an extended ideal if  $x$  is an extended ideal.

`GEN idealpow(GEN nf, GEN x, GEN n)` returns  $x^n$ . Returns an extended ideal if  $x$  is an extended ideal.

`GEN idealpows(GEN nf, GEN ideal, long n)` returns  $x^n$ . Returns an extended ideal if  $x$  is an extended ideal.

`GEN idealmulred(GEN nf, GEN x, GEN y)` returns an extended ideal equal to  $xy$ .

`GEN idealpowred(GEN nf, GEN x, GEN n)` returns an extended ideal equal to  $x^n$ .

More specialized routines suffer from various restrictions:

`GEN idealdivexact(GEN nf, GEN x, GEN y)` returns  $x/y$ , assuming that the quotient is an integral ideal. Much faster than `idealdiv` when the norm of the quotient is small compared to  $Nx$ . Strips the principal parts if either  $x$  or  $y$  is an extended ideal.

`GEN idealdivpowprime(GEN nf, GEN x, GEN pr, GEN n)` returns  $x\mathfrak{p}^{-n}$ , assuming  $x$  is an ideal in HNF or a rational number, and `pr` a *prid* attached to `p`. Not suitable for `gerepileupto` since it returns  $x$  when  $n = 0$ . The `nf` argument must be a true *nf* structure.

`GEN idealmulpowprime(GEN nf, GEN x, GEN pr, GEN n)` returns  $x\mathfrak{p}^n$ , assuming  $x$  is an ideal in HNF or a rational number, and `pr` a *prid* attached to `p`. Not suitable for `gerepileupto` since it returns  $x$  when  $n = 0$ . The `nf` argument must be a true *nf* structure.

`GEN idealprodprime(GEN nf, GEN v)` given a list  $v$  of prime ideals in *prid* form, return their product. Assume that `nf` is a true *nf* structure.

`GEN idealprod(GEN nf, GEN v)` given a list  $v$  of ideals, return their product.

`GEN idealprodval(GEN nf, GEN v, GEN pr)` given a list  $v$  of ideals return the valuation of their product at the prime ideal `pr`.

`GEN idealHNF_mul(GEN nf, GEN x, GEN y)` returns  $xy$ , assuming that `nf` is a true *nf*,  $x$  is an integral ideal in HNF and  $y$  is an integral ideal in HNF or precompiled form (see below). For maximal speed, the second ideal  $y$  may be given in precompiled form  $y = [a, b]$ , where  $a$  is a nonzero `t_INT` and  $b$  is an algebraic integer in regular representation (a `t_MAT` giving the multiplication table by the fixed element): very useful when many ideals  $x$  are going to be multiplied by the same ideal  $y$ . This essentially reduces each ideal multiplication to an  $N \times N$  matrix multiplication followed by a  $N \times 2N$  modular HNF reduction (modulo  $xy \cap \mathbf{Z}$ ).

`GEN idealHNF_inv(GEN nf, GEN I)` returns  $I^{-1}$ , assuming that `nf` is a true *nf* and  $x$  is a fractional ideal in HNF.

`GEN idealHNF_inv_Z(GEN nf, GEN I)` returns  $(I \cap \mathbf{Z}) \cdot I^{-1}$ , assuming that `nf` is a true *nf* and  $x$  is an integral fractional ideal in HNF. The result is an integral ideal in HNF.

`GEN ideals_by_norm(GEN nf, GEN N)` given a true *nf* structure and a integer  $N$ , which can also be given by a factorization matrix or (preferably) by a pair  $[N, \text{factor}(N)]$ , return all ideals of norm  $N$  in factored form. Not `gerepile` clean.

### Approximation.

`GEN idealaddtoone(GEN nf, GEN A, GEN B)` given to coprime integer ideals  $A, B$ , returns  $[a, b]$  with  $a \in A, b \in B$ , such that  $a + b = 1$ . The result is reduced mod  $AB$ , so  $a, b$  will be small.

`GEN idealaddtoone_i(GEN nf, GEN A, GEN B)` as `idealaddtoone` except that `nf` must be a true *nf*, and only  $a$  is returned.

`GEN idealaddtoone_raw(GEN nf, GEN A, GEN B)` as `idealaddtoone_i` except that the reduction mod  $AB$  is only performed modulo the lcm of  $A \cap \mathbf{Z}$  and  $B \cap \mathbf{Z}$ , which will increase the size of  $a$ .

`GEN zkchineseinit(GEN nf, GEN A, GEN B, GEN AB)` given two coprime integral ideals  $A$  and  $B$  (in any form, preferably HNF) and their product  $AB$  (in HNF form), initialize a solution to the Chinese remainder problem modulo  $AB$ . The `nf` argument must be a true *nf* structure.

`GEN zkchinese(GEN zkc, GEN x, GEN y)` given `zkc` from `zkchineseinit`, and  $x, y$  two integral elements given as `t_INT` or `ZC`, return a  $z$  modulo  $AB$  such that  $z = x \bmod A$  and  $z = y \bmod B$ .



GEN `zkchinese1`(GEN `zkc`, GEN `x`) as `zkchinese` for  $y = 1$ ; useful to lift elements in a nice way from  $(\mathbf{Z}_K/A_i)^*$  to  $(\mathbf{Z}_K/\prod_i A_i)^*$ .

GEN `hnfmerge_get_1`(GEN `A`, GEN `B`) given two square upper HNF integral matrices  $A, B$  of the same dimension  $n > 0$ , return  $a$  in the image of  $A$  such that  $1 - a$  is in the image of  $B$ . (By abuse of notation we denote  $1$  the column vector  $[1, 0, \dots, 0]$ .) If such an  $a$  does not exist, return `NULL`. This is the function underlying `idealaddtoone`.

GEN `idealaddmultoone`(GEN `nf`, GEN `v`) given a list of  $n$  (globally) coprime integer ideals  $(v[i])$  returns an  $n$ -dimensional vector  $a$  such that  $a[i] \in v[i]$  and  $\sum a[i] = 1$ . If  $[K : \mathbf{Q}] = N$ , this routine computes the HNF reduction (with  $Gl_{nN}(\mathbf{Z})$  base change) of an  $N \times nN$  matrix; so it is well worth pruning "useless" ideals from the list (as long as the ideals remain globally coprime).

GEN `idealapprfact`(GEN `nf`, GEN `fx`) as `idealappr`, except that  $x$  *must* be given in factored form. (This is unchecked.)

GEN `idealcoprime`(GEN `nf`, GEN `x`, GEN `y`). Given 2 integral ideals  $x$  and  $y$ , returns an algebraic number  $\alpha$  such that  $\alpha x$  is an integral ideal coprime to  $y$ .

GEN `idealcoprimefact`(GEN `nf`, GEN `x`, GEN `fy`) same as `idealcoprime`, except that  $y$  is given in factored form, as from `idealfactor`.

GEN `idealchinese`(GEN `nf`, GEN `x`, GEN `y`)

GEN `idealchineseinit`(GEN `nf`, GEN `x`)

### 13.1.16 Maximal ideals.

The PARI structure attached to maximal ideals is a *prid* (for *prime ideal*), usually produced by `idealprimedec` and `idealfactor`. In this section, we describe the format; other sections will deal with their daily use.

A *prid* attached to a maximal ideal  $\mathfrak{p}$  stores the following data: the underlying rational prime  $p$ , the ramification degree  $e \geq 1$ , the residue field degree  $f \geq 1$ , a  $p$ -uniformizer  $\pi$  with valuation 1 at  $\mathfrak{p}$  and valuation 0 at all other primes dividing  $p$  and a rescaled "anti-uniformizer"  $\tau$  used to compute valuations. This  $\tau$  is an algebraic integer such that  $\tau/p$  has valuation  $-1$  at  $\mathfrak{p}$  and is integral at all other primes; in particular, the valuation of  $x \in \mathbf{Z}_K$  is positive if and only if the algebraic integer  $x\tau$  is divisible by  $p$  (easy to check for elements in `t_COL` form).

GEN `pr_get_p`(GEN `pr`) returns  $p$ . Shallow function.

GEN `pr_get_gen`(GEN `pr`) returns  $\pi$ . Shallow function.

long `pr_get_e`(GEN `pr`) returns  $e$ .

long `pr_get_f`(GEN `pr`) returns  $f$ .

GEN `pr_get_tau`(GEN `pr`) returns `zk_scalar_or_multable(nf,  $\tau$ )`, which is the `t_INT` 1 iff  $p$  is inert, and a ZM otherwise. Shallow function.

int `pr_is_inert`(GEN `pr`) returns 1 if  $p$  is inert, 0 otherwise.

GEN `pr_norm`(GEN `pr`) returns the norm  $p^f$  of the maximal ideal.

ulong `upr_norm`(GEN `pr`) returns the norm  $p^f$  of the maximal ideal, as an `ulong`. Assume that the result does not overflow.

GEN `pr_hnf`(GEN `pr`) return the HNF of  $\mathfrak{p}$ .

GEN `pr_inv`(GEN `pr`) return the fractional ideal  $\mathfrak{p}^{-1}$ , in HNF.

GEN `pr_inv_p`(GEN `pr`) return the integral ideal  $p\mathfrak{p}^{-1}$ , in HNF.

GEN `idealprimedec`(GEN `nf`, GEN `p`) list of maximal ideals dividing the prime  $p$ .

GEN `idealprimedec_limit_f`(GEN `nf`, GEN `p`, long `f`) as `idealprimedec`, limiting the list to primes of residual degree  $\leq f$  if  $f$  is nonzero.

GEN `idealprimedec_limit_norm`(GEN `nf`, GEN `p`, GEN `B`) as `idealprimedec`, limiting the list to primes of norm  $\leq B$ , which must be a positive `t_INT`.

GEN `idealprimedec_galois`(GEN `nf`, GEN `p`) return a single prime ideal above  $p$ . The `nf` argument is a true *nf* structure.

GEN `idealprimedec_degrees`(GEN `nf`, GEN `p`) return a (sorted) `t_VECSMALL` containing the residue degrees  $f(\mathfrak{p}/p)$ . The `nf` argument is a true *nf* structure.

GEN `idealprimedec_kummer`(GEN `nf`, GEN `Ti`, long `ei`, GEN `p`) let *nf* (true *nf*) correspond to  $K = \mathbf{Q}[X]/(T)$  ( $T$  monic  $\mathbf{Z}[X]$ ). Let  $T \equiv \prod_i T_i^{e_i} \pmod{p}$  be the factorization of  $T$  and let  $(f, g, h)$  be as in Dedekind criterion for prime  $p$ :  $f \equiv \prod T_i$ ,  $g \equiv \prod T_i^{e_i-1}$ ,  $h = (T - fg)/p$ , and let  $D$  be the gcd of  $(f, g, h)$  in  $\mathbf{F}_p[X]$ . Let `Ti` (`FpX`) be one irreducible factor  $T_i$  not dividing  $D$ , with `ei` =  $e_i$ . This function returns the prime ideal attached to  $T_i$  by Kummer / Dedekind criterion, namely  $p\mathbf{Z}_K + T_i(\bar{X})\mathbf{Z}_K$ , which has ramification index  $e_i$  over  $p$ . The `nf` argument is a true *nf* structure. Shallow function.

GEN `idealfactor`(GEN `nf`, GEN `x`) factors the fractional (hence nonzero) ideal  $x$  into prime ideal powers; return the factorization matrix.

GEN `idealfactor_limit`(GEN `nf`, GEN `x`, ulong `lim`) as `idealfactor`, including only prime ideals above rational primes  $< \text{lim}$ .

GEN `idealfactor_partial`(GEN `nf`, GEN `x`, GEN `L`) return partial factorization of fractional ideal  $x$  as limited by argument  $L$ :

- $L = \text{NULL}$ : as `idealfactor`;
- $L$  a `t_INT`: as `idealfactor_limit`;
- $L$  a vector of prime ideals of *nf* and/or rational primes (standing for “all prime ideal divisors of given rational prime”) limit factorization to trial division by elements of  $L$ ; do not include the cofactor.

GEN `idealHNF_Z_factor`(GEN `x`, GEN `*pvN`, GEN `*pvZ`) given an integral (nonzero) ideal  $x$  in HNF, compute both the factorization of  $Nx$  and of  $x \cap \mathbf{Z}$ . This returns the vector of prime divisors of both and sets `*pvN` and `*pvZ` to the corresponding `t_VECSMALL` vector of exponents for the factorization for the Norm and intersection with  $\mathbf{Z}$  respectively.

GEN `idealHNF_Z_factor_i`(GEN `x`, GEN `fa`, GEN `*pvN`, GEN `*pvZ`) internal variant of `idealHNF_Z_factor` where `fa` is either a partial factorization of  $x \cap \mathbf{Z}$  ( $= x[1, 1]$ ) or `NULL`. Returns the prime divisors of  $x$  above the rational primes in `fa` and attached `vn` and `vZ`. If `fa` is `NULL`, use the full factorization, i.e. identical to `idealHNF_Z_factor`.

GEN `nf_pV_to_prV`(GEN `nf`, GEN `P`) given a vector of rational primes  $P$ , return the vector of all prime ideals above the  $P[i]$ .

**GEN nf\_deg1\_prime(GEN nf)** let  $nf$  be a true  $nf$ . This function returns a degree 1 (unramified) prime ideal not dividing  $nf.index$ . In fact it returns an ideal above the smallest prime  $p \geq [K : \mathbf{Q}]$  satisfying those conditions.

**GEN prV\_lcm\_capZ(GEN L)** given a vector  $L$  of  $prid$  (maximal ideals) return the squarefree positive integer generating their lcm intersected with  $\mathbf{Z}$ . Not **gerepile-safe**.

**GEN prV\_primes(GEN) GEN L** given a vector of  $prid$ , return the (sorted) list of rational primes  $P$  they divide. Not **gerepile-clean** but suitable for **gerepileupto**.

**GEN pr\_uniformizer(GEN pr, GEN F)** given a  $prid$  attached to  $\mathfrak{p}/p$  and  $F$  in  $\mathbf{Z}$  divisible exactly by  $p$ , return an  $F$ -uniformizer for  $\mathfrak{p}$ , i.e. a  $t$  in  $\mathbf{Z}_K$  such that  $v_{\mathfrak{p}}(t) = 1$  and  $(t, F/\mathfrak{p}) = 1$ . Not **gerepile-safe**.

### 13.1.17 Decomposition groups.

**GEN idealramfrobenius(GEN nf, GEN gal, GEN pr, GEN ram)** Let  $K$  be the number field defined by  $nf$  and assume  $K/\mathbf{Q}$  be a Galois extension with Galois group given  $gal=galoisinit(nf)$ , and that  $pr$  is the prime ideal  $\mathfrak{P}$  in  $prid$  format, and that  $\mathfrak{P}$  is ramified, and  $ram$  is its list of ramification groups as output by **idealramgroups**. This function returns a permutation of  $gal.group$  which defines an automorphism  $\sigma$  in the decomposition group of  $\mathfrak{P}$  such that if  $p$  is the unique prime number in  $\mathfrak{P}$ , then  $\sigma(x) \equiv x^p \pmod{\mathbf{P}}$  for all  $x \in \mathbf{Z}_K$ .

**GEN idealramfrobenius\_aut(GEN nf, GEN gal, GEN pr, GEN ram, GEN aut)** as **idealramfrobenius(nf, gal, pr, ram)**.

**GEN idealramgroups\_aut(GEN nf, GEN gal, GEN pr, GEN aut)** as **idealramgroups(nf, gal, pr)**.

**GEN idealfrobenius\_aut(GEN nf, GEN gal, GEN pr, GEN aut)** faster version of **idealfrobenius(nf, gal, pr)** where  $aut$  must be equal to **nfgaloispermtobasis(nf, gal)**.

### 13.1.18 Reducing modulo maximal ideals.

**GEN nfmodprinit(GEN nf, GEN pr)** returns an abstract **modpr** structure, attached to reduction modulo the maximal ideal  $\mathfrak{p}$ , in **idealprimedec** format. From this data we can quickly project any  $\mathfrak{p}$ -integral number field element to the residue field.

**GEN modpr\_get\_pr(GEN x)** return the  $\mathfrak{p}$  component from a **modpr** structure.

**GEN modpr\_get\_p(GEN x)** return the  $p$  component from a **modpr** structure (underlying rational prime).

**GEN modpr\_get\_T(GEN x)** return the  $T$  component from a **modpr** structure: either **NULL** (prime of degree 1) or an irreducible **FpX** defining the residue field over  $\mathbf{F}_p$ .

In library mode, it is often easier to use directly

**GEN nf\_to\_Fq\_init(GEN nf, GEN \*ppr, GEN \*pT, GEN \*pp)** concrete version of **nfmodprinit**:  $nf$  and  $*ppr$  are the inputs, the return value is a **modpr** and  $*ppr$ ,  $*pT$  and  $*pp$  are set as side effects.

The input  $*ppr$  is either a maximal ideal or already a **modpr** (in which case it is replaced by the underlying maximal ideal). The residue field is realized as  $\mathbf{F}_p[X]/(T)$  for some monic  $T \in \mathbf{F}_p[X]$ , and we set  $*pT$  to  $T$  and  $*pp$  to  $p$ . Set  $T = \mathbf{NULL}$  if the prime has degree 1 and the residue field is  $\mathbf{F}_p$ .

In short, this receives (or initializes) a `modpr` structure, and extracts from it  $T$ ,  $p$  and  $\mathfrak{p}$ .

`GEN nf_to_Fq(GEN nf, GEN x, GEN modpr)` returns an `Fq` congruent to  $x$  modulo the maximal ideal attached to `modpr`. The output is canonical: all elements in a given residue class are represented by the same `Fq`.

`GEN Fq_to_nf(GEN x, GEN modpr)` returns an `nf` element lifting the residue field element  $x$ , either a `t_INT` or an algebraic integer in `algtobasis` format.

`GEN modpr_genFq(GEN modpr)` Returns an `nf` element whose image by `nf_to_Fq` is  $X \pmod{T}$ , if  $\deg T > 1$ , else 1.

`GEN zkmodprinit(GEN nf, GEN pr)` as `nfmodprinit`, but we assume we will only reduce algebraic integers, hence do not initialize data allowing to remove denominators. More precisely, we can in fact still handle an  $x$  whose rational denominator is not 0 in the residue field (i.e. if the valuation of  $x$  is nonnegative at all primes dividing  $p$ ).

`GEN zk_to_Fq_init(GEN nf, GEN *pr, GEN *T, GEN *p)` as `nf_to_Fq_init`, able to reduce only  $p$ -integral elements.

`GEN zk_to_Fq(GEN x, GEN modpr)` as `nf_to_Fq`, for a  $p$ -integral  $x$ .

`GEN nfM_to_FqM(GEN M, GEN nf, GEN modpr)` reduces a matrix of `nf` elements to the residue field; returns an `FqM`.

`GEN FqM_to_nfM(GEN M, GEN modpr)` lifts an `FqM` to a matrix of `nf` elements.

`GEN nfV_to_FqV(GEN A, GEN nf, GEN modpr)` reduces a vector of `nf` elements to the residue field; returns an `FqV` with the same type as  $A$  (`t_VEC` or `t_COL`).

`GEN FqV_to_nfV(GEN A, GEN modpr)` lifts an `FqV` to a vector of `nf` elements (same type as  $A$ ).

`GEN nfX_to_FqX(GEN Q, GEN nf, GEN modpr)` reduces a polynomial with `nf` coefficients to the residue field; returns an `FqX`.

`GEN FqX_to_nfX(GEN Q, GEN modpr)` lifts an `FqX` to a polynomial with coefficients in `nf`.

The following functions are technical and avoid computing a true `nfmodpr`:

`GEN pr_basis_perm(GEN nf, GEN pr)` given a true `nf` structure and a prime ideal `pr` above  $p$ , return as a `t_VEC` the  $f(\mathfrak{p}/p)$  indices  $i$  such that the `nf.zk[i]` mod  $\mathfrak{p}$  form an  $\mathbf{F}_p$ -basis of the residue field.

`GEN QXQV_to_FpM(GEN v, GEN T, GEN p)` let  $p$  be a positive integer,  $v$  be a vector of  $n$  polynomials with rational coefficients whose denominators are coprime to  $p$ , and  $T$  be a `ZX` (preferably monic) of degree  $d$  whose leading coefficient is coprime to  $p$ . Return the  $d \times n$  `FpM` whose columns are the  $v[i] \pmod{T, p}$  in the canonical basis  $1, X, \dots, X^{d-1}$ , see `RgX_to_RgC`. This is for instance useful when  $v$  contains a  $\mathbf{Z}$ -basis of the maximal order of a number field  $\mathbf{Q}[X]/(P)$ ,  $p$  is a prime not dividing the index of  $P$  and  $T$  is an irreducible factor of  $P \pmod{p}$ , attached to a maximal ideal  $\mathfrak{p}$ : left-multiplication by the matrix maps number field elements (in basis form) to the residue field of  $\mathfrak{p}$ .

### 13.1.19 Valuations.

`long nfval(GEN nf, GEN x, GEN P)` return  $v_P(x)$

**Unsafe functions.** assume that  $P, Q$  are `prid`.

`long ZC_nfval(GEN x, GEN P)` returns  $v_P(x)$ , assuming  $x$  is a `ZC`, representing a nonzero algebraic integer.

`long ZC_nfvalrem(GEN x, GEN P, GEN *newx)` returns  $v = v_P(x)$ , assuming  $x$  is a `ZC`, representing a nonzero algebraic integer, and sets `*newx` to  $x\tau^v$  which is an algebraic integer coprime to  $p$ .

`int ZC_prdvd(GEN x, GEN P)` returns 1 if  $P$  divides  $x$  and 0 otherwise. Assumes that  $x$  is a `ZC`, representing an algebraic integer. Faster than computing  $v_P(x)$ .

`int pr_equal(GEN P, GEN Q)` returns 1 if  $P$  and  $Q$  represent the same maximal ideal: they must lie above the same  $p$  and share the same  $e, f$  invariants, but the  $p$ -uniformizer and  $\tau$  element may differ. Returns 0 otherwise.

### 13.1.20 Signatures.

“Signs” of the real embeddings of number field element are represented in additive notation, using the standard identification  $(\mathbf{Z}/2\mathbf{Z}, +) \rightarrow (\{-1, 1\}, \times)$ ,  $s \mapsto (-1)^s$ .

With respect to a fixed `nf` structure, a selection of real places (a divisor at infinity) is normally given as a `t_VECSMALL` of indices of the roots `nf.roots` of the defining polynomial for the number field. For compatibility reasons, in particular under `GP`, the (obsolete) `vec01` form is also accepted: a `t_VEC` with `gen_0` or `gen_1` entries.

The following internal functions go back and forth between the two representations for the Archimedean part of divisors (`GP`: 0/1 vectors, library: list of indices):

`GEN vec01_to_indices(GEN v)` given a `t_VEC`  $v$  with `t_INT` entries return as a `t_VECSMALL` the list of indices  $i$  such that  $v[i] \neq 0$ . (Typically used with 0, 1-vectors but not necessarily so.) If  $v$  is already a `t_VECSMALL`, return it: not suitable for `gerepile` in this case.

`GEN vecsmall01_to_indices(GEN v)` as

`vec01_to_indices(zv_to_ZV(v));`

`GEN indices_to_vec01(GEN p, long n)` return the 0/1 vector of length  $n$  with ones exactly at the positions  $p[1], p[2], \dots$

`GEN nfsign(GEN nf, GEN x)`  $x$  being a number field element and `nf` any form of number field, return the 0 – 1-vector giving the signs of the  $r_1$  real embeddings of  $x$ , as a `t_VECSMALL`. Linear algebra functions like `Flv_add_inplace` then allow keeping track of signs in series of multiplications. The argument `nf` is a true `nf` structure.

If  $x$  is a `t_VEC` of number field elements, return the matrix whose columns are the signs of the  $x[i]$ .

`GEN nfsign_arch(GEN nf, GEN x, GEN arch)` `arch` being a list of distinct real places, either in `vec01` (`t_VEC` with `gen_0` or `gen_1` entries) or `indices` (`t_VECSMALL`) form (see `vec01_to_indices`), returns the signs of  $x$  at the corresponding places. This is the low-level function underlying `nfsign`. The argument `nf` is a true `nf` structure.

`int nfchecksigns(GEN nf, GEN x, GEN pl)`  $pl$  is a `t_VECSMALL` with  $r_1$  components, all of which are in  $\{-1, 0, 1\}$ . Return 1 if  $\sigma_i(x)pl[i] \geq 0$  for all  $i$ , and 0 otherwise.

`GEN nfsign_units(GEN bnf, GEN archp, int add_tu)` `archp` being a divisor at infinity in `indices` form (or `NULL` for the divisor including all real places), return the signs at `archp` of a

**bnf.tu** and of system of fundamental units for the field **bnf.fu**, in that order if **add\_tu** is set; and in the same order as **bnf.fu** otherwise.

**GEN nfsign\_fu(GEN bnf, GEN archp)** returns the signs at **archp** of the fundamental units **bnf.fu**. This is an alias for **nfsign\_units** with **add\_tu** unset.

**GEN nfsign\_tu(GEN bnf, GEN archp)** returns the signs at **archp** of the torsion unit generator **bnf.tu**.

**GEN nfsign\_from\_logarch(GEN L, GEN invpi, GEN archp)** given  $L$  the vector of the  $\log \sigma(x)$ , where  $\sigma$  runs through the (real or complex) embeddings of some number field, **invpi** being a floating point approximation to  $1/\pi$ , and **archp** being a divisor at infinity in **indices** form, return the signs of  $x$  at the corresponding places. This is the low-level function underlying **nfsign\_units**; the latter is actually a trivial wrapper **bnf** structures include the  $\log \sigma(x)$  for a system of fundamental units of the field.

**GEN set\_sign\_mod\_divisor(GEN nf, GEN x, GEN y, GEN sarch)** let  $f = f_0 f_\infty$  be a divisor, let **sarch** be the output of **nfarchstar**(**nf**, **f0**, **finf**), let  $x$  encode a vector of signs at the places of  $f_\infty$  (see below), and let  $y$  be a nonzero number field element. Returns  $z$  congruent to  $y \bmod f_0$  (integral if  $y$  is) such that  $z$  and  $x$  have the same signs at  $f_\infty$ . The argument **nf** is a true  $nf$  structure.

The following formats are supported for  $x$ : a  $\{0,1\}$ -vector of signs as a **t\_VECSMALL** (0 for positive, 1 for negative); **NULL** for a totally positive element (only 0s); a number field element which is replaced by its signature at  $f_\infty$ .

**GEN nfarchstar(GEN nf, GEN f0, GEN finf)** for a divisor  $f = f_0 f_\infty$  represented by the integral ideal **f0** in HNF and the **finf** in **indices** form, returns  $(\mathbf{Z}_K/f_\infty)^*$  in a form suitable for computations mod  $f$ . See **set\_sign\_mod\_divisor**.

**GEN idealprincipalunits(GEN nf, GEN pr, long e)** returns the multiplicative group  $(1 + pr)/(1 + pr^e)$  as an abelian group. Faster than **idealstar** when the norm of  $pr$  is large, since it avoids (useless) work in the multiplicative group of the residue field.

### 13.1.21 Complex embeddings.

**GEN nfembed(GEN nf, GEN x, long k)** returns a floating point approximation of the  $k$ -th embedding of  $x$  (attached to the  $k$ -th complex root in **nf.roots**).

**GEN nf\_cxlog(GEN nf, GEN x, long prec)** return the vector of complex logarithmic embeddings  $(e_i \text{Log}(\sigma_i X))$  where  $e_i = 1$  if  $i \leq r_1$  and  $e_i = 2$  if  $r_1 < i \leq r_2$  of  $X = \mathbf{Q\_primpart}(x)$ . Returns **NULL** if loss of accuracy. Not **gerepile**-clean but suitable for **gerepileupto**. Allows  $x$  in compact representation, in which case **Q\_primpart** is taken componentwise.

**GEN nf\_cxlog\_normalize(GEN nf, GEN x, long prec)** an  $nf$  structure attached to a number field  $K$  and  $x$  from **nf\_cxlog**( $nf, X$ ) (a column vector of complex logarithmic embeddings with  $r_1 + r_2$  components) and let  $e = (e_1, \dots, e_{r_1+r_2})$ . Return

$$x - \frac{\log(N_{K/\mathbf{Q}} X)}{[K : \mathbf{Q}]} e$$

where the imaginary parts are further normalized modulo  $2\pi i \cdot e$ .

The composition **nf\_cxlog** followed by **nf\_cxlog\_normalize** is a morphism from  $(K^*/\mathbf{Q}_+^*, \times)$  to  $((\mathbf{C}/2\pi i \mathbf{Z})^{r_1} \times (\mathbf{C}/4\pi i \mathbf{Z})^{r_2}, +)$ . Its real part maps the units  $\mathbf{Z}_K^*$  to a lattice in the hyperplane  $\sum_i x_i = 0$  in  $\mathbf{R}^{r_1+r_2}$ .

`GEN nfV_cxlog(GEN nf, GEN x, long prec)` applies `nf_cxlog` to each component of the vector  $x$ . Returns NULL if loss of accuracy for even one component. Not `gerepile-clean`.

`GEN nflogembed(GEN nf, GEN x, GEN *emb, long prec)` return the vector of real logarithmic embeddings  $(e_i \text{Log}|\sigma_i x|)$  where  $e_i = 1$  if  $i \leq r_1$  and  $e_i = 2$  if  $r_1 < i \leq r_2$ . Returns NULL if loss of accuracy. Not `gerepile-clean`. If `emb` is non-NULL set it to  $(e_i \sigma_i x)$ . Allows  $x$  in compact representation, in which case `emb` is returned in compact representation as well, as a factorization matrix (expanding the factorization may overflow exponents).

### 13.1.22 Maximal order and discriminant, conversion to `nf` structure.

A number field  $K = \mathbf{Q}[X]/(T)$  is defined by a monic  $T \in \mathbf{Z}[X]$ . The low-level function computing a maximal order is

`void nfmaxord(nfmaxord_t *S, GEN T0, long flag)`, where the polynomial  $T_0$  is squarefree with integer coefficients. Let  $K$  be the étale algebra  $\mathbf{Q}[X]/(T_0)$  and let  $T = \text{ZX\_Q\_normalize}(T_0)$ , i.e.  $T = CT_0(X/L)$  is monic and integral for some  $C, Q \in \mathbf{Q}$ .

The structure `nfmaxord_t` is initialized by the call; it has the following fields:

```
GEN T0, T, dT, dK; /* T0, T, discriminants of T and K */
GEN unscale; /* the integer L */
GEN index; /* index of power basis in maximal order */
GEN dTP, dTE; /* factorization of |dT|, primes / exponents */
GEN dKP, dKE; /* factorization of |dK|, primes / exponents */
GEN basis; /* Z-basis for maximal order of Q[X]/(T) */
```

The exponent vectors are `t_VECSMALL`. The primes in `dTP` and `dKP` are pseudoprimes, not proven primes. We recommend restricting to  $T = T_0$ , i.e. either to pass the input polynomial through `ZX_Q_normalize` *before* the call, or to forget about  $T_0$  and go on with the polynomial  $T$ ; otherwise `unscale`  $\neq 1$ , all data is expressed in terms of  $T \neq T_0$ , and needs to be converted to  $T_0$ . For instance to convert the basis to  $\mathbf{Q}[X]/(T_0)$ :

```
RgXV_unscale(S.basis, S.unscale)
```

Instead of passing  $T$  (monic `ZX`), one can use the format  $[T, \text{list}P]$  as in `nfbasis` or `nfinit`, which computes an order which is maximal at a set of primes, but need not be the maximal order.

The `flag` is an or-ed combination of the binary flags, both of them deprecated:

`nf_PARTIALFACT`: do not try to fully factor `dT` and only look for primes less than `primelimit`. In that case, the elements in `dTP` and `dKP` need not all be primes. But the resulting `dK`, `index` and `basis` are correct provided there exists no prime  $p > \text{primelimit}$  such that  $p^2$  divides the field discriminant `dK`. This flag is *deprecated*: the  $[T, \text{list}P]$  format is safer and more flexible.

`nf_ROUND2`: this flag is *deprecated* and now ignored.

`void nfinit_basic(nfmaxord_t *S, GEN T0)` a wrapper around `nfmaxord` (without the deprecated `flag`) that also accepts number field structures (`nf`, `bnf`, ...) for  $T_0$ .

`GEN nfmaxord_to_nf(nfmaxord_t *S, GEN ro, long prec)` convert an `nfmaxord_t` to an `nf` structure at precision `prec`, where `ro` is NULL. The argument `ro` may also be set to a vector with  $r_1 + r_2$  components containing the roots of  $S \rightarrow T$  suitably ordered, i.e. first  $r_1$  `t_REAL` roots, then  $r_2$  `t_COMPLEX` representing the conjugate pairs, but this is *strongly discouraged*: the format is error-prone, and it is hard to compute the roots to the right accuracy in order to achieve `prec` accuracy

for the **nf**. This function uses the integer basis **S->basis** as is, *without* performing LLL-reduction. Unless the basis is already known to be reduced, use rather the following higher-level function:

**GEN nfinit\_complete(nfmaxord\_t \*S, long flag, long prec)** convert an **nfmaxord\_t** to an **nf** structure at precision **prec**. The **flag** has the same meaning as in **nfinit0**. If **S->basis** is known to be reduced, it will be faster to use **nfmaxord\_to\_nf**.

**GEN indexpartial(GEN T, GEN dT)**  $T$  a monic separable **ZX**, **dT** is either **NULL** (no information) or a multiple of the discriminant of  $T$ . Let  $K = \mathbf{Q}[X]/(T)$  and  $\mathbf{Z}_K$  its maximal order. Returns a multiple of the exponent of the quotient group  $\mathbf{Z}_K/(\mathbf{Z}[X]/(T))$ . In other word, a *denominator*  $d$  such that  $dx \in \mathbf{Z}[X]/(T)$  for all  $x \in \mathbf{Z}_K$ .

**GEN FpX\_gcd\_check(GEN x, GEN y, GEN D)** let  $x$  and  $y$  be two coprime polynomials with integer coefficients and let  $D$  be a factor of the resultant of  $x$  and  $y$ ; try to factor  $D$  by running the Euclidean algorithm on  $x$  and  $y$  modulo  $D$ . This returns **NULL** or a non trivial factor of  $D$ . This is the low-level function underlying **poldiscfactors** (applied to  $x$ , **ZX\_deriv**( $x$ ) and the discriminant of  $x$ ). It succeeds when  $D$  has at least two prime divisors  $p$  and  $q$  such that one sub-resultant of  $x$  and  $y$  is divisible by  $p$  but not by  $q$ .

### 13.1.23 Computing in the class group.

We compute with arbitrary ideal representatives (in any of the various formats seen above), and call

**GEN bnfisprincipal0(GEN bnf, GEN x, long flag)**. The **bnf** structure already contains information about the class group in the form  $\oplus_{i=1}^n (\mathbf{Z}/d_i\mathbf{Z})g_i$  for canonical integers  $d_i$  (with  $d_n \mid \dots \mid d_1$  all  $> 1$ ) and essentially random generators  $g_i$ , which are ideals in HNF. We normally do not need the value of the  $g_i$ , only that they are fixed once and for all and that any (nonzero) fractional ideal  $x$  can be expressed uniquely as  $x = (t) \prod_{i=1}^n g_i^{e_i}$ , where  $0 \leq e_i < d_i$ , and  $(t)$  is some principal ideal. Computing  $e$  is straightforward, but  $t$  may be very expensive to obtain explicitly. The routine returns (possibly partial) information about the pair  $[e, t]$ , depending on **flag**, which is an or-ed combination of the following symbolic flags:

- **nf\_GEN** tries to compute  $t$ . Returns  $[e, t]$ , with  $t$  an empty vector if the computation failed. This flag is normally useless in nontrivial situations since the next two serve analogous purposes in more efficient ways.

- **nf\_GENMAT** tries to compute  $t$  in factored form, which is much more efficient than **nf\_GEN** if the class group is moderately large; imagine a small ideal  $x = (t)g^{10000}$ : the norm of  $t$  has 10000 as many digits as the norm of  $g$ ; do we want to see it as a vector of huge meaningless integers? The idea is to compute  $e$  first, which is easy, then compute  $(t)$  as  $x \prod g_i^{-e_i}$  using successive **idealmulred**, where the ideal reduction extracts small principal ideals along the way, eventually raised to large powers because of the binary exponentiation technique; the point is to keep this principal part in factored *unexpanded* form. Returns  $[e, t]$ , with  $t$  an empty vector if the computation failed; this should be exceedingly rare, unless the initial accuracy to which **bnf** was computed was ridiculously low (and then **bnfinit** should not have succeeded either). Setting/unsetting **nf\_GEN** has no effect when this flag is set.

- **nf\_GEN\_IF\_PRINCIPAL** tries to compute  $t$  *only* if the ideal is principal ( $e = 0$ ). Returns **gen\_0** if the ideal is not principal. Setting/unsetting **nf\_GEN** has no effect when this flag is set, but setting/unsetting **nf\_GENMAT** is possible.

- **nf\_FORCE** in the above, insist on computing  $t$ , even if it requires recomputing a **bnf** from scratch. This is a last resort, and normally the accuracy of a **bnf** can be increased without trouble,



but it may be that some algebraic information simply cannot be recovered from what we have: see `bnfnewprec`. It should be very rare, though.

In simple cases where you do not care about  $t$ , you may use

`GEN isprincipal(GEN bnf, GEN x)`, which is a shortcut for `bnfisprincipal0(bnf, x, 0)`.

The following low-level functions are often more useful:

`GEN isprincipalfact(GEN bnf, GEN C, GEN L, GEN f, long flag)` is about the same as `bnfisprincipal0` applied to  $C \prod L[i]^{f[i]}$ , where the  $L[i]$  are ideals, the  $f[i]$  integers and  $C$  is either an ideal or `NULL` (omitted). Make sure to include `nf_GENMAT` in `flag`!

`GEN isprincipalfact_or_fail(GEN bnf, GEN C, GEN L, GEN f)` is for delicate cases, where we must be more clever than `nf_FORCE` (it is used when trying to increase the accuracy of a *bnf*, for instance). It performs

```
isprincipalfact(bnf,C, L, f, nf_GENMAT);
```

but if it fails to compute  $t$ , it just returns a `t_INT`, which is the estimated precision (in words, as usual) that would have been sufficient to complete the computation. The point is that `nf_FORCE` does exactly this internally, but goes on increasing the accuracy of the *bnf*, then discarding it, which is a major inefficiency if you intend to compute lots of discrete logs and have selected a precision which is just too low. (It is sometimes not so bad since most of the really expensive data is cached in *bnf* anyway, if all goes well.) With this function, the *caller* may decide to increase the accuracy using `bnfnewprec` (and keep the resulting *bnf*!), or avoid the computation altogether. In any case the decision can be taken at the place where it is most likely to be correct.

`void bnftestprimes(GEN bnf, GEN B)` is an ingredient to certify unconditionnally a *bnf* computed assuming GRH, cf. `bnfcertify`. Running this function successfully proves that the classes of all prime ideals of norm  $\leq B$  belong to the subgroup of the class group generated by the factorbase used to compute the *bnf* (equal to the class group under GRH). If the condition is not true, then (GRH is false and) the function will run forever.

If it is known that primes of norm less than  $B$  generate the class group (through variants of Minkowski's convex body or Zimmert's twin classes theorems), then the true class group is proven to be a quotient of `bnf.clgp`.

### 13.1.24 Floating point embeddings, the $T_2$ quadratic form.

We assume the *nf* is a true `nf` structure, attached to a number field  $K$  of degree  $n$  and signature  $(r_1, r_2)$ . We saw that

`GEN nf_get_M(GEN nf)` returns the  $(r_1 + r_2) \times n$  matrix  $M$  giving the embeddings of  $K$ , so that if  $v$  is an  $n$ -th dimensional `t_COL` representing the element  $\sum_{i=1}^n v[i]w_i$  of  $K$ , then `RgM_RgC_mul(M,v)` represents the embeddings of  $v$ . Its first  $r_1$  components are real numbers (`t_INT`, `t_FRAC` or `t_REAL`, usually the latter), and the last  $r_2$  are complex numbers (usually of `t_COMPLEX`, but not necessarily for embeddings of rational numbers).

`GEN embed_T2(GEN x, long r1)` assuming  $x$  is the vector of floating point embeddings of some algebraic number  $v$ , i.e.

```
x = RgM_RgC_mul(nf_get_M(nf), algtobasis(nf,v));
```

returns  $T_2(v)$ . If the floating point embeddings themselves are not needed, but only the values of  $T_2$ , it is more efficient to restrict to real arithmetic and use

`gnorml2( RgM_RgC_mul(nf_get_G(nf), algtobasis(nf,v)))`;

`GEN embednorm_T2(GEN x, long r1)` analogous to `embed_T2`, applied to the `gnorm` of the floating point embeddings. Assuming that

`x = gnorm( RgM_RgC_mul(nf_get_M(nf), algtobasis(nf,v)) )`;

returns  $T_2(v)$ .

`GEN embed_roots(GEN z, long r1)` given a vector  $z$  of  $r_1 + r_2$  complex embeddings of the algebraic number  $v$ , return the  $r_1 + 2r_2$  roots of its characteristic polynomial. Shallow function.

`GEN embed_disc(GEN z, long r1, long prec)` given a vector  $z$  of  $r_1 + r_2$  complex embeddings of the algebraic number  $v$ , return a floating point approximation of the discriminant of its characteristic polynomial as a `t_REAL` of precision `prec`.

`GEN embed_norm(GEN x, long r1)` given a vector  $z$  of  $r_1 + r_2$  complex embeddings of the algebraic number  $v$ , return (a floating point approximation of) the norm of  $v$ .

### 13.1.25 Ideal reduction, low level.

In the following routines  $nf$  is a true `nf`, attached to a number field  $K$  of degree  $n$ :

`GEN nf_get_Gtwist(GEN nf, GEN v)` assuming  $v$  is a `t_VEC`SMALL with  $r_1 + r_2$  entries, let

$$||x||_v^2 = \sum_{i=1}^{r_1+r_2} 2^{v_i} \varepsilon_i |\sigma_i(x)|^2,$$

where as usual the  $\sigma_i$  are the (real and) complex embeddings and  $\varepsilon_i = 1$ , resp. 2, for a real, resp. complex place. This is a twisted variant of the  $T_2$  quadratic form, the standard Euclidean form on  $K \otimes \mathbf{R}$ . In applications, only the relative size of the  $v_i$  will matter.

Let  $G_v \in M_n(\mathbf{R})$  be a square matrix such that if  $x \in K$  is represented by the column vector  $X$  in terms of the fixed  $\mathbf{Z}$ -basis of  $\mathbf{Z}_K$  in  $nf$ , then

$$||x||_v^2 = {}^t(G_v X) \cdot G_v X.$$

(This is a kind of Cholesky decomposition.) This function returns a rescaled copy of  $G_v$ , rounded to nearest integers, specifically `RM_round_maxrank( $G_v$ )`. Suitable for `gerepileupto`, but does not collect garbage. For convenience, also allow  $v = \text{NULL}$  (`nf_get_roundG`) and  $v$  a `t_MAT` as output from the function itself: in both these cases, shallow function.

`GEN nf_get_Gtwist1(GEN nf, long i)`. Simple special case. Returns the twisted  $G$  matrix attached to the vector  $v$  whose entries are all 0 except the  $i$ -th one, which is equal to 10.

`GEN idealpseudomin(GEN x, GEN G)`. Let  $x, G$  be two ZMs, such that the product  $Gx$  is well-defined. This returns a “small” integral linear combinations of the columns of  $x$ , given by the LLL-algorithm applied to the lattice  $Gx$ . Suitable for `gerepileupto`, but does not collect garbage.

In applications,  $x$  is an integral ideal,  $G$  approximates a Cholesky form for the  $T_2$  quadratic form as returned by `nf_get_Gtwist`, and we return a small element  $a$  in the lattice  $(x, T_2)$ . This is used to implement `idealred`.

`GEN idealpseudomin_nonscalar(GEN x, GEN G)`. As `idealpseudomin`, but we insist of returning a nonscalar  $a$  (`ZV_isscalar` is false), if the dimension of  $x$  is  $> 1$ .

In the interpretation where  $x$  defines an integral ideal on a fixed  $\mathbf{Z}_K$  basis whose first element is 1, this means that  $a$  is not rational.

GEN `idealpseudominvec`(GEN  $x$ , GEN  $G$ ). As `idealpseudomin_nonscalar`, but we return about  $n^2/2$  nonscalar elements in  $x$  with small  $T_2$ -norm, where the dimension of  $x$  is  $n$ .

GEN `idealpseudored`(GEN  $x$ , GEN  $G$ ). As `idealpseudomin` but we return the full reduced  $\mathbf{Z}$ -basis of  $x$  as a `t_MAT` instead of a single vector.

GEN `idealred_elt`(GEN  $nf$ , GEN  $x$ ) shortcut for

`idealpseudomin(x, nf_get_roundG(nf))`

### 13.1.26 Ideal reduction, high level.

Given an ideal  $x$  this means finding a “simpler” ideal in the same ideal class. The public GP function is of course available

GEN `idealred0`(GEN  $nf$ , GEN  $x$ , GEN  $v$ ) finds an  $a \in K^*$  such that  $(a)x$  is integral of small norm and returns it, as an ideal in HNF. What “small” means depends on the parameter  $v$ , see the GP description. More precisely,  $a$  is returned by `idealpseudomin` $((x_{\mathbf{Z}})x - 1, G)$  divided by  $x_{\mathbf{Z}}$ , where  $x_{\mathbf{Z}} = (x \cap \mathbf{Z})$  and where  $G$  is `nf_get_Gtwist`( $nf, v$ ) for  $v \neq \text{NULL}$  and `nf_get_roundG`( $nf$ ) otherwise.

Usually one sets  $v = \text{NULL}$  to obtain an element of small  $T_2$  norm in  $x$ :

GEN `idealred`(GEN  $nf$ , GEN  $x$ ) is a shortcut for `idealred0`( $nf, x, \text{NULL}$ ).

The function `idealred` remains complicated to use: in order not to lose information  $x$  must be an extended ideal, otherwise the value of  $a$  is lost. There is a subtlety here: the principal ideal  $(a)$  is easy to recover, but  $a$  itself is an instance of the principal ideal problem which is very difficult given only an  $nf$  (once a  $bnf$  structure is available, `bnfisprincipal0` will recover it).

GEN `idealmoddivisor`(GEN  $bnr$ , GEN  $x$ ) A proof-of-concept implementation, useless in practice. If  $bnr$  is attached to some modulus  $f$ , returns a “small” ideal in the same class as  $x$  in the ray class group modulo  $f$ . The reason why this is useless is that using extended ideals with principal part in a computation, there is a simple way to reduce them: simply reduce the generator of the principal part in  $(\mathbf{Z}_K/f)^*$ .

GEN `famat_to_nf_moddivisor`(GEN  $nf$ , GEN  $g$ , GEN  $e$ , GEN  $bid$ ) given a true  $nf$  attached to a number field  $K$ , a  $bid$  structure attached to a modulus  $f$ , and an algebraic number in factored form  $\prod g[i]^{e[i]}$ , such that  $(g[i], f) = 1$  for all  $i$ , returns a small element in  $\mathbf{Z}_K$  congruent to it mod  $f$ . Note that if  $f$  contains places at infinity, this includes sign conditions at the specified places.

A simpler case when the conductor has no place at infinity:

GEN `famat_to_nf_modideal_coprime`(GEN  $nf$ , GEN  $g$ , GEN  $e$ , GEN  $f$ , GEN  $expo$ ) as above except that the ideal  $f$  is now integral in HNF (no need for a full  $bid$ ), and we pass the exponent of the group  $(\mathbf{Z}_K/f)^*$  as  $expo$ ; any multiple will also do, at the expense of efficiency. Of course if a  $bid$  for  $f$  is available, it is easy to extract  $f$  and the exact value of  $expo$  from it (the latter is the first elementary divisor in the group structure). A useful trick: if you set  $expo$  to *any* positive integer, the result is correct up to  $expo$ -th powers, hence exact if  $expo$  is a multiple of the exponent; this is useful when trying to decide whether an element is a square in a residue field for instance! (take  $expo = 2$ ).

GEN `nf_to_Fp_coprime`(GEN  $nf$ , GEN  $x$ , GEN  $modpr$ ) this low-level function is variant of `famat_to_nf_modideal_coprime`:  $nf$  is a true  $nf$  structure,  $modpr$  is from `zkmodprinit` attached

to a prime of degree 1 above the prime number  $p$ , and  $x$  is either a number field element or a `famat` factorization matrix. We finally assume that no component of  $x$  has a denominator  $p$ .

What to do when the  $g[i]$  are not coprime to  $f$ , but only  $\prod g[i]^{e[i]}$  is? Then the situation is more complicated, and we advise to solve it one prime divisor of  $f$  at a time. Let  $v$  be the valuation attached to a maximal ideal `pr`:

`GEN famat_makecoprime(GEN nf, GEN g, GEN e, GEN pr, GEN prk, GEN expo)` returns an element in  $(\mathbf{Z}_K/\mathbf{pr}^k)^*$  congruent to the product  $\prod g[i]^{e[i]}$ , assumed to be globally coprime to `pr`. As above, `expo` is any positive multiple of the exponent of  $(\mathbf{Z}_K/\mathbf{pr}^k)^*$ , for instance  $(Nv - 1)p^{k-1}$ , if  $p$  is the underlying rational prime. You may use other values of `expo` (see the useful trick in `famat_to_nf_modideal_coprime`).

`GEN sunits_makecoprime(GEN g, GEN pr, GEN prk)` is a specialized variant that allows to precondition a vector of  $g[i]$  assumed to be integral primes or algebraic integers so that it becomes suitable for `famat_to_nf_modideal_coprime` modulo `pr`. This is in particular useful for the output of `bnf_get_sunits`.

`GEN Idealstarprk(GEN nf, GEN pr, long k, long flag)` same as `Idealstar` for  $I = \mathbf{pr}^k$ . The `nf` argument is a true `nf` structure.

### 13.1.27 Class field theory.

Under GP, a class-field theoretic description of a number field is given by a triple  $A, B, C$ , where the defining set  $[A, B, C]$  can have any of the following forms:  $[bnr]$ ,  $[bnr, subgroup]$ ,  $[bnf, modulus]$ ,  $[bnf, modulus, subgroup]$ . You can still use directly all of (`libpari`'s routines implementing) GP's functions as described in Chapter 3, but they are often awkward in the context of `libpari` programming. In particular, it does not make much sense to always input a triple  $A, B, C$  because of the fringe  $[bnf, modulus, subgroup]$ . The first routine to call, is thus

`GEN Buchray(GEN bnf, GEN mod, long flag)` initializes a `bnr` structure from `bnf` and modulus `mod`. `flag` is an or-ed combination of `nf_GEN` (include generators) and `nf_INIT` (if omitted, do not return a `bnr`, only the ray class group as an abelian group). In fact, the single most useful value of `flag` is `nf_INIT` to initialize a proper `bnr`: omitting `nf_GEN` saves a lot of time and will not adversely affect any class field theoretic function; adding `nf_GEN` makes debugging easier. The flag 0 allows to compute only the ray class group structure but will gain little time; if we only need the *order* of the ray class group, then `bnrclassno` is fastest.

Now we have a proper `bnr` encoding a `bnf` and a modulus, we no longer need the  $[bnf, modulus]$  and  $[bnf, modulus, subgroup]$  forms, which would internally call `Buchray` anyway. Recall that a subgroup  $H$  is given by a matrix in HNF, whose column express generators of  $H$  on the fixed generators of the ray class group that stored in our `bnr`. You may also code the trivial subgroup by `NULL`. It is also allowed to replace  $H$  by a character  $\chi$  of the ray class group modulo `mod`: it represents the subgroup  $\text{Ker}\chi$ .

`GEN bnr_subgroup_check(GEN bnr, GEN H, GEN *pdeg)` given a `bnr` attached to a modulus `mod`, check whether  $H$  represents a congruence subgroup (of the ray class group modulo `mod`) and returns a normalized representation: `NULL` for the trivial subgroup, or in HNF, reduced modulo the elementary divisors of the ray class group. In particular, if  $H$  is a character of the ray class group, the returned value is the character kernel. If `pdeg` is not `NULL`, `*pdeg` is set to the degree of the attached class field: the index of  $H$  in the ray class group.

`void bnr_subgroup_sanitise(GEN *pbnr, GEN *pH)` given a `bnr` and a congruence subgroup, make sanity checks and compute the subgroup conductor. Then replace the pair to match the

conductor: the *bnr* has the right conductor as modulus, and the subgroup is normalized. Instead of a *bnr*, this function also accepts a *bnf* (gets replaced by the *bnr* with trivial conductor). Instead of a subgroup, the function also accepts an integer  $N$  (replaced by  $\text{Cl}_f(K)^N$ ) or a character (replaced by its kernel).

`void bnr_char_sanitize(GEN *pbnr, GEN *pchi)` same idea as `bnr_subgroup_sanitize`: we are given a *bnr* and a ray class character, make sanity checks and update the data to use the conductor as modulus.

`GEN bnrconductor(GEN bnr, GEN H, long flag)` see the documentation of the GP function.

`GEN bnrconductor_factored(GEN bnr, GEN H)` return a pair  $[F, fa]$  where  $F$  is the conductor and  $fa$  is the factorization of the finite part of the conductor. Shallow function.

`GEN bnrconductor_raw(GEN bnr, GEN H)` return the conductor of  $H$ . Shallow function.

`long bnrisc conductor(GEN bnr, GEN H)` returns 1 if the class field defined by the subgroup  $H$  (of the ray class group mod  $f$  coded in *bnr*) has conductor  $f$ . Returns 0 otherwise.

`GEN ideallog_units(GEN bnf, GEN bid)` return the images of the units generators `bnf.tu` and `bnf.tu` in the finite abelian group  $(\mathbf{Z}_K/f)^*$  attached to *bid*.

`GEN ideallog_units0(GEN bnf, GEN bid, GEN N)` let  $G = (\mathbf{Z}_K/f)^*$  be the finite abelian group attached to *bid*. Return the images of the units generators `bnf.tu` and `bnf.tu` in  $G/G^N$ . If  $N$  is NULL, same as `ideallog_units`.

`GEN bnrchar_primitive(GEN bnr, GEN chi, GEN bnrc)` Given a normalized character  $\chi = [d, c]$  on `bnr.clgp` (see `char_normalize`) of conductor `bnrc.mod`, compute the primitive character  $\chi_{\text{ic}}$  on `bnrc.clgp` equivalent to  $\chi$ , given as a normalized character  $[D, C]$ : `chi_c(bnrc.gen[i])` is  $\zeta_D^{C[i]}$ , where  $D$  is minimal. It is easier to use `bnrconductor_i(bnr, chi, 2)`, but the latter recomputes `bnrc` for each new character.

`GEN bnrchar_primitive_raw(GEN bnr, GEN chi, GEN bnrc)` as `bnrchar_primitive`, with  $\chi$  a regular (unnormalized) character on `bnr.clgp` of conductor `bnrc.mod`. Return a regular (unnormalized) primitive character on `bnrc`.

`GEN bnrdisc(GEN bnr, GEN H, long flag)` returns the discriminant and signature of the class field defined by *bnr* and  $H$ . See the description of the GP function for details. *flag* is an or-ed combination of the flags `rnf_REL` (output relative data) and `rnf_COND` (return 0 unless the modulus is the conductor).

`GEN ABC_to_bnr(GEN A, GEN B, GEN C, GEN *H, int addgen)` This is a quick conversion function designed to go from the too general (inefficient)  $A, B, C$  form to the preferred *bnr*,  $H$  form for class fields. Given  $A, B, C$  as explained above (omitted entries coded by NULL), return the attached *bnr*, and set  $H$  to the attached subgroup. If `addgen` is 1, make sure that if the *bnr* needed to be computed, then it contains generators.

**13.1.28 Abelian maps.** A map  $f : A \rightarrow B$  between two abelian groups of finite type is given by a triple:  $[M, cyc_A, cyc_B]$ , where  $cyc_A = [a_1, \dots, a_m]$  and  $cyc_B = [b_1, \dots, b_n]$  are the elementary divisors for  $A$  and  $B$  (see `ZM_snf`) so that  $A = \oplus_{i \leq m} (\mathbf{Z}/a_i \mathbf{Z}) g_i$  and  $B = \sim \oplus_{j \leq n} (\mathbf{Z}/b_j \mathbf{Z}) G_j$ . The matrix  $M$  gives the image of the generators  $g_i$  in terms of the  $G_j$ :  $(f(g_i))_{i \leq m} = (G_j)_{j \leq n} \cdot M$ . The function `bnrmap` returns such a structure.

`GEN bnr surjection(GEN BNR, GEN bnr)` `BNR` and `bnr` defined over the same field  $K$ , for moduli  $F$  and  $f$  with  $f \mid F$ , returns the canonical surjection  $\text{Cl}_K(F) \rightarrow \text{Cl}_K(f)$  as an abelian map. I.e., a triple  $[M, cyc_F, cyc_f]$ .  $M$  gives the image of the fixed ray class group generators of `BNR` in terms of the ones in `bnr`,  $cyc_F$  and  $cyc_f$  are the cyclic structures of `BNR` and `bnr` respectively (as per `bnr_get_cyc`). Shallow function.

`GEN abmap_kernel(GEN S)` returns the kernel of the abelian map  $S$ , as a matrix  $H$  in HNF: the subgroup is  $(g_i) \cdot H$ .

`GEN abmap_subgroup_image(GEN S, GEN H)` given a subgroup  $H$  of  $A$  (its generators are the  $(g_i)H$ ); for efficiency,  $H$  should be given in canonical form, i.e., as an HNF left divisor of  $\text{diag}(a_1, \dots, a_m)$ . Returns the subgroup  $f(H)$  of  $B$ , as an HNF left divisor of  $\text{diag}(b_1, \dots, b_n)$ .

### 13.1.29 Grunwald–Wang theorem.

`GEN nfgwkummer(GEN nf, GEN Lpr, GEN Ld, GEN pl, long var)` low-level version of `nfgrunwaldwang`, assuming that `nf` contains suitable roots of unity, and directly using Kummer theory to construct the extension.

`GEN bnfgwgeneric(GEN bnf, GEN Lpr, GEN Ld, GEN pl, long var)` low-level version of `nfgrunwaldwang`, assuming that `bnf` is a `bnfinit` structure, and calling `rnfkummer` to construct the extension.

### 13.1.30 Relative equations, Galois conjugates.

`GEN nfissquarefree(GEN nf, GEN P)` given  $P$  a polynomial with coefficients in  $nf$ , return 1 if  $P$  is squarefree, and 0 otherwise. It is allowed (though less efficient) to replace  $nf$  by a monic `ZX` defining the field.

`GEN rnfequationall(GEN A, GEN B, long *pk, GEN *pLPRS)`  $A$  is either an  $nf$  type (corresponding to a number field  $K$ ) or an irreducible `ZX` defining a number field  $K$ .  $B$  is an irreducible polynomial in  $K[X]$ . Returns an absolute equation  $C$  (over  $\mathbf{Q}$ ) for the number field  $K[X]/(B)$ .  $C$  is the characteristic polynomial of  $b + ka$  for some roots  $a$  of  $A$  and  $b$  of  $B$ , and  $k$  is a small rational integer. Set `*pk` to  $k$ .

If `pLPRS` is not `NULL` set it to  $[h_0, h_1]$ ,  $h_i \in \mathbf{Q}[X]$ , where  $h_0 + h_1 Y$  is the last nonconstant polynomial in the pseudo-Euclidean remainder sequence attached to  $A(Y)$  and  $B(X - kY)$ , leading to  $C = \text{Res}_Y(A(Y), B(X - kY))$ . In particular  $a := -h_0/h_1$  is a root of  $A$  in  $\mathbf{Q}[X]/(C)$ , and  $X - ka$  is a root of  $B$ .

`GEN nf_rnfeq(GEN A, GEN B)` wrapper around `rnfequationall` to allow mapping  $K \rightarrow L$  (`eltup`) and converting elements of  $L$  between absolute and relative form (`reltoabs`, `abstorel`), without computing a full  $rnf$  structure, which is useful if the relative integral basis is not required. In fact, since  $A$  may be a `t_POL` or an  $nf$ , the integral basis of the base field is not needed either. The return value is the same as `rnf_get_map`. Shallow function.

`GEN nf_rnfeqsimple(GEN A, GEN B)` as `nf_rnfeq` except some fields are omitted, so that only the `abstorel` operation is supported. Shallow function.

GEN `eltabstorel`(GEN `rnfeq`, GEN `x`) `rnfeq` is as given by `rnf_get_map` (but in this case `rnfeltabstorel` is more robust), `nf_rnfeq` or `nf_rnfeqsimple`, return  $x$  as an element of  $L/K$ , i.e. as a `t_POLMOD` with `t_POLMOD` coefficients. Shallow function.

GEN `eltabstorel_lift`(GEN `rnfeq`, GEN `x`) same as `eltabstorel`, except that  $x$  is returned in partially lifted form, i.e. as a `t_POL` with `t_POLMOD` coefficients.

GEN `eltreltoabs`(GEN `rnfeq`, GEN `x`) `rnfeq` is as given by `rnf_get_map` (but in this case `rnfeltreltoabs` is more robust) or `nf_rnfeq`, return  $x$  in absolute form.

GEN `nf_nfzk`(GEN `nf`, GEN `rnfeq`) `rnfeq` as given by `nf_rnfeq`, `nf` a true *nf* structure, return a suitable representation of `nf.zk` allowing quick computation of the map  $K \rightarrow L$  by the function `nfeltup`, *without* computing a full *rnfeq* structure, which is useful if the relative integral basis is not required. The computed value is the same as in `rnf_get_nfzk`. Shallow function.

GEN `nfeltup`(GEN `nf`, GEN `x`, GEN `zknf`) `zknf` and is initialized by `nf_nfzk` or `rnf_get_nfzk` (but in this case `nfeltup` is more robust); `nf` is a true *nf* structure for  $K$ , returns  $x \in K$  as a (lifted) element of  $L$ , in absolute form.

GEN `rnfdisc_factored`(GEN `nf`, GEN `pol`, GEN `*pd`) variant of `rnfdisc` returning the relative discriminant ideal *factorization*, and setting `*pd` to the discriminant as an element in  $K^*/(K^*)^2$ . Shallow function. The argument `nf` is a true *nf* structure.

GEN `Rg_nffix`(const char `*f`, GEN `T`, GEN `c`, int `lift`) given a ZX  $T$  and a “coefficient”  $c$  supposedly belonging to  $\mathbf{Q}[y]/(T)$ , check whether this is the case and return a cleaned up version of  $c$ . The string  $f$  is the calling function name, used to report errors.

This means that  $c$  must be one of `t_INT`, `t_FRAC`, `t_POL` in the variable  $y$  with rational coefficients, or `t_POLMOD` modulo  $T$  which lift to a rational `t_POL` as above. The cleanup consists in the following improvements:

- `t_POL` coefficients are reduced modulo  $T$ .
- `t_POL` and `t_POLMOD` belonging to  $\mathbf{Q}$  are converted to rationals, `t_INT` or `t_FRAC`.
- if `lift` is nonzero, convert `t_POLMOD` to `t_POL`, and otherwise convert `t_POL` to `t_POLMODs` modulo  $T$ .

GEN `RgX_nffix`(const char `*f`, GEN `T`, GEN `P`, int `lift`) check whether  $P$  is a polynomials with coefficients in the number field defined by the absolute equation  $T(y) = 0$ , where  $T$  is a ZX and returns a cleaned up version of  $P$ . This checks whether  $P$  is indeed a `t_POL` with variable compatible with coefficients in  $\mathbf{Q}[y]/(T)$ , i.e.

$$\text{varncmp}(\text{varn}(P), \text{varn}(T)) < 0$$

and applies `Rg_nffix` to each coefficient.

GEN `RgV_nffix`(const char `*f`, GEN `T`, GEN `P`, int `lift`) as `RgX_nffix` for a vector of coefficients.

GEN `polmod_nffix`(const char `*f`, GEN `rnf`, GEN `x`, int `lift`) given a `t_POLMOD`  $x$  supposedly defining an element of *rnfeq*, check this and perform `Rg_nffix` cleanups.

GEN `polmod_nffix2`(const char `*f`, GEN `T`, GEN `P`, GEN `x`, int `lift`) as in `polmod_nffix`, where the relative extension is explicitly defined as  $L = (\mathbf{Q}[y]/(T))[x]/(P)$ , instead of by an *rnfeq* structure.

`long numberofconjugates(GEN T, long pinit)` returns a quick multiple for the number of  $\mathbf{Q}$ -automorphism of the (integral, monic)  $\mathbf{t\_POL}$   $T$ , from modular factorizations, starting from prime `pinit` (you can set it to 2). This upper bounds often coincides with the actual number of conjugates. Of course, you should use `nfgaloisconj` to be sure.

`GEN nfroots_if_split(GEN *pt, GEN T)` let `*pt` point either to a number field structure or an irreducible  $\mathbf{ZX}$ , defining a number field  $K$ . Given  $T$  a monic squarefree polynomial with coefficients in  $\mathbf{Z}_K$ , return the list of roots of `pol` in  $K$  if the polynomial splits completely, and `NULL` otherwise. In other words, this checks whether  $K[X]/(T)$  is normal over  $K$  (hence Galois since  $T$  is separable by assumption).

In the case where `*pt` is a  $\mathbf{ZX}$ , the function has to compute internally a conditional `nf` attached to  $K$ , whose `nf.zk` may not define the maximal order  $\mathbf{Z}_K$  (see `nfroots`); `*pt` is then replaced by the conditional `nf` to avoid losing that information.

`GEN rnfabelianconjgen(GEN nf, GEN P)`  $nf$  being a number field structure attached to  $K$  and  $P$  being an irreducible polynomial in  $K[X]$ . This function returns `gen_0` if  $L = K[X]/(P)$  is not abelian over  $K$ , else it returns a pair  $(g, o)$  where  $g$  is a vector of  $K$ -automorphisms of  $L$  generating the abelian group  $G = \text{Gal}(L/K)$  and  $o$  is a `t_VECSMALL` of the same length giving the relative orders of the  $g_i$ :  $o[1]$  is the order of  $g_1$  and for  $i \geq 2$ ,  $o[i]$  is the order of  $g_i$  in  $G/(g_1, \dots, g_{i-1})$ . The length need not be minimal: the  $o[i]$  need not be the elementary divisors of  $G$ .

### 13.1.31 Units.

`GEN nfrootsof1(GEN nf)` returns a two-component vector  $[w, z]$  where  $w$  is the number of roots of unity in the number field  $nf$ , and  $z$  is a primitive  $w$ -th root of unity.

`GEN nfcyclotomicunits(GEN nf, GEN zu)` where `zu` is as output by `nfrootsof1(nf)`, return the vector of the cyclotomic units in  $nf$  expressed over the integral basis. If  $\zeta = \zeta_n$  belongs to the base field ( $n$  maximal), this function returns

- (when  $n$  is not a prime power) the  $\zeta^a - 1$ , for all  $1 \leq a < n/2$  such that  $n/(a, n)$  is not a prime power and  $a$  is a strict divisor of  $n$ .
- (all  $n$ ) for  $p$  prime,  $v_p(n) = k > 0$ , the  $(z^a - 1)/(z - 1)$ , where  $z = \zeta^{n/p^k}$ , for all  $1 < a \leq (p^k - 1)/2$ ,  $(p, a) = 1$ .

These are independent modulo torsion if  $n$  is a prime power, but not necessarily so otherwise.

`GEN sunits_mod_units(GEN bnf, GEN S)` return independent generators for  $U_S(K)/U(K)$ .

### 13.1.32 Obsolete routines.

Still provided for backward compatibility, but should not be used in new programs. They will eventually disappear.

`GEN zidealstar(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_GEN)`

`GEN zidealstarinit(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_INIT)`

`GEN zidealstarinitgen(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_GEN|nf_INIT)`

`GEN idealstar0(GEN nf, GEN x, long flag)` short for `idealstarmod(nf, ideal, flag, NULL)`. Use `Idealstarmod` or `Idealstar`.

`GEN bnrinit0(GEN bnf, GEN ideal, long flag)` short for `bnrinitmod(bnf,ideal,flag,NULL)`. Use `Buchray` or `Buchraymod`.



GEN buchimag(GEN D, GEN c1, GEN c2, GEN gC0) short for

Buchquad(D,gtodouble(c1),gtodouble(c2), /\*ignored\*/0)

GEN buchreal(GEN D, GEN gsens, GEN c1, GEN c2, GEN RELSUP, long prec) short for

Buchquad(D,gtodouble(c1),gtodouble(c2), prec)

The following use a naming scheme which is error-prone and not easily extensible; besides, they compute generators as per `nf_GEN` and not `nf_GENMAT`. Don't use them:

GEN isprincipalforce(GEN bnf, GEN x)

GEN isprincipalgen(GEN bnf, GEN x)

GEN isprincipalgenforce(GEN bnf, GEN x)

GEN isprincipalraygen(GEN bnr, GEN x), use `bnrisprincipal`.

Variants on `polred`: use `polredbest`.

GEN factoredpolred(GEN x, GEN fa)

GEN factoredpolred2(GEN x, GEN fa)

GEN smallpolred(GEN x)

GEN smallpolred2(GEN x), use `Polred`.

GEN polred0(GEN x, long flag, GEN p)

GEN polredabs(GEN x)

GEN polredabs2(GEN x)

GEN polredabsall(GEN x, long flun)

Superseded by `bnrdiscclist0`:

GEN discrayabslist(GEN bnf, GEN L)

GEN discrayabslistarch(GEN bnf, GEN arch, long bound)

Superseded by `idealappr` (*flag* is ignored)

GEN idealappr0(GEN nf, GEN x, long flag)

Superseded by `bnrconductor_raw` or `bnrconductormod`:

GEN bnrconductor\_i(GEN bnr, GEN H, long flag) shallow variant of `bnrconductor`.

GEN bnrconductorofchar(GEN bnr, GEN chi)

## 13.2 Galois extensions of $\mathbb{Q}$ .

This section describes the data structure output by the function `galoisinit`. This will be called a `gal` structure in the following.

### 13.2.1 Extracting info from a `gal` structure.

The functions below expect a `gal` structure and are shallow. See the documentation of `galoisinit` for the meaning of the member functions.

`GEN gal_get_pol(GEN gal)` returns `gal.pol`

`GEN gal_get_p(GEN gal)` returns `gal.p`

`GEN gal_get_e(GEN gal)` returns the integer  $e$  such that `gal.mod==gal.pe`.

`GEN gal_get_mod(GEN gal)` returns `gal.mod`.

`GEN gal_get_roots(GEN gal)` returns `gal.roots`.

`GEN gal_get_invvdm(GEN gal)` `gal[4]`.

`GEN gal_get_den(GEN gal)` return `gal[5]`.

`GEN gal_get_group(GEN gal)` returns `gal.group`.

`GEN gal_get_gen(GEN gal)` returns `gal.gen`.

`GEN gal_get_orders(GEN gal)` returns `gal.orders`.

### 13.2.2 Miscellaneous functions.

`GEN nfgaloispermtobasis(GEN nf, GEN gal)` return the images of the field generator by the automorphisms `gal.orders` expressed on the integral basis `nf.zk`.

`GEN nfgaloismatrix(GEN nf, GEN s)` returns the ZM attached to the automorphism  $s$ , seen as a linear operator expressed on the number field integer basis. This allows to use

```
M = nfgaloismatrix(nf, s);
sx = ZM_ZC_mul(M, x);    /* or RgM_RgC_mul(M, x) if x is not integral */
```

instead of

```
sx = nfgaloisapply(nf, s, x);
```

for an algebraic integer  $x$ .

`GEN nfgaloismatrixapply(GEN nf, GEN M, GEN x)` given an automorphism  $M$  in `nfgaloismatrix` form, return the image of  $x$  under the automorphism. Variant of `galoisapply`.

## 13.3 Quadratic number fields and quadratic forms.

### 13.3.1 Checks.

`void check_quaddisc(GEN x, long *s, long *mod4, const char *f)` checks whether the GEN  $x$  is a quadratic discriminant (`t_INT`, not a square, congruent to 0, 1 modulo 4), and raise an exception otherwise. Set  $*s$  to the sign of  $x$  and  $*mod4$  to  $x$  modulo 4 (0 or 1), unless  $mod4$  is NULL.

`void check_quaddisc_real(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is positive.

`void check_quaddisc_imag(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is negative.

### 13.3.2 Class number.

Given a  $D$  congruent to 0 or 1 modulo 4, let  $h(D)$  denote the class number of the order of discriminant  $D$ . The function `quadclassunit` uses index calculus and computes  $h(D)$  in subexponential time in  $\log |D|$  but it assumes the truth of the GRH. For imaginary quadratic orders, it is also comparatively slow for *small* values, say  $|D| \leq 10^{18}$ . Here are some alternatives:

`GEN classno(GEN D)` corresponds to `qfbclassno(D,0)` and is only useful for  $D < 0$ , uses a baby-step giant-step technique and runs in time  $O(D^{1/4})$ . The result is guaranteed correct for  $|D| < 2 \cdot 10^{10}$  and fastest in that range. For larger values of  $|D|$ , the algorithm is no longer rigorous and may give incorrect results (we know no concrete example); it also becomes relatively less interesting compared to `quadclassunit`.

`GEN classno2(GEN D)` corresponds to `qfbclassno(D,1)` and runs in time  $O(D^{1/2})$ ; the function is provided for testing purposes only since it is never competitive.

`GEN quadclassnoF(GEN D, GEN *pd)` returns  $h(D)/h(d)$  where  $d$  is the fundamental discriminant attached to  $D$ . If  $pd$  is not NULL, set  $*pd$  to  $d$ .

`GEN quadclassno(GEN D)` returns  $h(D)$  using Buchmann's algorithm on the order of discriminant  $D$ . If  $D$  is not fundamental, it will usually be faster to call `coredisc2_fact` and `quadclassnoF_fact` to reduce to this case first.

`long quadclassnos(long D)` returns  $h(D)$  using Buchmann's algorithm on the order of discriminant  $D$ .

`ulong unegquadclassnoF(ulong x, long *pd)` returns  $h(-x)/h(d)$ . Set  $*pd$  to  $d$ .

`ulong uposquadclassnoF(ulong x, long *pd)` returns  $h(x)/h(d)$ . Set  $*pd$  to  $d$ .

`GEN quadclassnoF_fact(GEN D, GEN P, GEN E)` let  $D$  be a fundamental discriminant, and  $f = \prod_i P[i]^{E[i]}$  be a positive conductor for the order of discriminant  $Df^2$  ( $P$  is a ZV and  $E$  is a ZV or zv). Returns

$$[O_D^\times : O_{Df^2}^\times] \cdot h(Df^2)/h(d) = f \prod_{p|f} (1 - (D/p)p^{-1}).$$

`ulong uquadclassnoF_fact(ulong d, long s, GEN P, GEN E)` let  $s = 1$  or  $-1$  be a sign,  $D = sd$  be a fundamental discriminant, and  $f = \prod_i P[i]^{E[i]}$  be a positive conductor for the order of discriminant  $Df^2$  ( $P$  and  $E$  are `t_VECSMALL`). Returns

$$[O_D^\times : O_{Df^2}^\times] \cdot h(Df^2)/h(d) = f \prod_{p|f} (1 - (D/p)p^{-1}).$$

GEN `hclassno`(GEN `d`) returns the Hurwitz-Kronecker class number  $H(d)$ . These play a central role in trace formulas and are usually needed for many consecutive values of  $d$ . Thus, the function uses a cache so that later calls for *small* consecutive values of  $d$  are instantaneous, see `getcache`. Large values of  $d$  ( $d > 500000$ ) call `quadclassunit` individually and are not memoized.

GEN `hclassnoF_fact`(GEN `P`, GEN `E`, GEN `D`) return  $H(Df^2)/H(D)$  assuming  $D$  is a negative fundamental discriminant, where the conductor  $f$  is given in factored form:  $P$  (ZV) is the list of prime divisors of  $f$  and  $E$  (t\_VECSMALL) their multiplicities.

long `uhclassnoF_fact`(GEN `faf`, long `D`) return  $H(Df^2)/H(D)$  assuming  $D$  is a negative fundamental discriminant and  $d = Df^2$  is an `ulong` and `faf` is `factoru(d)`.

GEN `hclassno6`(GEN `d`) assuming  $d > 0$ , returns the integer  $6H(d)$ . This is a low-level function behind `hclassno`.

`ulong hclassno6u(ulong d)` assuming  $d > 0$ , returns the integer  $6H(d)$ . Using this function creates (or extends) caches of Hurwitz class numbers and Corediscs of negative integers to speed up consecutive or repeated calls (see `getcache`). If this is a problem, use:

`ulong hclassno6u_no_cache(ulong d)` as `hclassno6u` without creating caches. Existing caches will be used.

### 13.3.3 t\_QFB.

The functions in this section operate on binary quadratic forms of type `t_QFB`. When specified, a `t_QFB` argument  $q$  attached to an indefinite form can be replaced by the pair  $[q, d]$  where the `t_REAL`  $d$  is Shanks's distance.

GEN `qfb_1`(GEN `q`) given a `t_QFB`  $q$ , return the unit form  $q^0$ .

int `qfb_equal1`(GEN `q`) returns 1 if the `t_QFB`  $q$  is the unit form.

#### 13.3.3.1 Reduction.

GEN `qfbred`(GEN `x`) reduction of a `t_QFB`  $x$ . Also allow extended indefinite forms.

GEN `qfbred_i`(GEN `x`) internal version of `qfbred`: assume  $x$  is a `t_QFB`.

#### 13.3.3.2 Composition.

GEN `qfbcomp`(GEN `x`, GEN `y`) compose the two `t_QFB`  $x$  and  $y$  (with same discriminant), then reduce the result. This is the same as `gmul(x,y)`. Also allow extended indefinite forms.

GEN `qfbcomp_i`(GEN `x`, GEN `y`) internal version of `qfbcomp`: assume  $x$  and  $y$  are `t_QFB` of the same discriminant.

GEN `qfbsqr`(GEN `x`) as `qfbcomp(x,x)`.

GEN `qfbsqr_i`(GEN `x`) as `qfbcomp_i(x,y)`.

Same as above, *without* reducing the result:

GEN `qfbcompraw`(GEN `x`, GEN `y`) compose two `t_QFB`s, without reducing the result. Also allow extended indefinite forms.

GEN `qfbcompraw_i`(GEN `x`, GEN `y`) internal version of `qfbcompraw`: assume  $x$  and  $y$  are `t_QFB` of the same discriminant.

### 13.3.3.3 Powering.

GEN `qfbpow`(GEN `x`, GEN `n`) computes  $x^n$  and reduce the result. Also allow extended indefinite forms.

GEN `qfbpows`(GEN `x`, long `n`) computes  $x^n$  and reduce the result. Also allow extended indefinite forms.

GEN `qfbpow_i`(GEN `x`, GEN `n`) internal version of `qfbcomp`. Assume  $x$  is a QFB.

GEN `qfbpowraw`(GEN `x`, long `n`) compute  $x^n$  (pure composition, no reduction), for a `t_QFB`  $x$ . Also allow indefinite forms.

### 13.3.3.4 Order, discrete log.

GEN `qfi_order`(GEN `q`, GEN `o`) assuming that the imaginary `t_QFB`  $q$  has order dividing  $o$ , compute its order in the class group. The order can be given in all formats allowed by generic discrete log functions, the preferred format being [`ord`, `fa`] (`t_INT` and its factorization).

GEN `qfi_log`(GEN `a`, GEN `g`, GEN `o`) given an imaginary `t_QFB`  $a$  and assuming that the `t_QFB`  $g$  has order  $o$ , compute an integer  $k$  such that  $a^k = g$ . Return `cgetg(1, t_VEC)` if there are no solutions. Uses a generic Pollig-Hellman algorithm, then either Shanks (small  $o$ ) or Pollard rho (large  $o$ ) method. The order can be given in all formats allowed by generic discrete log functions, the preferred format being [`ord`, `fa`] (`t_INT` and its factorization).

GEN `qfi_Shanks`(GEN `a`, GEN `g`, long `n`) given an imaginary `t_QFB`  $a$  and assuming that the `t_QFB`  $g$  has (small) order  $n$ , compute an integer  $k$  such that  $a^k = g$ . Return `cgetg(1, t_VEC)` if there are no solutions. Directly uses Shanks algorithm, which is inefficient when  $n$  is composite.

### 13.3.3.5 Solve, Cornacchia.

The following functions underly `qfbsolve`;  $p$  denotes a prime number.

GEN `qfisolvep`(GEN `Q`, GEN `p`) solves  $Q(x, y) = p$  over the integers, for an imaginary `t_QFB`  $Q$ . Return `gen_0` if there are no solutions.

GEN `qfrsolvep`(GEN `Q`, GEN `p`) solves  $Q(x, y) = p$  over the integers, for a real `t_QFB`  $Q$ . Return `gen_0` if there are no solutions.

long `cornacchia`(GEN `d`, GEN `p`, GEN `*px`, GEN `*py`) solves  $x^2 + dy^2 = p$  over the integers, where  $d > 0$  is congruent to 0 or 3 modulo 4. Return 1 if there is a solution (and store it in `*x` and `*y`), 0 otherwise.

long `cornacchia2`(GEN `d`, GEN `p`, GEN `*px`, GEN `*py`) as `cornacchia`, for the equation  $x^2 + dy^2 = 4p$ .

long `cornacchia2_sqrt`(GEN `d`, GEN `p`, GEN `b`, GEN `*px`, GEN `*py`) as `cornacchia2`, where  $p > 2$  and  $b$  is the smallest squareroot of  $d$  modulo  $p$ .

### 13.3.3.6 Prime forms.

GEN `primeform_u`(GEN `D`, ulong `p`) `t_QFB` of discriminant  $D$  whose first coefficient is the prime  $p$ , assuming  $(D/p) \geq 0$ .

GEN `primeform`(GEN `D`, GEN `p`)

**13.3.4 Efficient real quadratic forms.** Unfortunately, real `t_QFBs` are very inefficient, and are only provided for backward compatibility.

- they do not contain needed quantities, which are thus constantly recomputed (the discriminant square root  $\sqrt{D}$  and its integer part),
- the distance component is stored in logarithmic form, which involves computing one extra logarithm per operation. It is much more efficient to store its exponential, computed from ordinary multiplications and divisions (taking exponent overflow into account), and compute its logarithm at the very end.

Internally, we have two representations for real quadratic forms:

- `qfr3`, a container  $[a, b, c]$  with at least 3 entries: the three coefficients; the idea is to ignore the distance component.
- `qfr5`, a container with at least 5 entries  $[a, b, c, e, d]$ : the three coefficients a `t_REAL`  $d$  and a `t_INT`  $e$  coding the distance component  $2^{Ne}d$ , in exponential form, for some large fixed  $N$ .

It is a feature that `qfr3` and `qfr5` have no specified length or type. It implies that a `qfr5` or `t_QFB` will do whenever a `qfr3` is expected. Routines using these objects require a global context, provided by a `struct qfr_data *`:

```
struct qfr_data {
    GEN D;          /* discriminant, t_INT */
    GEN sqrtD;      /* sqrt(D), t_REAL */
    GEN isqrtD;     /* floor(sqrt(D)), t_INT */
};
```

`void qfr_data_init(GEN D, long prec, struct qfr_data *S)` given a discriminant  $D > 0$ , initialize  $S$  for computations at precision `prec` ( $\sqrt{D}$  is computed to that initial accuracy).

All functions below are shallow, and not stack clean.

`GEN qfr3_comp(GEN x, GEN y, struct qfr_data *S)` compose two `qfr3`, reducing the result.

`GEN qfr3_compraw(GEN x, GEN y)` as `qfr3_comp`, without reducing the result.

`GEN qfr3_pow(GEN x, GEN n, struct qfr_data *S)` compute  $x^n$ , reducing along the way.

`GEN qfr3_red(GEN x, struct qfr_data *S)` reduce  $x$ .

`GEN qfr3_rho(GEN x, struct qfr_data *S)` perform one reduction step; `qfr3_red` just performs reduction steps until we hit a reduced form.

`GEN qfr3_to_qfr(GEN x, GEN d)` recover an ordinary `t_QFB` from the `qfr3`  $x$ , adding discriminant component  $d$ .

Before we explain `qfr5`, recall that it corresponds to an ideal, that reduction corresponds to multiplying by a principal ideal, and that the distance component is a clever way to keep track of these principal ideals. More precisely, reduction consists in a number of reduction steps, going from the form  $(a, b, c)$  to  $\rho(a, b, c) = (c, -b \bmod 2c, *)$ ; the distance component is multiplied by (a floating point approximation to)  $(b + \sqrt{D})/(b - \sqrt{D})$ .

`GEN qfr5_comp(GEN x, GEN y, struct qfr_data *S)` compose two `qfr5`, reducing the result, and updating the distance component.

`GEN qfr5_compraw(GEN x, GEN y)` as `qfr5_comp`, without reducing the result.

GEN `qfr5_pow`(GEN `x`, GEN `n`, struct `qfr_data *S`) compute  $x^n$ , reducing along the way.

GEN `qfr5_red`(GEN `x`, struct `qfr_data *S`) reduce  $x$ .

GEN `qfr5_rho`(GEN `x`, struct `qfr_data *S`) perform one reduction step.

GEN `qfr5_dist`(GEN `e`, GEN `d`, long `prec`) decode the distance component from exponential (qfr5-specific) to logarithmic form (true Shanks's distance).

GEN `qfr_to_qfr5`(GEN `x`, long `prec`) convert a real `t_QFB` to a `qfr5` with initial trivial distance component (= 1).

GEN `qfr5_to_qfr`(GEN `x`, GEN `d`), assume  $x$  is a `qfr5` and  $d$  is NULL or the original distance component of some real `t_QFB`. Convert  $x$  to a `t_QFB`, with the correct (logarithmic) distance component if  $d$  is not NULL.

## 13.4 Linear algebra over $\mathbb{Z}$ .

### 13.4.1 Hermite and Smith Normal Forms.

GEN `ZM_hnf`(GEN `x`) returns the upper triangular Hermite Normal Form of the ZM  $x$  (removing 0 columns), using the `ZM_hnfall` algorithm. If you want the true HNF, use `ZM_hnfall(x, NULL, 0)`.

GEN `ZM_hnfmod`(GEN `x`, GEN `d`) returns the HNF of the ZM  $x$  (removing 0 columns), assuming the `t_INT`  $d$  is a multiple of the determinant of  $x$ . This is usually faster than `ZM_hnf` (and uses less memory) if the dimension is large, > 50 say.

GEN `ZM_hnfmodid`(GEN `x`, GEN `d`) returns the HNF of the ZM  $x$  concatenated with the diagonal matrix with diagonal  $d$ , where  $d$  is a vector of integers of compatible dimension. Variant: if  $d$  is a `t_INT`, then concatenate  $d\text{Id}$ .

GEN `ZM_hnfmodprime`(GEN `x`, GEN `p`) returns the HNF of the matrix  $(x \mid p\text{Id})$  (removing 0 columns), for a ZM  $x$  and a prime number  $p$ . The algorithm involves only  $\mathbb{F}_p$ -linear algebra and is faster than `ZM_hnfmodid` (which will call it when  $d$  is prime).

GEN `ZM_hnfmodall`(GEN `x`, GEN `d`, long `flag`) low-level function underlying the `ZM_hnfmod` variants. If `flag` is 0, calls `ZM_hnfmod(x,d)`; `flag` is an or-ed combination of:

- `hnf_MODID` call `ZM_hnfmodid` instead of `ZM_hnfmod`,
- `hnf_PART` return as soon as we obtain an upper triangular matrix, saving time. The pivots are nonnegative and give the diagonal of the true HNF, but the entries to the right of the pivots need not be reduced, i.e. they may be large or negative.
- `hnf_CENTER` returns the centered HNF, where the entries to the right of a pivot  $p$  are centered residues in  $[-p/2, p/2[$ , hence smallest possible in absolute value, but possibly negative.

GEN `ZM_hnfmodall_i`(GEN `x`, GEN `d`, long `flag`) as `ZM_hnfmodall` without final garbage collection. Not `gerepile`-safe.

GEN `ZM_hnfall`(GEN `x`, GEN `*U`, long `remove`) returns the upper triangular HNF  $H$  of the ZM  $x$ ; if  $U$  is not NULL, set it to the matrix  $U$  such that  $xU = H$ . If `remove` = 0,  $H$  is the true HNF, including 0 columns; if `remove` = 1, delete the 0 columns from  $H$  but do not update  $U$  accordingly (so that the integer kernel may still be recovered): we no longer have  $xU = H$ ; if `remove` = 2,

remove 0 columns from  $H$  and update  $U$  so that  $xU = H$ . The matrix  $U$  is square and invertible unless `remove = 2`.

This routine uses a naive algorithm which is potentially exponential in the dimension (due to coefficient explosion) but is fast in practice, although it may require lots of memory. The base change matrix  $U$  may be very large, when the kernel is large.

GEN `ZM_hnfall_i`(GEN  $x$ , GEN  $*U$ , long `remove`) as `ZM_hnfall` without final garbage collection. Not `gerepile`-safe.

GEN `ZM_hnfperm`(GEN  $A$ , GEN  $*ptU$ , GEN  $*ptperm$ ) returns the hnf  $H = PAU$  of the matrix  $PA$ , where  $P$  is a suitable permutation matrix, and  $U \in \text{Gl}_n(\mathbf{Z})$ .  $P$  is chosen so as to (heuristically) minimize the size of  $U$ ; in this respect it is less efficient than `ZM_hnfl11` but usually faster. Set  $*ptU$  to  $U$  and  $*ptperm$  to a `t_VECSMALL` representing the row permutation attached to  $P = (\delta_{i, \text{perm}[i]})$ . If  $ptU$  is set to `NULL`,  $U$  is not computed, saving some time; although useless, setting  $ptperm$  to `NULL` is also allowed.

GEN `ZM_hnf_knapsack`(GEN  $x$ ) given a ZM  $x$ , compute its HNF  $h$ . Return  $h$  if it has the knapsack property: every column contains only zeroes and ones and each row contains a single 1; return `NULL` otherwise. Not suitable for `gerepile`.

GEN `ZM_hnfl11`(GEN  $x$ , GEN  $*U$ , int `remove`) returns the HNF  $H$  of the ZM  $x$ ; if  $U$  is not `NULL`, set it to the matrix  $U$  such that  $xU = H$ . The meaning of `remove` is the same as in `ZM_hnfall`.

This routine uses the LLL variant of Havas, Majewski and Mathews, which is polynomial time, but rather slow in practice because it uses an exact LLL over the integers instead of a floating point variant; it uses polynomial space but lots of memory is needed for large dimensions, say larger than 300. On the other hand, the base change matrix  $U$  is essentially optimally small with respect to the  $L_2$  norm.

GEN `ZM_hnfcenter`(GEN  $M$ ). Given a ZM in HNF  $M$ , update it in place so that nondiagonal entries belong to a system of *centered* residues. Not suitable for `gerepile`.

Some direct applications: the following routines apply to upper triangular integral matrices; in practice, these come from HNF algorithms.

GEN `hnf_divscale`(GEN  $A$ , GEN  $B$ , GEN  $t$ )  $A$  an upper triangular ZM,  $B$  a ZM,  $t$  an integer, such that  $C := tA^{-1}B$  is integral. Return  $C$ .

GEN `hnf_invscale`(GEN  $A$ , GEN  $t$ )  $A$  an upper triangular ZM,  $t$  an integer such that  $C := tA^{-1}$  is integral. Return  $C$ . Special case of `hnf_divscale` when  $B$  is the identity matrix.

GEN `hnf_solve`(GEN  $A$ , GEN  $B$ )  $A$  a ZM in upper HNF (not necessarily square),  $B$  a ZM or ZC. Return  $A^{-1}B$  if it is integral, and `NULL` if it is not.

GEN `hnf_invimage`(GEN  $A$ , GEN  $b$ )  $A$  a ZM in upper HNF (not necessarily square),  $b$  a ZC. Return  $A^{-1}B$  if it is integral, and `NULL` if it is not.

int `hnfdivide`(GEN  $A$ , GEN  $B$ )  $A$  and  $B$  are two upper triangular ZM. Return 1 if  $A^{-1}B$  is integral, and 0 otherwise.



## Smith Normal Form.

GEN ZM\_snf(GEN x) returns the Smith Normal Form (vector of elementary divisors) of the ZM  $x$ .

GEN ZM\_snfall(GEN x, GEN \*U, GEN \*V) returns ZM\_snf(x) and sets  $U$  and  $V$  to unimodular matrices such that  $UxV = D$  (diagonal matrix of elementary divisors). Either (or both)  $U$  or  $V$  may be NULL in which case the corresponding matrix is not computed.

GEN ZV\_snfall(GEN d, GEN \*U, GEN \*V) here  $d$  is a ZV; same as ZM\_snfall applied to diagonal(d), but faster.

GEN ZM\_snfall\_i(GEN x, GEN \*U, GEN \*V, long flag) low level version of ZM\_snfall:

- if the first bit of *flag* is 0, return a diagonal matrix (as in ZM\_snfall), else a vector of elementary divisors (as in ZM\_snf).

- if the second bit of *flag* is 1, assume that  $x$  is invertible and allow  $U$  and  $V$  to have determinant congruent to 1 modulo  $d$ , where  $d$  is the largest elementary divisor of  $x$ . Rationale: the finite group  $G = \mathbf{Z}^n / \mathfrak{S}x$  has exponent  $d$  and we are only interested in the action of  $U, V$  as they act on  $G$  not in genuine unimodular matrices. (See ZM\_snf\_group.)

void ZM\_snfclean(GEN d, GEN U, GEN V) assuming  $d, U, V$  come from  $d = \text{ZM\_snfall}(x, \&U, \&V)$ , where  $U$  or  $V$  may be NULL, cleans up the output in place. This means that elementary divisors equal to 1 are deleted and  $U, V$  are updated. This also works when  $d$  is a t\_VEC of elementary divisors. The output is not suitable for gerepileupto.

void ZV\_snfclean(GEN d) assuming  $d$  is a t\_VEC of elementary divisors, return a shortened version where divisors equal to 1 are deleted. The output is not suitable for gerepileupto; we return  $d$  itself if no divisor is 1.

void ZV\_snf\_trunc(GEN D) given a vector  $D$  of elementary divisors (i.e. a ZV such that  $d_i \mid d_{i+1}$ ), truncate it *in place* to leave out the trivial divisors (equal to 1).

GEN ZM\_snf\_group(GEN H, GEN \*U, GEN \*Uinv) this function computes data to go back and forth between an abelian group (of finite type) given by generators and relations, and its canonical SNF form. Given an abstract abelian group with generators  $g = (g_1, \dots, g_n)$  and a vector  $X = (x_i) \in \mathbf{Z}^n$ , we write  $gX$  for the group element  $\sum_i x_i g_i$ ; analogously if  $M$  is an  $n \times r$  integer matrix  $gM$  is a vector containing  $r$  group elements. The group neutral element is 0; by abuse of notation, we still write 0 for a vector of group elements all equal to the neutral element. The input is a full relation matrix  $H$  among the generators, i.e. a ZM (not necessarily square) such that  $gX = 0$  for some  $X \in \mathbf{Z}^n$  if and only if  $X$  is in the integer image of  $H$ , so that the abelian group is isomorphic to  $\mathbf{Z}^n / \text{Im}H$ . The routine assumes that  $H$  is in HNF; replace it by its HNF if it is not the case. (Of course this defines the same group.)

Let  $G$  a minimal system of generators in SNF for our abstract group: if the  $d_i$  are the elementary divisors ( $\dots \mid d_2 \mid d_1$ ), each  $G_i$  has either infinite order ( $d_i = 0$ ) or order  $d_i > 1$ . Let  $D$  the matrix with diagonal  $(d_i)$ , then

$$GD = 0, \quad G = gU_{\text{inv}}, \quad g = GU,$$

for some integer matrices  $U$  and  $U_{\text{inv}}$ . Note that these are not even square in general; even if square, there is no guarantee that these are unimodular: they are chosen to have minimal entries given the known relations in the group and only satisfy  $D \mid (UU_{\text{inv}} - \text{Id})$  and  $H \mid (U_{\text{inv}}U - \text{Id})$ .

The function returns the vector of elementary divisors  $(d_i)$ ; if  $U$  is not NULL, it is set to  $U$ ; if  $U_{\text{inv}}$  is not NULL it is set to  $U_{\text{inv}}$ . The function is not memory clean.

GEN ZV\_snf\_group(GEN d, GEN \*newU, GEN \*newUi), here  $d$  is a ZV; same as ZM\_snf\_group applied to diagonal(d), but faster.

GEN ZV\_snf\_gcd(GEN v, GEN N) given a vector  $v$  of integers and a positive integer  $N$ , return the vector whose entries are the gcds  $(v[i], N)$ . Use case: if  $v$  gives the cyclic components for some abelian group  $G$  of finite type, then this returns the structure of the finite groupe  $G/G^N$ .

The following functions compute the  $p^n$ -rank of abelian groups given a vector of elementary divisors and underly snfrank:

long ZV\_snf\_rank(GEN D, GEN p) assume  $D$  is a ZV and  $p$  is a t\_INT.

long ZV\_snf\_rank\_u(GEN D, ulong p) assume  $D$  is a ZV.

long zv\_snf\_rank(GEN D, ulong p) assume  $D$  is a zv.

The following routines underly the various matrixqz variants. In all case the  $m \times n$  t\_MAT  $x$  is assumed to have rational (t\_INT and t\_FRAC) coefficients

GEN QM\_ImQ(GEN x) returns a basis for  $\text{Im}_{\mathbf{Q}}x \cap \mathbf{Z}^n$ .

GEN QM\_ImZ(GEN x) returns a basis for  $\text{Im}_{\mathbf{Z}}x \cap \mathbf{Z}^n$ .

GEN QM\_ImQ\_hnf(GEN x) returns an HNF basis for  $\text{Im}_{\mathbf{Q}}x \cap \mathbf{Z}^n$ .

GEN QM\_ImZ\_hnf(GEN x) returns an HNF basis for  $\text{Im}_{\mathbf{Z}}x \cap \mathbf{Z}^n$ .

GEN QM\_ImQ\_hnfall(GEN A, GEN \*pB, long remove) as QM\_ImQ\_hnf, further returning the transformation matrix as in ZM\_hnfall.

GEN QM\_ImZ\_hnfall(GEN A, GEN \*pB, long remove) as QM\_ImZ\_hnf, further returning the transformation matrix as in ZM\_hnfall.

GEN QM\_ImQ\_all(GEN A, GEN \*pB, long remove, long hnf) as QM\_ImQ, further returning the transformation matrix as in ZM\_hnfall, and returning an HNF basis if hnf is nonzero.

GEN QM\_ImZ\_all(GEN A, GEN \*pB, long remove, long hnf) as QM\_ImZ, further returning the transformation matrix as in ZM\_hnfall, and returning an HNF basis if hnf is nonzero.

GEN QM\_minors\_coprime(GEN x, GEN D), assumes  $m \geq n$ , and returns a matrix in  $M_{m,n}(\mathbf{Z})$  with the same  $\mathbf{Q}$ -image as  $x$ , such that the GCD of all  $n \times n$  minors is coprime to  $D$ ; if  $D$  is NULL, we want the GCD to be 1.

The following routines are simple wrappers around the above ones and are normally useless in library mode:

GEN hnf(GEN x) checks whether  $x$  is a ZM, then calls ZM\_hnf. Normally useless in library mode.

GEN hnfmmod(GEN x, GEN d) checks whether  $x$  is a ZM, then calls ZM\_hnfmmod. Normally useless in library mode.

GEN hnfmmodid(GEN x, GEN d) checks whether  $x$  is a ZM, then calls ZM\_hnfmmodid. Normally useless in library mode.

GEN hnfall(GEN x) calls ZM\_hnfall(x, &U, 1) and returns  $[H, U]$ . Normally useless in library mode.

GEN hnfl1l(GEN x) calls ZM\_hnfl1l(x, &U, 1) and returns  $[H, U]$ . Normally useless in library mode.

GEN `hnfperm`(GEN `x`) calls `ZM_hnfperm(x, &U, &P)` and returns  $[H, U, P]$ . Normally useless in library mode.

GEN `smith`(GEN `x`) checks whether  $x$  is a ZM, then calls `ZM_snf`. Normally useless in library mode.

GEN `smithall`(GEN `x`) checks whether  $x$  is a ZM, then calls `ZM_snfall(x, &U, &V)` and returns  $[U, V, D]$ . Normally useless in library mode.

Some related functions over  $K[X]$ ,  $K$  a field:

GEN `gsmith`(GEN `A`) the input matrix must be square, returns the elementary divisors.

GEN `gsmithall`(GEN `A`) the input matrix must be square, returns the  $[U, V, D]$ ,  $D$  diagonal, such that  $UAV = D$ .

GEN `RgM_hnfall`(GEN `A`, GEN `*pB`, long `remove`) analogous to `ZM_hnfall`.

GEN `smithclean`(GEN `z`) cleanup the output of `smithall` or `gsmithall` (delete elementary divisors equal to 1, updating base change matrices).

### 13.4.2 The LLL algorithm.

The basic GP functions and their immediate variants are normally not very useful in library mode. We briefly list them here for completeness, see the documentation of `qflll` and `qflllgram` for details:

- GEN `qflll0`(GEN `x`, long `flag`)

GEN `lll`(GEN `x`) *flag*= 0

GEN `lllint`(GEN `x`) *flag*= 1

GEN `lllkerim`(GEN `x`) *flag*= 4

GEN `lllkerimgen`(GEN `x`) *flag*= 5

GEN `lllgen`(GEN `x`) *flag*= 8

- GEN `qflllgram0`(GEN `x`, long `flag`)

GEN `lllgram`(GEN `x`) *flag*= 0

GEN `lllgramint`(GEN `x`) *flag*= 1

GEN `lllgramkerim`(GEN `x`) *flag*= 4

GEN `lllgramkerimgen`(GEN `x`) *flag*= 5

GEN `lllgramgen`(GEN `x`) *flag*= 8

The basic workhorse underlying all integral and floating point LLLs is

GEN `ZM_lll`(GEN `x`, double `D`, long `flag`), where  $x$  is a ZM;  $D \in ]1/4, 1[$  is the Lovász constant determining the frequency of swaps during the algorithm: a larger values means better guarantees for the basis (in principle smaller basis vectors) but longer running times (suggested value:  $D = 0.99$ ).

**Important.** This function does not collect garbage and its output is not suitable for either `gerepile` or `gerepileupto`. We expect the caller to do something simple with the output (e.g. matrix multiplication), then collect garbage immediately.

`flag` is an or-ed combination of the following flags:

- **LLL\_GRAM.** If set, the input matrix  $x$  is the Gram matrix  ${}^t v v$  of some lattice vectors  $v$ .
- **LLL\_INPLACE.** Incompatible with **LLL\_GRAM**. If unset, we return the base change matrix  $U$ , otherwise the transformed matrix  $xU$ . Implies **LLL\_IM** (see below).
- **LLL\_KEEP\_FIRST.** The first vector in the output basis is the same one as was originally input. Provided this is a shortest nonzero vector of the lattice, the output basis is still LLL-reduced. This is used to reduce maximal orders of number fields with respect to the  $T_2$  quadratic form, to ensure that the first vector in the output basis corresponds to 1 (which is a shortest vector).
- **LLL\_COMPATIBLE.** DEPRECATED. This is now a no-op.

The last three flags are mutually exclusive, either 0 or a single one must be set:

- **LLL\_KER** If set, only return a kernel basis  $K$  (not LLL-reduced).
- **LLL\_IM** If set, only return an LLL-reduced lattice basis  $T$ . (This is implied by **LLL\_INPLACE**).
- **LLL\_ALL** If set, returns a 2-component vector  $[K, T]$  corresponding to both kernel and image.

`GEN lllfp(GEN x, double D, long flag)` is a variant for matrices with inexact entries:  $x$  is a matrix with real coefficients (types `t_INT`, `t_FRAC` and `t_REAL`),  $D$  and  $flag$  are as in `ZM_lll`. The matrix is rescaled, rounded to nearest integers, then fed to `ZM_lll`. The flag **LLL\_INPLACE** is still accepted but less useful (it returns an LLL-reduced basis attached to rounded input, instead of an exact base change matrix).

`GEN ZM_lll_norms(GEN x, double D, long flag, GEN *ptB)` slightly more general version of `ZM_lll`, setting `*ptB` to a vector containing the squared norms of the Gram-Schmidt vectors  $(b_i^*)$  attached to the output basis  $(b_i)$ ,  $b_i^* = b_i + \sum_{j < i} \mu_{i,j} b_j^*$ .

`GEN lllintpartial_inplace(GEN x)` given a `ZM`  $x$  of maximal rank, returns a partially reduced basis  $(b_i)$  for the space spanned by the columns of  $x$ :  $|b_i \pm b_j| \geq |b_i|$  for any two distinct basis vectors  $b_i, b_j$ . This is faster than the LLL algorithm, but produces much larger bases.

`GEN lllintpartial(GEN x)` as `lllintpartial_inplace`, but returns the base change matrix  $U$  from the canonical basis to the  $b_i$ , i.e.  $xU$  is the output of `lllintpartial_inplace`.

`GEN RM_round_maxrank(GEN G)` given a matrix  $G$  with real floating point entries and independent columns, let  $G_e$  be the rescaled matrix  $2^e G$  rounded to nearest integers, for  $e \geq 0$ . Finds a small  $e$  such that the rank of  $G_e$  is equal to the rank of  $G$  (its number of columns) and return  $G_e$ . This is useful as a preconditioning step to speed up LLL reductions, see `nf_get_Gtwist`. Suitable for `gerepileupto`, but does not collect garbage.

`GEN Hermite_bound(long n, long prec)` return a majoration of  $\gamma_n^n$  where  $\gamma_n$  is the Hermite constant for lattices of dimension  $n$ . The bound is sharp in dimension  $n \leq 8$  and  $n = 24$ .

### 13.4.3 Linear dependencies.

The following functions underly the `lindep` GP function:

GEN `lindep`(GEN `v`) real/complex entries, guess that about only the 80% leading bits of the input are correct.

GEN `lindep_bit`(GEN `v`, long `b`) real/complex entries, explicit form of the above: multiply the input by  $2^b$  and round to nearest integer before looking for a linear dependency. Truncating dubious bits allows to find better relations.

GEN `lindepfull_bit`(GEN `v`, long `b`) as `lindep_bit` but return a matrix  $M$  with  $n = \#v$  columns and  $r$  rows, with  $r = n + 1$  (if  $v$  is real) or  $n + 2$  (general case) which is an LLL-reduced basis of the lattice formed by concatenating vertically an identity matrix and the floor of  $2^b \text{real}(v)$  and  $2^b \text{imag}(v)$  if  $r = n + 2$ . The first  $n$  rows of  $M$  potentially correspond to relations: whenever the last  $r - n$  entries of a column are small. The function `lindep_bit` essentially returns the first column of  $M$  truncated to  $n$  components.

GEN `lindep_padic`(GEN `v`)  $p$ -adic entries.

GEN `lindep_Xadic`(GEN `v`) polynomial entries.

GEN `deplin`(GEN `v`) returns a nonzero kernel vector for a `t_MAT` input.

Deprecated routine:

GEN `lindep2`(GEN `x`, long `dig`) analogous to `lindep_bit`, with `dig` counting decimal digits.

### 13.4.4 Reduction modulo matrices.

GEN `ZC_hnfremdiv`(GEN `x`, GEN `y`, GEN `*Q`) assuming  $y$  is an invertible ZM in HNF and  $x$  is a ZC, returns the ZC  $R$  equal to  $x \bmod y$  (whose  $i$ -th entry belongs to  $[-y_{i,i}/2, y_{i,i}/2[)$ . Stack clean *unless*  $x$  is already reduced (in which case, returns  $x$  itself, not a copy). If  $Q$  is not NULL, set it to the ZC such that  $x = yQ + R$ .

GEN `ZM_hnfdivrem`(GEN `x`, GEN `y`, GEN `*Q`) reduce each column of the ZM  $x$  using `ZC_hnfremdiv`. If  $Q$  is not NULL, set it to the ZM such that  $x = yQ + R$ .

GEN `ZC_hnfrem`(GEN `x`, GEN `y`) alias for `ZC_hnfremdiv(x,y,NULL)`.

GEN `ZM_hnfrem`(GEN `x`, GEN `y`) alias for `ZM_hnfremdiv(x,y,NULL)`.

GEN `ZC_reducemodmatrix`(GEN `v`, GEN `y`) Let  $y$  be a ZM, not necessarily square, which is assumed to be LLL-reduced (otherwise, very poor reduction is expected). Size-reduces the ZC  $v$  modulo the  $\mathbf{Z}$ -module  $Y$  spanned by  $y$ : if the columns of  $y$  are denoted by  $(y_1, \dots, y_{n-1})$ , we return  $y_n \equiv v$  modulo  $Y$ , such that the Gram-Schmidt coefficients  $\mu_{n,j}$  are less than  $1/2$  in absolute value for all  $j < n$ . In short,  $y_n$  is almost orthogonal to  $Y$ .

GEN `ZM_reducemodmatrix`(GEN `v`, GEN `y`) Let  $y$  be as in `ZC_reducemodmatrix`, and  $v$  be a ZM. This returns a matrix  $v$  which is congruent to  $v$  modulo the  $\mathbf{Z}$ -module spanned by  $y$ , whose columns are size-reduced. This is faster than repeatedly calling `ZC_reducemodmatrix` on the columns since most of the Gram-Schmidt coefficients can be reused.

GEN `ZC_reducemodlll`(GEN `v`, GEN `y`) Let  $y$  be an arbitrary ZM, LLL-reduce it then call `ZC_reducemodmatrix`.

GEN `ZM_reducemodlll`(GEN `v`, GEN `y`) Let  $y$  be an arbitrary ZM, LLL-reduce it then call `ZM_reducemodmatrix`.

Besides the above functions, which were specific to integral input, we also have:

**GEN reducemodinvertible**(GEN *x*, GEN *y*) *y* is an invertible matrix and *x* a **t\_COL** or **t\_MAT** of compatible dimension. Returns  $x - y[y^{-1}x]$ , which has small entries and differs from *x* by an integral linear combination of the columns of *y*. Suitable for **gerepileupto**, but does not collect garbage.

**GEN closemodinvertible**(GEN *x*, GEN *y*) returns  $x - \text{reducemodinvertible}(x, y)$ , i.e. an integral linear combination of the columns of *y*, which is close to *x*.

**GEN reducemodlll**(GEN *x*, GEN *y*) LLL-reduce the nonsingular ZM *y* and call **reducemodinvertible** to find a small representative of  $x \bmod y\mathbf{Z}^n$ . Suitable for **gerepileupto**, but does not collect garbage.

## 13.5 Finite abelian groups and characters.

### 13.5.1 Abstract groups.

A finite abelian group *G* in GP format is given by its Smith Normal Form as a pair  $[h, d]$  or triple  $[h, d, g]$ . Here *h* is the cardinality of *G*,  $(d_i)$  is the vector of elementary divisors, and  $(g_i)$  is a vector of generators. In short,  $G = \oplus_{i \leq n} (\mathbf{Z}/d_i\mathbf{Z})g_i$ , with  $d_n \mid \dots \mid d_2 \mid d_1$  and  $\prod d_i = h$ .

Let  $e(x) := \exp(2i\pi x)$ . For ease of exposition, we restrict to complex-valued characters, but everything applies to more general fields *K* where *e* denotes a morphism  $(\mathbf{Q}, +) \rightarrow (K^*, \times)$  such that  $e(a/b)$  denotes a *b*-th root of unity.

A *character* on the abelian group  $\oplus (\mathbf{Z}/d_j\mathbf{Z})g_j$  is given by a row vector  $\chi = [a_1, \dots, a_n]$  such that  $\chi(\prod g_j^{n_j}) = e(\sum a_j n_j / d_j)$ .

**GEN cyc\_normalize**(GEN *d*) shallow function. Given a vector  $(d_i)_{i \leq n}$  of elementary divisors for a finite group (no  $d_i$  vanish), returns the vector  $D = [1]$  if  $n = 0$  (trivial group) and  $[d_1, d_1/d_2, \dots, d_1/d_n]$  otherwise. This will allow to define characters as  $\chi(\prod g_j^{x_j}) = e(\sum_j x_j a_j D_j / D_1)$ , see **char\_normalize**.

**GEN char\_normalize**(GEN *chi*, GEN *ncyc*) shallow function. Given a character **chi** =  $(a_j)$  and *ncyc* from **cyc\_normalize** above, returns the normalized representation  $[d, (n_j)]$ , such that  $\chi(\prod g_j^{x_j}) = \zeta_d^{\sum_j n_j x_j}$ , where  $\zeta_d = e(1/d)$  and *d* is *minimal*. In particular, *d* is the order of **chi**. Shallow function.

**GEN char\_simplify**(GEN *D*, GEN *N*) given a quasi-normalized character  $[D, (N_j)]$  such that  $\chi(\prod g_j^{x_j}) = \zeta_D^{\sum_j N_j x_j}$ , but where we only assume that *D* is a multiple of the character order, return a normalized character  $[d, (n_j)]$  with *d* *minimal*. Shallow function.

**GEN char\_denormalize**(GEN *cyc*, GEN *d*, GEN *n*) given a normalized representation  $[d, n]$  (where *d* need not be minimal) of a character on the abelian group with abelian divisors **cyc**, return the attached character (where the image of each generator  $g_i$  is given in terms of roots of unity of different orders **cyc**[*i*]).

**GEN charconj**(GEN *cyc*, GEN *chi*) return the complex conjugate of **chi**.

**GEN charmul**(GEN *cyc*, GEN *a*, GEN *b*) return the product character  $a \times b$ .

**GEN chardiv**(GEN *cyc*, GEN *a*, GEN *b*) returns the character  $a/b = a \times \bar{b}$ .

`int char_check(GEN cyc, GEN chi)` return 1 if `chi` is a character compatible with cyclic factors `cyc`, and 0 otherwise.

`GEN cyc2elts(GEN d)` given a `t_VEC`  $d = (d_1, \dots, d_n)$  of nonnegative integers, return the vector of all `t_VECSMALLs` of length  $n$  whose  $i$ -th entry lies in  $[0, d_i[$ . Assumes that the product of the  $d_i$  fits in a `long`.

`long zv_cyc_minimize(GEN d, GEN c, GEN coprime)` given  $d = (d_1, \dots, d_n)$ ,  $d_n \mid \dots \mid d_1 \neq 0$  a list of elementary divisors for a finite abelian group as a `t_VECSMALL`, given  $c = [g_1, \dots, g_n]$  representing an element in the group, and given a mask `coprime` (as from `coprimes.zv(o)`) representing a list of forbidden congruence classes modulo  $o$ , return an integer  $k$  such that `coprime[k%o]` is nonzero and  $k \cdot c$  is lexicographically minimal. For instance, if  $c$  is attached to a Dirichlet character  $\chi$  of order  $o$  via the usual identification  $\chi(g_i) = \zeta_{g_i}^{c_i}$ , then  $\chi^k$  is a “canonical” representative in the Galois orbit of  $\chi$ .

`long zv_cyc_minimal(GEN d, GEN c, GEN coprime)` return 1 if `zv_cyc_minimize` would return  $k = 1$ , i.e.  $c$  is already the canonical representative for the attached character orbit.

### 13.5.2 Dirichlet characters.

The functions in this section are specific to characters on  $(\mathbf{Z}/N\mathbf{Z})^*$ . The argument  $G$  is a special `bid` structure as returned by `znstar0(N, nf_INIT)`. In this case, there are additional ways to input character via Conrey’s representation. The character `chi` is either a `t_INT` (Conrey label), a `t_COL` (a Conrey logarithm) or a `t_VEC` (generic character on `bid.gen` as explained in the previous subsection). The following low-level functions are called by GP’s generic character functions.

`int zncharcheck(GEN G, GEN chi)` return 1 if `chi` is a valid character and 0 otherwise.

`GEN zncharconj(GEN G, GEN chi)` as `charconj`.

`GEN znchardiv(GEN G, GEN a, GEN b)` as `chardiv`.

`GEN zncharker(GEN G, GEN chi)` as `charker`.

`GEN znchareval(GEN G, GEN chi, GEN n, GEN z)` as `chareval`.

`GEN zncharmulo(GEN G, GEN a, GEN b)` as `charmulo`.

`GEN zncharpow(GEN G, GEN a, GEN n)` as `charpow`.

`GEN zncharorder(GEN G, GEN chi)` as `charorder`.

The following functions handle characters in Conrey notation (attached to Conrey generators, not `G.gen`):

`int znconrey_check(GEN cyc, GEN chi)` return 1 if `chi` is a valid Conrey logarithm and 0 otherwise.

`GEN znconrey_normalized(GEN G, GEN chi)` return normalized character attached to `chi`, as in `char_normalize` but on Conrey generators.

`GEN znconreyfromchar(GEN G, GEN chi)` return Conrey logarithm attached to the generic (`t_VEC`, on `G.gen`)

`GEN znconreyfromchar_normalized(GEN G, GEN chi)` return normalized Conrey character attached to the generic (`t_VEC`, on `G.gen`) character `chi`.

`GEN znconreylog_normalize(GEN G, GEN m)` given a Conrey logarithm  $m$  (`t_COL`), return the attached normalized Conrey character, as in `char_normalize` but on Conrey generators.

GEN `znchar_quad`(GEN `G`, GEN `D`) given a nonzero `t_INT` `D` congruent to 0, 1 mod 4, return  $(D/.)$  as a character modulo  $N$ , given by a Conrey logarithm (`t_COL`). Assume that  $|D|$  divides  $N$ .

GEN `Zideallog`(GEN `G`, GEN `x`) return the `znconreylog` of  $x$  expressed on `G.gen`, i.e. the ordinary discrete logarithm from `ideallog`.

GEN `ncharvecexpo`(GEN `G`, GEN `nchi`) given `nchi` =  $[d, n]$  a quasi-normalized character ( $d$  may be a multiple of the character order), i.e.  $\chi(g_i) = e(n[i]/d)$  for all Conrey or SNF generators  $g_i$  (as usual, we use SNF generators if  $n$  is a `t_VEC` and the Conrey generators otherwise). Return a `t_VECSMALL`  $v$  such that  $v[i] = -1$  if  $(i, N) > 1$  else  $\chi(i) = e(v[i]/d)$ ,  $1 \leq i \leq N$ .

## 13.6 Hecke characters.

The functions in this section are specific to Hecke characters. The argument `gc` is a `gchar` structure as returned by `gcharinit(bnf, mod)`, and the character `chi` is a `t_COL` of components on the SNF generators of `gc`.

GEN `eulerf_gchar`(GEN `an`, GEN `p`, long `prec`) `an` being the first component of a Hecke L-function `Ldata` (as output by `lfungchar`) and  $p$  a prime number, return the Euler factor at  $p$ .

GEN `gchari_lfun`(GEN `gc`, GEN `chi`, GEN `w`) `chi` being a `t_VEC` describing a Hecke character encoded on the internal basis `gc[1]`, return the `Ldata` structure corresponding to the Hecke L-function associated to `chi`.

int `is_gchar_group`(GEN `gc`) return 1 if `gc` is a valid `gchar` structure and 0 otherwise.

GEN `lfungchar`(GEN `gc`, GEN `chi`) return the `Ldata` structure corresponding to the Hecke L-function associated to `chi`.

GEN `vecan_gchar`(GEN `an`, long `n`, long `prec`) `an` being the first component of a Hecke L-function `Ldata` (as output by `lfungchar`), return a `t_VEC` of length  $n$  containing the first  $n$  Dirichlet coefficients of this L-function, computed to absolute precision `prec`.

## 13.7 Central simple algebras.

### 13.7.1 Initialization.

Low-level routines underlying `alginit`; argument `rnf` (resp. `nf`) must be true *rnf* (resp. *nf*) structure.

GEN `alg_csa_table`(GEN `nf`, GEN `mt`, long `v`, long `maxord`) algebra defined by a multiplication table.

GEN `alg_cyclic`(GEN `rnf`, GEN `aut`, GEN `b`, long `maxord`) cyclic algebra  $(L/K, \sigma, b)$ .

GEN `alg_hasse`(GEN `nf`, long `d`, GEN `hi`, GEN `hf`, long `v`, long `maxord`) algebra defined by local Hasse invariants.

GEN `alg_hilbert`(GEN `nf`, GEN `a`, GEN `b`, long `v`, long `maxord`) quaternion algebra.

GEN `alg_matrix`(GEN `nf`, long `n`, long `v`, GEN `L`, long `maxord`) matrix algebra.

GEN `alg_complete`(GEN `rnf`, GEN `aut`, GEN `hf`, GEN `hi`, long `maxord`) cyclic algebra  $(L/K, \sigma, b)$  with  $b$  computed from the Hasse invariants.

GEN `alg_changeorder`(GEN `alg`, GEN `ord`) return the algebra with the integral basis replaced by `ord` (a `t_MAT` expressing the basis of the new order in terms of the integral basis of `alg`). No checks are performed.



### 13.7.2 Type checks.

`void checkalg(GEN a)` raise an exception if  $a$  was not initialized by `alginit`.

`void checklat(GEN al, GEN lat)` raise an exception if `lat` is not a valid full lattice in the algebra `al`.

`void checkhasse(GEN nf, GEN hi, GEN hf, long n)` raise an exception if  $(hi, hf)$  do not describe valid Hasse invariants of a central simple algebra of degree  $n$  over  $nf$ .

`long alg_type(GEN al)` internal function called by `algtype`: assume `al` was created by `alginit` (thereby saving a call to `checkalg`). Return values are symbolic rather than numeric:

- `al_NULL`: not a valid algebra.
- `al_TABLE`: table algebra output by `altableinit`.
- `al_CSA`: central simple algebra output by `alginit` and represented by a multiplication table over its center.
- `al_CYCLIC`: central simple algebra output by `alginit` and represented by a cyclic algebra.

`long alg_model(GEN al, GEN x)` given an element  $x$  in algebra  $al$ , check for inconsistencies (raise a type error) and return the representation model used for  $x$ :

- `al_ALGEBRAIC`: `basistoalg` form, algebraic representation.
- `al_BASIS`: `algtobasis` form, column vector on the integral basis.
- `al_MATRIX`: matrix with coefficients in an algebra.
- `al_TRIVIAL`: trivial algebra of degree 1; can be understood as both basis or algebraic form (since  $e_1 = 1$ ).

### 13.7.3 Shallow accessors.

All these routines assume their argument was initialized by `alginit` and provide minor speedups compared to the GP equivalent. The routines returning a `GEN` are shallow.

`long alg_get_absdim(GEN al)` low-level version of `algabsdim`.

`long alg_get_dim(GEN al)` low-level version of `algdim`.

`long alg_get_degree(GEN al)` low-level version of `algdegree`.

`GEN alg_get_aut(GEN al)` low-level version of `algaut`.

`GEN alg_get_auts(GEN al)`, given a cyclic algebra  $al = (L/K, \sigma, b)$  of degree  $n$ , returns the vector of  $\sigma^i$ ,  $1 \leq i < n$ .

`GEN alg_get_b(GEN al)` low-level version of `algb`.

`GEN alg_get_basis(GEN al)` low-level version of `algbasis`.

`GEN alg_get_center(GEN al)` low-level version of `algcenter`.

`GEN alg_get_char(GEN al)` low-level version of `algchar`.

`GEN alg_get_hasse_f(GEN al)` low-level version of `alghassef`.

`GEN alg_get_hasse_i(GEN al)` low-level version of `alghassei`.

GEN `alg_get_invbasis`(GEN `al`) low-level version of `alginvbasis`.  
 GEN `alg_get_multable`(GEN `al`) low-level version of `algmultable`.  
 GEN `alg_get_relmultable`(GEN `al`) low-level version of `algrelmultable`.  
 GEN `alg_get_splittingfield`(GEN `al`) low-level version of `algsplittingfield`.  
 GEN `alg_get_abssplitting`(GEN `al`) returns the absolute *nf* structure attached to the *nf* returned by `algsplittingfield`.  
 GEN `alg_get_splitpol`(GEN `al`) returns the relative polynomial defining the *nf* returned by `algsplittingfield`.  
 GEN `alg_get_splittingdata`(GEN `al`) low-level version of `algsplittingdata`.  
 GEN `alg_get_splittingbasis`(GEN `al`) the matrix *Lbas* from `algsplittingdata`.  
 GEN `alg_get_splittingbasisinv`(GEN `al`) the matrix *Lbasinv* from `algsplittingdata`.  
 GEN `alg_get_tracebasis`(GEN `al`) returns the traces of the basis elements; used by `algtrace`.  
 GEN `alglat_get_primbasis`(GEN `lat`) from the description of `lat` as  $\lambda L$  with  $L \subset \mathcal{O}_0$  and  $\lambda \in \mathbf{Q}$ , returns a basis of *L*.  
 GEN `alglat_get_scalar`(GEN `lat`) from the description of `lat` as  $\lambda L$  with  $L \subset \mathcal{O}_0$  and  $\lambda \in \mathbf{Q}$ , returns  $\lambda$ .

#### 13.7.4 Other low-level functions.

GEN `conjclasses_algcenter`(GEN `cc`, GEN `p`) low-level function underlying `alggroupcenter`, where `cc` is the output of `groupelts_to_conjclasses`, and *p* is either NULL or a prime number. Not stack clean.  
 GEN `algsimpledec_ss`(GEN `al`, long `maps`) assuming that `al` is semisimple, returns the second component of `algsimpledec(al,maps)`.

## Chapter 14:

### Elliptic curves and arithmetic geometry

This chapter is quite short, but is added as a placeholder, since we expect the library to expand in that direction.

#### 14.1 Elliptic curves.

Elliptic curves are represented in the Weierstrass model

$$(E) : y^2z + a_1xyz + a_3yz = x^3 + a_2x^2z + a_4xz^2 + a_6z^3,$$

by the 5-tuple  $[a_1, a_2, a_3, a_4, a_6]$ . Points in the projective plane are represented as follows: the point at infinity  $(0 : 1 : 0)$  is coded as `[0]`, a finite point  $(x : y : 1)$  outside the projective line at infinity  $z = 0$  is coded as  $[x, y]$ . Note that other points at infinity than  $(0 : 1 : 0)$  cannot be represented; this is harmless, since they do not belong to any of the elliptic curves  $E$  above.

*Points on the curve* are just projective points as described above, they are not tied to a curve in any way: the same point may be used in conjunction with different curves, provided it satisfies their equations (if it does not, the result is usually undefined). In particular, the point at infinity belongs to all elliptic curves.

As with `factor` for polynomial factorization, the 5-tuple  $[a_1, a_2, a_3, a_4, a_6]$  implicitly defines a base ring over which the curve is defined. Point coordinates must be operation-compatible with this base ring (`gadd`, `gmul`, `gdiv` involving them should not give errors).

##### 14.1.1 Types of elliptic curves.

We call a 5-tuple as above an `ell5`; most functions require an `ell` structure, as returned by `ellinit`, which contains additional data (usually dynamically computed as needed), depending on the base field.

`GEN ellinit(GEN E, GEN D, long prec)`, returns an `ell` structure, attached to the elliptic curve  $E$ : either an `ell5`, a pair  $[a_4, a_6]$  or a `t_STR` in Cremona's notation, e.g. "11a1". The optional  $D$  (NULL to omit) describes the domain over which the curve is defined.

##### 14.1.2 Type checking.

`void checkell(GEN e)` raise an error unless  $e$  is an `ell`.

`int checkell_i(GEN e)` return 1 if  $e$  is an `ell` and 0 otherwise.

`void checkell5(GEN e)` raise an error unless  $e$  is an `ell` or an `ell5`.

`void checkellpt(GEN z)` raise an error unless  $z$  is a point (either finite or at infinity).

`long ell_get_type(GEN e)` returns the domain type over which the curve is defined, one of

`t_ELL_Q` the field of rational numbers;

`t_ELL_NF` a number field;

`t_ELL_Qp` the field of  $p$ -adic numbers, for some prime  $p$ ;

`t_ELL_Fp` a prime finite field, base field elements are represented as  $\mathbb{F}_p$ , i.e. a `t_INT` reduced modulo  $p$ ;

`t_ELL_Fq` a nonprime finite field (a prime finite field can also be represented by this subtype, but this is inefficient), base field elements are represented as `t_FFELT`;

`t_ELL_Rg` none of the above.

`void checkell_Fq(GEN e)` checks whether  $e$  is an `ell`, defined over a finite field (either prime or nonprime). Otherwise the function raises a `pari_err_TYPE` exception.

`void checkell_Q(GEN e)` checks whether  $e$  is an `ell`, defined over  $\mathbb{Q}$ . Otherwise the function raises a `pari_err_TYPE` exception.

`void checkell_Qp(GEN e)` checks whether  $e$  is an `ell`, defined over some  $\mathbb{Q}_p$ . Otherwise the function raises a `pari_err_TYPE` exception.

`void checkellisog(GEN v)` raise an error unless  $v$  is an isogeny, from `ellisogeny`.

### 14.1.3 Extracting info from an `ell` structure.

These functions expect an `ell` argument. If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

#### 14.1.3.1 All domains.

`GEN ell_get_a1(GEN e)`

`GEN ell_get_a2(GEN e)`

`GEN ell_get_a3(GEN e)`

`GEN ell_get_a4(GEN e)`

`GEN ell_get_a6(GEN e)`

`GEN ell_get_b2(GEN e)`

`GEN ell_get_b4(GEN e)`

`GEN ell_get_b6(GEN e)`

`GEN ell_get_b8(GEN e)`

`GEN ell_get_c4(GEN e)`

`GEN ell_get_c6(GEN e)`

`GEN ell_get_disc(GEN e)`

`GEN ell_get_j(GEN e)`

### 14.1.3.2 Curves over $\mathbf{Q}$ .

`GEN ellQ_get_N(GEN e)` returns the curve conductor

`void ellQ_get_Nfa(GEN e, GEN *N, GEN *faN)` sets  $N$  to the conductor and  $faN$  to its factorization

`int ell_is_integral(GEN e)` return 1 if  $e$  is given by an integral model, and 0 otherwise.

`long ellQ_get_CM(GEN e)` if  $e$  has CM by a principal imaginary quadratic order, return its discriminant. Else return 0.

`long ellap_CM_fast(GEN e, ulong p, long CM)` assuming that  $p$  does not divide the discriminant of  $E$  (in particular,  $E$  has good reduction at  $p$ ), and that  $CM$  is as given by `ellQ_get_CM`, return the trace of Frobenius for  $E/\mathbf{F}_p$ . This is meant to quickly compute lots of  $a_p$ , esp. when  $e$  has CM by a principal quadratic order.

`long ellrootno_global(GEN e)` returns the global root number  $c \in \{-1, 1\}$ .

`GEN ellheightoo(GEN E, GEN P, long prec)` given  $P = [x, y]$  an affine point on  $E$ , return

$$\lambda_{\infty}(P) + \frac{1}{12} \log |\text{disc} E| = \frac{1}{2} \text{real}(z\eta(z)) - \log |\sigma(E, z)| \in \mathbf{R},$$

where  $\lambda_{\infty}(P)$  is the canonical local height at infinity and  $z$  is `ellpointtoz`( $E, P$ ). This is computed using Mestre's (quadratically convergent) AGM algorithm.

`long ellorder_Q(GEN E, GEN P)` return the order of  $P \in E(\mathbf{Q})$ , using the impossible value 0 for a point of infinite order. Ultimately called by the generic `ellorder` function.

`GEN point_to_a4a6(GEN E, GEN P, GEN p, GEN *a4)` given  $E/\mathbf{Q}$ ,  $p \neq 2, 3$  not dividing the discriminant of  $E$  and  $P \in E(\mathbf{Q})$  outside the kernel of reduction, return the image of  $P$  on the short Weierstrass model  $y^2 = x^3 + a_4x + a_6$  isomorphic to the reduction  $E_p$  of  $E$  at  $p$ . Also set `a4` to the  $a_4$  coefficient in the above model. This function allows quick computations modulo varying primes  $p$ , avoiding the overhead of `ellinit`( $E, p$ ), followed by a change of coordinates. It produces data suitable for `FpE` routines.

`GEN point_to_a4a6_Fl(GEN E, GEN P, ulong p, ulong *pa4)` as `point_to_a4a6`, returning a `Fle`.

`GEN elldatagenerators(GEN E)` returns generators for  $E(\mathbf{Q})$  extracted from Cremona's table.

`GEN ellanal_globalred(GEN e, GEN *v)` takes an *ell* over  $\mathbf{Q}$  and returns a global minimal model  $E$  (in `ellinit` form, over  $\mathbf{Q}$ ) for  $e$  suitable for analytic computations related to the curve  $L$  series: it contains `ellglobalred` data, as well as global and local root numbers. If `v` is not `NULL`, set `*v` to the needed change of variable: `NULL` if  $e$  was already the standard minimal model, such that  $E = \text{ellchangecurve}(e, v)$  otherwise. Compared to the direct use of `ellchangecurve` followed by `ellrootno`, this function avoids converting unneeded dynamic data and avoids potential memory leaks (the changed curve would have had to be deleted using `obj_free`). The original curve  $e$  is updated as well with the same information.

`GEN ellanal_globalred_all(GEN e, GEN *v, GEN *N, GEN *tam)` as `ellanal_globalred`; further set `*N` to the curve conductor and `*tam` to the product of the local Tamagawa numbers, including the factor at infinity (multiply by the number of connected components of  $e(\mathbf{R})$ ).

`GEN ellintegralmodel(GEN e, GEN *pv)` return an integral model for  $e$  (in `ellinit` form, over  $\mathbf{Q}$ ). Set  $v = \text{NULL}$  (already integral, we returned  $e$  itself), else to the variable change  $[u, 0, 0, 0]$  making  $e$  integral. We have  $u = 1/t$ ,  $t > 1$ .

GEN `ellintegralmodel_i`(GEN `e`, GEN `*pv`) shallow version of `ellintegralmodel`.

GEN `ellQtwist_bsdperiod`(GEN `E`, long `s`) let  $E$  be a rational elliptic curve given by a minimal model,  $\Lambda_E$  its period lattice, and  $s \in \{-1, 1\}$ . Let  $\Omega_E^\pm$  be the canonical periods in  $\sqrt{\pm 1}\mathbf{R}^+$  generating  $\Lambda_E \cap \sqrt{\pm 1}\mathbf{R}$ . Return  $\Omega_E^+$  if  $s = 1$  and  $\Omega_E^-$  if  $s = -1$ .

GEN `elltors_psylo`(GEN `e`, ulong `p`) as `elltors`, but return the  $p$ -Sylow subgroup of the torsion group.

GEN `elleulerf`(GEN `E`, GEN `p`) returns the Euler factor at  $p$  of the  $L$ -function associated to the curve  $E$  defined over a number field.

### Deprecated routines.

GEN `elltors0`(GEN `e`, long `flag`) this function is deprecated; use `elltors`

#### 14.1.3.3 Curves over a number field $nf$ .

Let  $K$  be the number field over which  $E$  is defined, given by a *nf* or *bnf* structure.

GEN `ellnf_get_nf`(GEN `E`) returns the underlying *nf*.

GEN `ellnf_get_bnf`(GEN `x`) returns NULL if  $K$  does not contain a *bnf* structure, else return the *bnf*.

GEN `ellnf_vecarea`(GEN `E`) returns the vector of the period lattices areas of all the complex embeddings of  $E$  in the same order as `E.nf.roots`.

GEN `ellnf_veceta`(GEN `E`) returns the vector of the quasi-periods of all the complex embeddings of  $E$  in the same order as `E.nf.roots`.

GEN `ellnf_vecomega`(GEN `E`) returns the vector of the periods of all the complex embeddings of  $E$  in the same order as `E.nf.roots`.

#### 14.1.3.4 Curves over $\mathbf{Q}_p$ .

GEN `ellQp_get_p`(GEN `E`) returns  $p$

long `ellQp_get_prec`(GEN `E`) returns the default  $p$ -adic accuracy to which we must compute approximate results attached to  $E$ .

GEN `ellQp_get_zero`(GEN `x`) returns  $O(p^n)$ , where  $n$  is the default  $p$ -adic accuracy as above.

The following functions are only defined when  $E$  has multiplicative reduction (Tate curves):

GEN `ellQp_Tate_uniformization`(GEN `E`, long `prec`) returns a `t_VEC` containing  $u^2, u, q, [a, b]$ , at  $p$ -adic precision `prec`.

GEN `ellQp_u`(GEN `E`, long `prec`) returns  $u$ .

GEN `ellQp_u2`(GEN `E`, long `prec`) returns  $u^2$ .

GEN `ellQp_q`(GEN `E`, long `prec`) returns the Tate period  $q$ .

GEN `ellQp_ab`(GEN `E`, long `prec`) returns  $[a, b]$ .

GEN `ellQp_AGM`(GEN `E`, long `prec`) returns  $[a, b, R, v]$ , where  $v$  is an integer,  $a, b, R$  are vectors describing the sequence of 2-isogenous curves  $E_i : y^2 = x(x + A_i)(x + A_i - B_i)$ ,  $i \geq 1$  converging to the singular curve  $E_\infty : y^2 = x^2(x + M)$ . We have  $a[i] = A[i]p^v$ ,  $b[i] = B[i]p^v$ ,  $R[i] = A_i - B_i$ . These are used in `ellpointtoz` and `ellztopoint`.

GEN `ellQp_L`(GEN `E`, long `prec`) returns the  $\mathcal{L}$ -invariant  $L$ .

GEN `ellQp_root`(GEN `E`, long `prec`) returns  $e_1$ .

#### 14.1.3.5 Curves over a finite field $\mathbf{F}_q$ .

`GEN ellff_get_p(GEN E)` returns the characteristic

`GEN ellff_get_field(GEN E)` returns  $p$  if  $\mathbf{F}_q$  is a prime field, and a `t_FFELT` belonging to  $\mathbf{F}_q$  otherwise.

`GEN ellff_get_card(GEN E)` returns  $\#E(\mathbf{F}_q)$

`GEN ellff_get_gens(GEN E)` returns a minimal set of generators for  $E(\mathbf{F}_q)$ .

`GEN ellff_get_group(GEN E)` returns `ellgroup(E)`.

`GEN ellff_get_m(GEN E)` returns the `t_INT`  $m$  as needed by the `gen_ellgroup` function (the order of the pairing required to verify a generating set).

`GEN ellff_get_o(GEN E)` returns  $[d, \text{factord}]$ , where  $d$  is the exponent of  $E(\mathbf{F}_q)$ .

`GEN ellff_get_D(GEN E)` returns the elementary divisors for  $E(\mathbf{F}_q)$  in a form suitable for `gen_ellgens`: either  $[d_1]$  or  $[d_1, d_2]$ , where  $d_1$  is in `ellff_get_o` format.

$[d, \text{factord}]$ , where  $d$  is the exponent of  $E(\mathbf{F}_q)$ .

`GEN ellff_get_a4a6(GEN E)` returns a canonical “short model” for  $E$ , and the corresponding change of variable  $[u, r, s, t]$ . For  $p \neq 2, 3$ , this is  $[A_4, A_6, [u, r, s, t]]$ , corresponding to  $y^2 = x^3 + A_4x + A_6$ , where  $A_4 = -27c_4$ ,  $A_6 = -54c_6$ ,  $[u, r, s, t] = [6, 3b_2, 3a_1, 108a_3]$ .

- If  $p = 3$  and the curve is ordinary ( $b_2 \neq 0$ ), this is  $[[b_2], A_6, [1, v, -a_1, -a_3]]$ , corresponding to

$$y^2 = x^3 + b_2x^2 + A_6,$$

where  $v = b_4/b_2$ ,  $A_6 = b_6 - v(b_4 + v^2)$ .

- If  $p = 3$  and the curve is supersingular ( $b_2 = 0$ ), this is  $[-b_4, b_6, [1, 0, -a_1, -a_3]]$ , corresponding to

$$y^2 = x^3 + 2b_4x + b_6.$$

- If  $p = 2$  and the curve is ordinary ( $a_1 \neq 0$ ), return  $[A_2, A_6, [a_1^{-1}, da_1^{-2}, 0, (a_4 + d^2)a_1^{-1}]]$ , corresponding to

$$y^2 + xy = x^3 + A_2x^2 + A_6,$$

where  $d = a_3/a_1$ ,  $a_1^2A_2 = (a_2 + d)$  and

$$a_1^6A_6 = d^3 + a_2d^2 + a_4d + a_6 + (a_4^2 + d^4)a_1^{-2}.$$

- If  $p = 2$  and the curve is supersingular ( $a_1 = 0$ ,  $a_3 \neq 0$ ), return  $[[a_3, A_4, 1/a_3], A_6, [1, a_2, 0, 0]]$ , corresponding to

$$y^2 + a_3y = x^3 + A_4x + A_6,$$

where  $A_4 = a_2^2 + a_4$ ,  $A_6 = a_2a_4 + a_6$ . The value  $1/a_3$  is included in the vector since it is frequently needed in computations.

#### 14.1.3.6 Curves over $\mathbf{C}$ . (This includes curves over $\mathbf{Q}$ !)

`long ellR_get_prec(GEN E)` return the default accuracy to which we must compute approximate results attached to  $E$ .

`GEN ellR_ab(GEN E, long prec)` return  $[a, b]$

`GEN ellR_omega(GEN x, long prec)` return periods  $[\omega_1, \omega_2]$ .

`GEN ellR_eta(GEN E, long prec)` return quasi-periods  $[\eta_1, \eta_2]$ .

`GEN ellR_area(GEN x, long prec)` return the area  $(\Im(\omega_1 \overline{\omega_2}))$ .

`GEN ellR_roots(GEN E, long prec)` return  $[e_1, e_2, e_3]$ . If  $E$  is defined over  $\mathbf{R}$ , then  $e_1$  is real. If furthermore  $\text{disc}E > 0$ , then  $e_1 > e_2 > e_3$ .

`long ellR_get_sign(GEN E)` if  $E$  is defined over  $\mathbf{R}$  returns the signe of its discriminant, otherwise return 0.

#### 14.1.4 Points.

`int ell_is_inf(GEN z)` tests whether the point  $z$  is the point at infinity.

`GEN ellinf()` returns the point at infinity  $[0]$ .

#### 14.1.5 Change of variables.

`GEN ellchangeinvert(GEN w)` given a change of variables  $w = [u, r, s, t]$ , returns the inverse change of variables  $w'$ , such that if  $E' = \text{ellchangecurve}(E, w)$ , then  $E = \text{ellchangecurve}(E', w')$ .

#### 14.1.6 Generic helper functions.

The naming scheme assumes an affine equation  $F(x, y) = f(x) - (y^2 + h(x)y) = 0$  in standard Weierstrass form:  $f = x^3 + a_2x^2 + a_4x + a_6$ ,  $h = a_1x + a_3$ . Unless mentionned otherwise, these routine assume that all arguments are compatible with generic functions of `gadd` or `gmul` type. In particular they do not handle elements in number field in `nfalgtobasis` format.

`GEN ellbasechar(GEN E)` returns the characteristic of the base ring over which  $E$  is defined.

`GEN ec_bmodel(GEN E)` returns the polynomial  $4x^3 + b_2x^2 + 2b_4x + b_6$ .

`GEN ec_phi2(GEN E)` returns the polynomial  $x^4 - b_4x^2 - 2b_6 * X - b_8$ .

`GEN ec_f_evalx(GEN E, GEN x)` returns  $f(x)$ .

`GEN ec_h_evalx(GEN E, GEN x)` returns  $h(x)$ .

`GEN ec_dFdx_evalQ(GEN E, GEN Q)` returns  $3x^2 + 2a_2x + a_4 - a_1y$ , where  $Q = [x, y]$ .

`GEN ec_dFdy_evalQ(GEN E, GEN Q)` returns  $-(2y + a_1x + a_3)$ , where  $Q = [x, y]$ .

`GEN ec_dmFdy_evalQ(GEN e, GEN Q)` returns  $2y + a_1x + a_3$ , where  $Q = [x, y]$ .

`GEN ec_2divpol_evalx(GEN E, GEN x)` returns  $4x^3 + b_2x^2 + 2b_4x + b_6$ . This function supports inputs in `nfalgtobasis` format.

`GEN ec_half_deriv_2divpol_evalx(GEN E, GEN x)` returns  $6x^2 + b_2x + b_4$ .

`GEN ec_3divpol_evalx(GEN E, GEN x)` returns  $3x^4 + b_2x^2 + 3b_4x^2 + 3b_6x + b_8$ .



### 14.1.7 Functions to handle elliptic curves over finite fields.

#### 14.1.7.1 Tolerant routines.

`GEN ellap(GEN E, GEN p)` given a prime number  $p$  and an elliptic curve defined over  $\mathbf{Q}$  or  $\mathbf{Q}_p$  (assumed integral and minimal at  $p$ ), computes the trace of Frobenius  $a_p = p + 1 - \#E(\mathbf{F}_p)$ . If  $E$  is defined over a nonprime finite field  $\mathbf{F}_q$ , ignore  $p$  and return  $q + 1 - \#E(\mathbf{F}_q)$ . When  $p$  is implied ( $E$  defined over  $\mathbf{Q}_p$  or a finite field),  $p$  can be omitted (set to `NULL`).

**14.1.7.2 Curves defined a nonprime finite field.** In this subsection, we assume that `ell_get_type( $E$ )` is `t_ELL_Fq`. (As noted above, a curve defined over  $\mathbf{Z}/p\mathbf{Z}$  can be represented as a `t_ELL_Fq`.)

`GEN FF_elltwist(GEN E)` returns the coefficients  $[a_1, a_2, a_3, a_4, a_6]$  of the quadratic twist of  $E$ .

`GEN FF_ellmul(GEN E, GEN P, GEN n)` returns  $[n]P$  where  $n$  is an integer and  $P$  is a point on the curve  $E$ .

`GEN FF_ellrandom(GEN E)` returns a random point in  $E(\mathbf{F}_q)$ . This function never returns the point at infinity, unless this is the only point on the curve.

`GEN FF_ellorder(GEN E, GEN P, GEN o)` returns the order of the point  $P$ , where  $o$  is a multiple of the order of  $P$ , or its factorization.

`GEN FF_ellcard(GEN E)` returns  $\#E(\mathbf{F}_q)$ .

`GEN FF_ellcard_SEA(GEN E, long s)` This function returns  $\#E(\mathbf{F}_q)$ , using the Schoof-Elkies-Atkin algorithm. Assume  $p \neq 2, 3$ . The parameter  $s$  has the same meaning as in `Fp_ellcard_SEA`.

`GEN FF_ellgens(GEN E)` returns the generators of the group  $E(\mathbf{F}_q)$ .

`GEN FF_elllog(GEN E, GEN P, GEN G, GEN o)` Let  $G$  be a point of order  $o$ , return  $e$  such that  $[e]P = G$ . If  $e$  does not exists, the result is undefined.

`GEN FF_ellgroup(GEN E, GEN *pm)` returns the structure of the Abelian group  $E(\mathbf{F}_q)$  and set `*pm` to  $m$  (see `gen_ellgens`).

`GEN FF_ellweilpairing(GEN E, GEN P, GEN Q, GEN m)` returns the Weil pairing of the points of  $m$ -torsion  $P$  and  $Q$ .

`GEN FF_elltatepairing(GEN E, GEN P, GEN Q, GEN m)` returns the Tate pairing of  $P$  and  $Q$ , where  $[m]P = 0$ .

## 14.2 Arithmetic on elliptic curve over a finite field in simple form.

The functions in this section no longer operate on elliptic curve structures, as seen up to now. They are used to implement those higher-level functions without using cached information and thus require suitable explicitly enumerated data.

### 14.2.1 Helper functions.

`GEN elltrace_extension(GEN t, long n, GEN q)` Let  $E$  some elliptic curve over  $\mathbf{F}_q$  such that the trace of the Frobenius is  $t$ , returns the trace of the Frobenius over  $\mathbf{F}_q^n$ .

### 14.2.2 Elliptic curves over $\mathbf{F}_p$ , $p > 3$ .

Let  $p$  a prime number and  $E$  the elliptic curve given by the equation  $E: y^2 = x^3 + a_4x + a_6$ , with  $a_4$  and  $a_6$  in  $\mathbf{F}_p$ . A **FpE** is a point of  $E(\mathbf{F}_p)$ . Since an affine point and  $a_4$  determine an unique  $a_6$ , most functions do not take  $a_6$  as an argument. A **FpE** is either the point at infinity (**ellinf()**) or a **FpV** with two components. The parameters  $a_4$  and  $a_6$  are given as **t\_INTs** when required.

**GEN Fp\_ellj**(**GEN a4**, **GEN a6**, **GEN p**) returns the  $j$ -invariant of the curve  $E$ .

**int Fp\_elljissupersingular**(**GEN j**, **GEN p**) returns 1 if  $j$  is the  $j$ -invariant of a supersingular curve over  $\mathbf{F}_p$ , 0 otherwise.

**GEN Fp\_ellcard**(**GEN a4**, **GEN a6**, **GEN p**) returns the cardinality of the group  $E(\mathbf{F}_p)$ .

**GEN Fp\_ellcard\_SEA**(**GEN a4**, **GEN a6**, **GEN p**, **long s**) This function returns  $\#E(\mathbf{F}_p)$ , using the Schoof-Elkies-Atkin algorithm. If the **seadata** package is installed, the function will be faster.

The extra flag **s**, if set to a nonzero value, causes the computation to return **gen\_0** (an impossible cardinality) if one of the small primes  $\ell$  divides the curve order but does not divide  $s$ . For cryptographic applications, where one is usually interested in curves of prime order, setting  $s = 1$  efficiently weeds out most uninteresting curves; if curves of order a power of 2 times a prime are acceptable, set  $s = 2$ . If moreover **s** is negative, similar checks are performed for the twist of the curve.

**GEN Fp\_ffellcard**(**GEN a4**, **GEN a6**, **GEN q**, **long n**, **GEN p**) returns the cardinality of the group  $E(\mathbf{F}_q)$  where  $q = p^n$ .

**GEN Fp\_ellgroup**(**GEN a4**, **GEN a6**, **GEN N**, **GEN p**, **GEN \*pm**) returns the group structure  $D$  of the group  $E(\mathbf{F}_p)$ , which is assumed to be of order  $N$  and set **\*pm** to  $m$ .

**GEN Fp\_ellgens**(**GEN a4**, **GEN a6**, **GEN ch**, **GEN D**, **GEN m**, **GEN p**) returns generators of the group  $E(\mathbf{F}_p)$  with the base change **ch** (see **FpE.changepoint**), where  $D$  and  $m$  are as returned by **Fp\_ellgroup**.

**GEN Fp\_elldivpol**(**GEN a4**, **GEN a6**, **long n**, **GEN p**) returns the  $n$ -division polynomial of the elliptic curve  $E$ .

**void Fp\_elltwist**(**GEN a4**, **GEN a6**, **GEN p**, **GEN \*pA4**, **GEN \*pA6**) sets **\*pA4** and **\*pA6** to the corresponding parameters for the quadratic twist of  $E$ .

### 14.2.3 FpE.

**GEN FpE\_add**(**GEN P**, **GEN Q**, **GEN a4**, **GEN p**) returns the sum  $P + Q$  in the group  $E(\mathbf{F}_p)$ , where  $E$  is defined by  $E: y^2 = x^3 + a_4x + a_6$ , for any value of  $a_6$  compatible with the points given.

**GEN FpE\_sub**(**GEN P**, **GEN Q**, **GEN a4**, **GEN p**) returns  $P - Q$ .

**GEN FpE\_dbl**(**GEN P**, **GEN a4**, **GEN p**) returns  $2.P$ .

**GEN FpE\_neg**(**GEN P**, **GEN p**) returns  $-P$ .

**GEN FpE\_mul**(**GEN P**, **GEN n**, **GEN a4**, **GEN p**) return  $n.P$ .

**GEN FpE\_changepoint**(**GEN P**, **GEN m**, **GEN a4**, **GEN p**) returns the image  $Q$  of the point  $P$  on the curve  $E: y^2 = x^3 + a_4x + a_6$  by the coordinate change  $m$  (which is a **FpV**).

**GEN FpE\_changepointinv**(**GEN P**, **GEN m**, **GEN a4**, **GEN p**) returns the image  $Q$  on the curve  $E: y^2 = x^3 + a_4x + a_6$  of the point  $P$  by the inverse of the coordinate change  $m$  (which is a **FpV**).

GEN random\_FpE(GEN a4, GEN a6, GEN p) returns a random point on  $E(\mathbf{F}_p)$ , where  $E$  is defined by  $E : y^2 = x^3 + a_4x + a_6$ .

GEN FpE\_order(GEN P, GEN o, GEN a4, GEN p) returns the order of  $P$  in the group  $E(\mathbf{F}_p)$ , where  $o$  is a multiple of the order of  $P$ , or its factorization.

GEN FpE\_log(GEN P, GEN G, GEN o, GEN a4, GEN p) Let  $G$  be a point of order  $o$ , return  $e$  such that  $e.P = G$ . If  $e$  does not exist, the result is currently undefined.

GEN FpE\_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN p) returns the Tate pairing of the point of  $m$ -torsion  $P$  and the point  $Q$ .

GEN FpE\_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN p) returns the Weil pairing of the points of  $m$ -torsion  $P$  and  $Q$ .

GEN FpE\_to\_mod(GEN P, GEN p) returns  $P$  as a vector of `t_INTMODs`.

GEN RgE\_to\_FpE(GEN P, GEN p) returns the FpE obtained by applying `Rg_to_Fp` coefficientwise.

**14.2.4 Fle.** Let  $p$  be a prime `ulong`, and  $E$  the elliptic curve given by the equation  $E : y^2 = x^3 + a_4x + a_6$ , where  $a_4$  and  $a_6$  are `ulong`. A `Fle` is either the point at infinity (`ellinf()`), or a `Flv` with two components  $[x, y]$ .

`long Fl_elltrace(ulong a4, ulong a6, ulong p)` returns the trace  $t$  of the Frobenius of  $E(\mathbf{F}_p)$ . The cardinality of  $E(\mathbf{F}_p)$  is thus  $p + 1 - t$ , which might not fit in an `ulong`.

`long Fl_elltrace_CM(long CM, ulong a4, ulong a6, ulong p)` as `Fl_elltrace`. If  $CM$  is 0, use the standard algorithm; otherwise assume the curve has CM by a principal imaginary quadratic order of discriminant  $CM$  and use a faster algorithm. Useful when the curve is the reduction of  $E/\mathbf{Q}$ , which has CM by a principal order, and we need the trace of Frobenius for many distinct  $p$ , see `ellQ_get_CM`.

`ulong Fl_elldisc(ulong a4, ulong a6, ulong p)` returns the discriminant of the curve  $E$ .

`ulong Fl_elldisc_pre(ulong a4, ulong a6, ulong p, ulong pi)` returns the discriminant of the curve  $E$ , assuming  $pi$  is the pseudoinverse of  $p$ .

`ulong Fl_ellj(ulong a4, ulong a6, ulong p)` returns the  $j$ -invariant of the curve  $E$ .

`ulong Fl_ellj_pre(ulong a4, ulong a6, ulong p, ulong pi)` returns the  $j$ -invariant of the curve  $E$ , assuming  $pi$  is the pseudoinverse of  $p$ .

`void Fl_ellj_to_a4a6(ulong j, ulong p, ulong *pa4, ulong *pa6)` sets  $*pa4$  to  $a_4$  and  $*pa6$  to  $a_6$  where  $a_4$  and  $a_6$  define a fixed elliptic curve with  $j$ -invariant  $j$ .

`void Fl_elltwist(ulong a4, ulong a6, ulong p, ulong *pA4, ulong *pA6)` set  $*pA4$  to  $A_4$  and  $*pA6$  to  $A_6$  where  $A_4$  and  $A_6$  define the twist of  $E$ .

`void Fl_elltwist_disc(ulong a4, ulong a6, ulong D, ulong p, ulong *pA4, ulong *pA6)` sets  $*pA4$  to  $A_4$  and  $*pA6$  to  $A_6$  where  $A_4$  and  $A_6$  define the twist of  $E$  by the discriminant  $D$ .

GEN Fl\_ellptors(ulong l, ulong N, ulong a4, ulong a6, ulong p) return a basis of the  $l$ -torsion subgroup of  $E$ .

GEN Fle\_add(GEN P, GEN Q, ulong a4, ulong p)

GEN Fledbl(GEN P, ulong a4, ulong p)

GEN Fle\_sub(GEN P, GEN Q, ulong a4, ulong p)

```

GEN Fle_mul(GEN P, GEN n, ulong a4, ulong p)
GEN Fle_mulu(GEN P, ulong n, ulong a4, ulong p)
GEN Fle_order(GEN P, GEN o, ulong a4, ulong p)
GEN Fle_log(GEN P, GEN G, GEN o, ulong a4, ulong p)
GEN Fle_tatepairing(GEN P, GEN Q, ulong m, ulong a4, ulong p)
GEN Fle_weilpairing(GEN P, GEN Q, ulong m, ulong a4, ulong p)
GEN random_Fle(ulong a4, ulong a6, ulong p)
GEN random_Fle_pre(ulong a4, ulong a6, ulong p, ulong pi)
GEN Fle_changepoint(GEN x, GEN ch, ulong p), ch is assumed to give the change of coordinates
[u, r, s, t] as a t_VECSMALL.
GEN Fle_changepointinv(GEN x, GEN ch, ulong p), as Fle_changepoint

```

#### 14.2.5 FpJ.

Let  $p > 3$  be a prime  $t\_INT$ , and  $E$  the elliptic curve given by the equation  $E : y^2 = x^3 + a_4 \times x + a_6$ , where  $a_4$  and  $a_6$  are  $t\_INT$ . A  $FpJ$  is a  $FpV$  with three components  $[x, y, z]$ , representing the affine point  $[x/z^2, y/z^3]$  in Jacobian coordinates, the point at infinity being represented by  $[1, 1, 0]$ . The following must holds:  $y^2 = x^3 + a_4 x z^4 + a_6 z^6$ . For all nonzero  $u$ , the points  $[u^2 x, u^3 y, u z]$  and  $[x, y, z]$  are representing the same affine point.

```

GEN FpJ_add(GEN P, GEN Q, GEN a4, GEN p)
GEN FpJ_dbl(GEN P, GEN a4, GEN p)
GEN FpJ_mul(GEN P, GEN n, GEN a4, GEN p);
GEN FpJ_neg(GEN P, GEN p) return  $-P$ .
GEN FpJ_to_FpE(GEN P, GEN p) return the corresponding FpE.
GEN FpE_to_FpJ(GEN P) return the corresponding FpJ.

```

#### 14.2.6 Flj.

Let  $p > 3$  be a prime. Below,  $pi$  is assumed to be the pseudoinverse of  $p$  (see `get_Fl_red`).

```

GEN Fle_to_Flj(GEN P) convert a Fle to an equivalent Flj.
GEN Flj_to_Fle(GEN P, ulong p) convert a Flj to the equivalent Fle.
GEN Flj_to_Fle_pre(GEN P, ulong p, ulong pi) convert a Flj to the equivalent Fle.
GEN Flj_add_pre(GEN P, GEN Q, ulong a4, ulong p, ulong pi)
GEN Flj_dbl_pre(GEN P, ulong a4, ulong p, ulong pi)
GEN Flj_neg(GEN P, ulong p) return  $-P$ .
GEN Flj_mulu_pre(GEN P, ulong n, ulong a4, ulong p, ulong pi)
GEN random_Flj_pre(ulong a4, ulong a6, ulong p, ulong pi)
GEN Flj_changepointinv_pre(GEN P, GEN ch, ulong p, ulong pi) where ch is the Flv
[u, r, s, t].
GEN FljV_factorback_pre(GEN P, GEN L, ulong p, ulong pi)

```

**14.2.7 Elliptic curves over  $\mathbf{F}_{2^n}$ .** Let  $T$  be an irreducible  $\mathbf{F}_2[x]$  and  $E$  the elliptic curve given by either the equation  $E : y^2 + x * y = x^3 + a_2 x^2 + a_6$ , where  $a_2, a_6$  are  $\mathbf{F}_2[x]$  in  $\mathbf{F}_2[X]/(T)$  (ordinary case) or  $E : y^2 + a_3 * y = x^3 + a_4 x + a_6$ , where  $a_3, a_4, a_6$  are  $\mathbf{F}_2[x]$  in  $\mathbf{F}_2[X]/(T)$  (supersingular case).

A  $\mathbf{F}_2[x]E$  is a point of  $E(\mathbf{F}_2[X]/(T))$ . In the supersingular case, the parameter  $a_2$  is actually the  $\mathbf{t\_VEC} [a_3, a_4, a_3^{-1}]$ .

`GEN F2xq_ellcard(GEN a2, GEN a6, GEN T)` Return the order of the group  $E(\mathbf{F}_2[X]/(T))$ .

`GEN F2xq_ellgroup(GEN a2, GEN a6, GEN N, GEN T, GEN *pm)` Return the group structure  $D$  of the group  $E(\mathbf{F}_2[X]/(T))$ , which is assumed to be of order  $N$  and set  $*pm$  to  $m$ .

`GEN F2xq_ellgens(GEN a2, GEN a6, GEN ch, GEN D, GEN m, GEN T)` Returns generators of the group  $E(\mathbf{F}_2[X]/(T))$  with the base change  $ch$  (see `F2xqE_changepoint`), where  $D$  and  $m$  are as returned by `F2xq_ellgroup`.

`void F2xq_ellt twist(GEN a4, GEN a6, GEN T, GEN *a4t, GEN *a6t)` sets  $*a4t$  and  $*a6t$  to the parameters of the quadratic twist of  $E$ .

#### 14.2.8 $\mathbf{F}_2[x]E$ .

`GEN F2xqE_changepoint(GEN P, GEN m, GEN a2, GEN T)` returns the image  $Q$  of the point  $P$  on the curve  $E : y^2 + x * y = x^3 + a_2 x^2 + a_6$  by the coordinate change  $m$  (which is a  $\mathbf{F}_2[x]V$ ).

`GEN F2xqE_changepointinv(GEN P, GEN m, GEN a2, GEN T)` returns the image  $Q$  on the curve  $E : y^2 = x^3 + a_4 x + a_6$  of the point  $P$  by the inverse of the coordinate change  $m$  (which is a  $\mathbf{F}_2[x]V$ ).

`GEN F2xqE_add(GEN P, GEN Q, GEN a2, GEN T)`

`GEN F2xqE_sub(GEN P, GEN Q, GEN a2, GEN T)`

`GEN F2xqE_dbl(GEN P, GEN a2, GEN T)`

`GEN F2xqE_neg(GEN P, GEN a2, GEN T)`

`GEN F2xqE_mul(GEN P, GEN n, GEN a2, GEN T)`

`GEN random_F2xqE(GEN a2, GEN a6, GEN T)`

`GEN F2xqE_order(GEN P, GEN o, GEN a2, GEN T)` returns the order of  $P$  in the group  $E(\mathbf{F}_2[X]/(T))$ , where  $o$  is a multiple of the order of  $P$ , or its factorization.

`GEN F2xqE_log(GEN P, GEN G, GEN o, GEN a2, GEN T)` Let  $G$  be a point of order  $o$ , return  $e$  such that  $e.P = G$ . If  $e$  does not exist, the result is currently undefined.

`GEN F2xqE_tatepairing(GEN P, GEN Q, GEN m, GEN a2, GEN T)` returns the Tate pairing of the point of  $m$ -torsion  $P$  and the point  $Q$ .

`GEN F2xqE_weilpairing(GEN P, GEN Q, GEN m, GEN a2, GEN T)` returns the Weil pairing of the points of  $m$ -torsion  $P$  and  $Q$ .

`GEN RgE_to_F2xqE(GEN P, GEN T)` returns the  $\mathbf{F}_2[x]E$  obtained by applying `RgE_to_F2xqE` coefficient-wise.

**14.2.9 Elliptic curves over  $\mathbf{F}_q$ , small characteristic  $p > 2$ .** Let  $p > 2$  be a prime `ulong`,  $T$  an irreducible `Flx` mod  $p$ , and  $E$  the elliptic curve given by the equation  $E : y^2 = x^3 + a_4x + a_6$ , where  $a_4$  and  $a_6$  are `Flx` in  $\mathbf{F}_p[X]/(T)$ . A `FlxqE` is a point of  $E(\mathbf{F}_p[X]/(T))$ .

In the special case  $p = 3$ , ordinary elliptic curves ( $j(E) \neq 0$ ) cannot be represented as above, but admit a model  $E : y^2 = x^3 + a_2x^2 + a_6$  with  $a_2$  and  $a_6$  being `Flx` in  $\mathbf{F}_3[X]/(T)$ . In that case, the parameter `a2` is actually stored as a `t_VEC`,  $[a_2]$ , to avoid ambiguities.

`GEN Flxq_ellj(GEN a4, GEN a6, GEN T, ulong p)` returns the  $j$ -invariant of the curve  $E$ .

`void Flxq_ellj_to_a4a6(GEN j, GEN T, ulong p, GEN *pa4, GEN *pa6)` sets `*pa4` to  $a_4$  and `*pa6` to  $a_6$  where  $a_4$  and  $a_6$  define a fixed elliptic curve with  $j$ -invariant  $j$ .

`GEN Flxq_ellcard(GEN a4, GEN a6, GEN T, ulong p)` returns the order of  $E(\mathbf{F}_p[X]/(T))$ .

`GEN Flxq_ellgroup(GEN a4, GEN a6, GEN N, GEN T, ulong p, GEN *pm)` returns the group structure  $D$  of the group  $E(\mathbf{F}_p[X]/(T))$ , which is assumed to be of order  $N$  and sets `*pm` to  $m$ .

`GEN Flxq_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, ulong p)` returns generators of the group  $E(\mathbf{F}_p[X]/(T))$  with the base change `ch` (see `FlxqE_changepoint`), where  $D$  and  $m$  are as returned by `Flxq_ellgroup`.

`void Flxq_elltwist(GEN a4, GEN a6, GEN T, ulong p, GEN *pA4, GEN *pA6)` sets `*pA4` and `*pA6` to the corresponding parameters for the quadratic twist of  $E$ .

#### 14.2.10 `FlxqE`.

Let  $p > 2$  be a prime number.

`GEN FlxqE_changepoint(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image  $Q$  of the point  $P$  on the curve  $E : y^2 = x^3 + a_4x + a_6$  by the coordinate change  $m$  (which is a `FlxqV`).

`GEN FlxqE_changepointinv(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image  $Q$  on the curve  $E : y^2 = x^3 + a_4x + a_6$  of the point  $P$  by the inverse of the coordinate change  $m$  (which is a `FlxqV`).

`GEN FlxqE_add(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_sub(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_dbl(GEN P, GEN a4, GEN T, ulong p)`

`GEN FlxqE_neg(GEN P, GEN T, ulong p)`

`GEN FlxqE_mul(GEN P, GEN n, GEN a4, GEN T, ulong p)`

`GEN random_FlxqE(GEN a4, GEN a6, GEN T, ulong p)`

`GEN FlxqE_order(GEN P, GEN o, GEN a4, GEN T, ulong p)` returns the order of  $P$  in the group  $E(\mathbf{F}_p[X]/(T))$ , where  $o$  is a multiple of the order of  $P$ , or its factorization.

`GEN FlxqE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, ulong p)` Let  $G$  be a point of order  $o$ , return  $e$  such that  $e.P = G$ . If  $e$  does not exist, the result is currently undefined.

`GEN FlxqE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p)` returns the Tate pairing of the point of  $m$ -torsion  $P$  and the point  $Q$ .

`GEN FlxqE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p)` returns the Weil pairing of the points of  $m$ -torsion  $P$  and  $Q$ .

GEN FlxqE\_weilpairing\_pre(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p, ulong pi)  
, where  $pi$  is a pseudoinverse of  $p$ , or 0 in which case we assume SMALL\_ULONG( $p$ ).

GEN RgE\_to\_FlxqE(GEN P, GEN T, ulong p) returns the FlxqE obtained by applying Rg\_to\_Flxq coefficientwise.

#### 14.2.11 Elliptic curves over $\mathbf{F}_q$ , large characteristic .

Let  $p > 3$  be a prime number,  $T$  an irreducible polynomial mod  $p$ , and  $E$  the elliptic curve given by the equation  $E: y^2 = x^3 + a_4x + a_6$  with  $a_4$  and  $a_6$  in  $\mathbf{F}_p[X]/(T)$ . A FpXQE is a point of  $E(\mathbf{F}_p[X]/(T))$ .

GEN FpXQ\_ellj(GEN a4, GEN a6, GEN T, GEN p) returns the  $j$ -invariant of the curve  $E$ .

int FpXQ\_elljissupersingular(GEN j, GEN T, GEN p) returns 1 if  $j$  is the  $j$ -invariant of a supersingular curve over  $\mathbf{F}_p[X]/(T)$ , 0 otherwise.

GEN FpXQ\_ellcard(GEN a4, GEN a6, GEN T, GEN p) returns the order of  $E(\mathbf{F}_p[X]/(T))$ .

GEN Fq\_ellcard\_SEA(GEN a4, GEN a6, GEN q, GEN T, GEN p, long s) This function returns  $\#E(\mathbf{F}_p[X]/(T))$ , using the Schoof-Elkies-Atkin algorithm. Assume  $p \neq 2, 3$ , and  $q$  is the cardinality of  $\mathbf{F}_p[X]/(T)$ . The parameter  $s$  has the same meaning as in Fp\_ellcard\_SEA. If the seadata package is installed, the function will be faster.

GEN FpXQ\_ellgroup(GEN a4, GEN a6, GEN N, GEN T, GEN p, GEN \*pm) Return the group structure  $D$  of the group  $E(\mathbf{F}_p[X]/(T))$ , which is assumed to be of order  $N$  and set  $*pm$  to  $m$ .

GEN FpXQ\_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, GEN p) Returns generators of the group  $E(\mathbf{F}_p[X]/(T))$  with the base change  $ch$  (see FpXQE\_changepoint), where  $D$  and  $m$  are as returned by FpXQ\_ellgroup.

GEN FpXQ\_elldivpol(GEN a4, GEN a6, long n, GEN T, GEN p) returns the  $n$ -division polynomial of the elliptic curve  $E$ .

GEN Fq\_elldivpolmod(GEN a4, GEN a6, long n, GEN h, GEN T, GEN p) returns the  $n$ -division polynomial of the elliptic curve  $E$  modulo the polynomial  $h$ .

void FpXQ\_elltwist(GEN a4, GEN a6, GEN T, GEN p, GEN \*pA4, GEN \*pA6) sets  $*pA4$  and  $*pA6$  to the corresponding parameters for the quadratic twist of  $E$ .

#### 14.2.12 FpXQE.

GEN FpXQE\_changepoint(GEN P, GEN m, GEN a4, GEN T, GEN p) returns the image  $Q$  of the point  $P$  on the curve  $E: y^2 = x^3 + a_4x + a_6$  by the coordinate change  $m$  (which is a FpXQV).

GEN FpXQE\_changepointinv(GEN P, GEN m, GEN a4, GEN T, GEN p) returns the image  $Q$  on the curve  $E: y^2 = x^3 + a_4x + a_6$  of the point  $P$  by the inverse of the coordinate change  $m$  (which is a FpXQV).

GEN FpXQE\_add(GEN P, GEN Q, GEN a4, GEN T, GEN p)

GEN FpXQE\_sub(GEN P, GEN Q, GEN a4, GEN T, GEN p)

GEN FpXQE\_dbl(GEN P, GEN a4, GEN T, GEN p)

GEN FpXQE\_neg(GEN P, GEN T, GEN p)

GEN FpXQE\_mul(GEN P, GEN n, GEN a4, GEN T, GEN p)

`GEN random_FpXQE(GEN a4, GEN a6, GEN T, GEN p)`

`GEN FpXQE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, GEN p)` Let  $G$  be a point of order  $o$ , return  $e$  such that  $e.P = G$ . If  $e$  does not exist, the result is currently undefined.

`GEN FpXQE_order(GEN P, GEN o, GEN a4, GEN T, GEN p)` returns the order of  $P$  in the group  $E(\mathbf{F}_p[X]/(T))$ , where  $o$  is a multiple of the order of  $P$ , or its factorization.

`GEN FpXQE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, GEN p)` returns the Tate pairing of the point of  $m$ -torsion  $P$  and the point  $Q$ .

`GEN FpXQE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, GEN p)` returns the Weil pairing of the points of  $m$ -torsion  $P$  and  $Q$ .

`GEN RgE_to_FpXQE(GEN P, GEN T, GEN p)` returns the `FpXQE` obtained by applying `RgE_to_FpXQ` coefficientwise.

### 14.3 Functions related to modular polynomials.

Variants of `polmodular`, returning the modular polynomial of prime level  $L$  for the invariant coded by `inv` (0:  $j$ , 1: Weber- $f$ , see `polclass` for the full list).

`GEN polmodular_ZXX(long L, long inv, long vx, long vy)` returns a bivariate polynomial in variables  $vx$  and  $vy$ .

`GEN polmodular_ZM(long L, long inv)` returns a matrix of (integral) coefficients.

`GEN Fp_polmodular_evalx(long L, long inv, GEN J, GEN p, long v, int derivs)` returns the modular polynomial evaluated at  $J$  modulo the prime  $p$  in the variable  $v$  (if `derivs` is nonzero, returns a vector containing the modular polynomial and its first and second derivatives, all evaluated at  $J$  modulo  $p$ ).

#### 14.3.1 Functions related to modular invariants.

`void check_modinv(long inv)` report an error if `inv` is not a valid code for a modular invariant.

`int modinv_good_disc(long inv, long D)` test whether the invariant `inv` is defined for the discriminant  $D$ .

`int modinv_good_prime(long inv, long p)` test whether the invariant `inv` is defined for the prime  $p$ .

`long modinv_height_factor(long inv)` return the height factor of the modular invariant `inv` with respect to the  $j$ -invariant. This is an integer  $n$  such that the  $j$ -invariant is asymptotically of the order of the  $n$ -th power of the invariant `inv`.

`long modinv_is_Weber(long inv)` test whether the invariant `inv` is a power of Weber  $f$ .

`long modinv_is_double_eta(long inv)` test whether the invariant `inv` is a double  $\eta$  quotient.

`long disc_best_modinv(long D)` the integer  $D$  being a negative discriminant, return the modular invariant compatible with  $D$  with the highest height factor.

`GEN Fp_modinv_to_j(GEN x, long inv, GEN p)` Let  $\Phi$  the modular equation between  $j$  and the modular invariant `inv`, return  $y$  such that  $\Phi(y, x) = 0 \pmod{p}$ .



## 14.4 Other curves.

The following functions deal with hyperelliptic curves in weighted projective space  $\mathbf{P}_{(1,d,1)}$ , with coordinates  $(x, y, z)$  and a model of the form  $y^2 = T(x, z)$ , where  $T$  is homogeneous of degree  $2d$ , and squarefree. Thus the curve is nonsingular of genus  $d - 1$ .

`long hyperell_locally_soluble(GEN T, GEN p)` assumes that  $T \in \mathbf{Z}[X]$  is integral. Returns 1 if the curve is locally soluble over  $\mathbf{Q}_p$ , 0 otherwise.

`long nf_hyperell_locally_soluble(GEN nf, GEN T, GEN pr)` let  $K$  be a number field, attached to `nf`, `pr` a *prid* attached to some maximal ideal  $\mathfrak{p}$ ; assumes that  $T \in \mathbf{Z}_K[X]$  is integral. Returns 1 if the curve is locally soluble over  $K_{\mathfrak{p}}$ . The argument `nf` is a true *nf* structure.



## Chapter 15:

### *L*-functions

#### 15.1 Accessors.

```
long is_linit(GEN data)
GEN ldata_get_an(GEN ldata)
GEN ldata_get_dual(GEN ldata)
long ldata_isreal(GEN ldata)
GEN ldata_get_gammavec(GEN ldata)
long ldata_get_degree(GEN ldata)
GEN ldata_get_k(GEN ldata)
GEN ldata_get_k1(GEN ldata)
GEN ldata_get_conductor(GEN ldata)
GEN ldata_get_rootno(GEN ldata)
GEN ldata_get_residue(GEN ldata)
long ldata_get_type(GEN ldata)
long linit_get_type(GEN linit)
GEN linit_get_ldata(GEN linit)
GEN linit_get_tech(GEN linit)
GEN lfun_get_domain(GEN tech)
GEN lfun_get_dom(GEN tech)
long lfun_get_bitprec(GEN tech)
GEN lfun_get_factgammavec(GEN tech)
GEN lfun_get_step(GEN tech)
GEN lfun_get_pol(GEN tech)
GEN lfun_get_Residue(GEN tech)
GEN lfun_get_k2(GEN tech)
GEN lfun_get_w2(GEN tech)
GEN lfun_get_expot(GEN tech)
long lfun_get_bitprec(GEN tech)
```

```

GEN lfunprod_get_fact(GEN tech)
GEN theta_get_an(GEN tdata)
GEN theta_get_K(GEN tdata)
GEN theta_get_R(GEN tdata)
long theta_get_bitprec(GEN tdata)
long theta_get_m(GEN tdata)
GEN theta_get_tdom(GEN tdata)
GEN theta_get_isqrtN(GEN tdata)

```

## 15.2 Conversions and constructors.

GEN lfunmisc\_to\_ldata(GEN obj) converts obj to Ldata format. Exception if obj cannot be converted.

GEN lfunmisc\_to\_ldata\_shallow(GEN obj) as lfunmisc\_to\_ldata, shallow result. Exception if obj cannot be converted.

GEN lfunmisc\_to\_ldata\_shallow\_i(GEN obj) as lfunmisc\_to\_ldata\_shallow, returning NULL on failure.

```
GEN lfunrtopoles(GEN r)
```

```
int sdomain_isincl(double k, GEN dom, GEN dom0)
```

GEN ldata\_vecan(GEN ldata, long N, long prec) return the vector of coefficients of indices 1 to  $N$  to precision `prec`. The output is allowed to be a `t_VECSMALL` when the coefficients are known to be all integral and fit into a `long`; for instance the Dirichlet  $L$  function of a real character or the  $L$ -function of a rational elliptic curve.

GEN ldata\_newprec(GEN ldata, long prec) return a shallow copy of `ldata` with fields accurate to precision `prec`.

long etaquotient(GEN \*peta, GEN \*pN, GEN \*pk, GEN \*pCHI, long \*pv, long \*psd, long \*pcusp) Let `eta` be the integer matrix factorization supposedly attached to an  $\eta$ -quotient  $f(z) = \prod_i \eta(n_i z)^{e_i}$ . Assuming `*peta` is initially set to `eta`, this function returns 0 if there is a type error or this does not define a function on some  $X_0(N)$ . Else it returns 1 and sets

- `*peta` to a normalized factorization (as would be returned by `factor`),
- `*pN` to the level  $N$  of  $f$ ,
- `*pk` to the modular weight  $k$  of  $f$ ,
- `*pCHI` to the Nebentypus of  $f$  (quadratic character) as an integer,
- `*pv` to the valuation at infinity  $v_q(f)$ ,
- `*psd` to 1 if and only if  $f$  is self-dual,
- `*pcusp` to 1 if  $f$  is cuspidal, else to 0 if  $f$  holomorphic at all cusps, else to  $-1$ .

The last three arguments `pCHI`, `pv` and `pcusp` can be set to `NULL`, in which case the relevant information is not computed, which saves time.

### 15.3 Variants of GP functions.

GEN lfun(GEN ldata, GEN s, long bitprec)  
GEN lfuninit(GEN ldata, GEN dom, long der, long bitprec)  
GEN lfuninit\_make(long t, GEN ldata, GEN tech, GEN domain)  
GEN lfunlambda(GEN ldata, GEN s, long bitprec)  
GEN lfunquadneg(long D, long k) for  $L(\chi_D, k)$ ,  $D$  fundamental discriminant and  $k \geq 0$ .  
long lfunthetacost(GEN ldata, GEN tdom, long m, long bitprec): lfunthetacost0 when the first argument is known to be an Ldata.  
GEN lfunthetacheckinit(GEN data, GEN tinf, long m, long bitprec)  
GEN lfunrootno(GEN data, long bitprec)  
GEN lfunzetakinit(GEN nf, GEN dom, long der, long bitprec) where nf is a true  $nf$  structure.  
GEN lfunellmfpeters(GEN E, long bitprec)  
GEN ellanalyticrank(GEN E, long prec) DEPRECATED.  
GEN ellL1(GEN E, long prec) DEPRECATED.

### 15.4 Inverse Mellin transforms of Gamma products.

GEN gammamellininv(GEN Vga, GEN s, long m, long bitprec)  
GEN gammamellininvinit(GEN Vga, long m, long bitprec)  
GEN gammamellininvrt(GEN K, GEN s, long bitprec) no GC-clean, but suitable for gerepile-upto.  
int Vgaeasytheta(GEN Vga) return 1 if the inverse Mellin transform is an exponential and 0 otherwise.  
double dbllemma526(double a, double b, double c, long B)  
double dblcoro526(double a, double c, long B)



## Chapter 16: Modular symbols

`void checkms(GEN W)` raise an exception if  $W$  is not an *ms* structure from `msinit`.

`void checkmspadic(GEN W)` raise an exception if  $W$  is not an *mspadic* structure from `mspadicinit`.

`GEN mseval2_ooQ(GEN W, GEN phi, GEN c)` let  $W$  be a `msinit` structure for  $k = 2$ ,  $\phi$  be a modular symbol with integral values and  $c$  be a rational number. Return the integer  $\phi(p)$ , where  $p$  is the path  $\{\infty, c\}$ .

`void mspadic_parse_chi(GEN s, GEN *s1, GEN *s2)` see `mspadicL`; let  $\chi$  be the cyclotomic character from  $\text{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$  to  $\mathbf{Z}_p^*$  and  $\tau$  be the Teichmüller character for  $p > 2$  and the character of order 2 on  $(\mathbf{Z}/4\mathbf{Z})^*$  if  $p = 2$ . Let  $s$  encode the  $p$ -adic character  $\chi^s := \langle \chi \rangle^{s_1} \tau^{s_2}$ ; set `*s1` and `*s2` to the integers  $s_1$  and  $s_2$ .

`GEN mspadic_unit_eigenvalue(GEN ap, long k, GEN p, long n)` let  $p$  be a prime not dividing the trace of Frobenius `ap`, return the unit root of  $x^2 - ap * x + p^{(k-1)}$  to  $p$ -adic accuracy  $p^n$ .

Variants of `mfnumcusps` :

`ulong mfnumcuspsu(ulong n)`

`GEN mfnumcusps_fact(GEN fa)` where `fa` is `factor(n)`.

`ulong mfnumcuspsu_fact(GEN fa)` where `fa` is `factoru(n)`.





## Chapter 17: Modular forms

### 17.1 Implementation of public data structures.

`void checkMF(GEN mf)` raise an exception if the argument is not a modular form space.

`GEN checkMF_i(GEN mf)` return the underlying modular form space if `mf` is either directly a modular form space from `mfinit` or a symbol from `mfsymbol`. Return `NULL` otherwise.

`int checkmf_i(GEN mf)` return 1 if the argument is a modular form and 0 otherwise.

`int checkfarey_i(GEN F)` return 1 if the argument is a Farey symbol (from `mspolygon` or `msfarey`) and 0 otherwise.

#### 17.1.1 Accessors for modular form spaces.

Shallow functions; assume that their argument is a modular form space is created by `mfinit` and checked using `checkMF`.

`GEN MF_get_gN(GEN mf)` return the level  $N$  as a `t_INT`.

`long MF_get_N(GEN mf)` return the level  $N$  as a `long`.

`GEN MF_get_gk(GEN mf)` return the level  $k$  as a `t_INT`.

`long MF_get_k(GEN mf)` return the level  $k$  as a `long`.

`long MF_get_r(GEN mf)` assuming the level is a half-integer, return the integer  $r = k - (1/2)$ .

`GEN MF_get_CHI(GEN mf)` return the nebentypus  $\chi$ , which is a special form of character structure attached to Dirichlet characters (see next section). Its values are given as algebraic numbers: either  $\pm 1$  or `t_POLMOD` in  $t$ .

`long MF_get_space(GEN mf)` returns the space type, corresponding to `mfinit`'s `space` flag. The current list is

`mf_NEW, mf_CUSP, mf_OLD, mf_EISEN, mf_FULL`

`GEN MF_get_basis(GEN mf)` return the  $\mathbf{Q}$ -basis of the space, concatenation of `MF_get_E` and `MF_get_S`, in this order; the forms have coefficients in  $\mathbf{Q}(\chi)$ . Low-level version of `mbasis`.

`long MF_get_dim(GEN mf)` returns the dimension  $d$  of the space. It is the cardinality of `MF_get_basis`.

`GEN MF_get_E(GEN mf)` returns a  $\mathbf{Q}$ -basis for the subspace spanned by Eisenstein series in the space; the forms have coefficients in  $\mathbf{Q}(\chi)$ .

`GEN MF_get_S(GEN mf)` returns a  $\mathbf{Q}$ -basis for the cuspidal subspace in the space; the forms have coefficients in  $\mathbf{Q}(\chi)$ .

GEN `MF_get_fields`(GEN `mf`) returns the vector of polynomials defining each Galois orbit of newforms over  $\mathbf{Q}(\chi)$ . Uses memoization: a first call splits the space and may be costly; subsequent calls return the cached result.

GEN `MF_get_newforms`(GEN `mf`) returns a vector `vF` containing the coordinates of the eigenforms on `MF_get_basis` (`mftobasis` form). Low-level version of `mfeigenbasis`, whose elements are recovered as `mflinear`(`mf`, `gel`(`vF`, `i`)). Uses memoization, sharing the same data as `MF_get_fields`. Note that it is much more efficient to use `mfcoefs`(`mf`,) then multiply by this vector than to compute the coefficients of eigenforms from `mfeigenbasis` individually.

The following accessors are technical,

GEN `MF_get_M`(GEN `mf`) the  $(1+m) \times d$  matrix whose  $j$ -th column contain the coefficients of the  $j$ -th entry in `MF_get_basis`,  $m$  is the optimal “Sturm bound” for the space: the maximum of the  $v_\infty(f)$  over nonzero forms. It has entries in  $\mathbf{Q}(\chi)$ .

GEN `MF_get_Mindex`(GEN `mf`) is a `t_VECSMALL` containing  $d$  row indices, the corresponding rows of  $M$  form an invertible matrix  $M_0$ .

GEN `MF_get_Minv`(GEN `mf`) the inverse of  $M_0$  in a form suitable for fast multiplication.

GEN `MFcusp_get_vMjd`(GEN `mf`) valid only for a full *cuspidal* space. Then the functions in `MF_get_S` are of the form  $B_d T_j Tr_M^{new}$ . This returns the vector of triples (`t_VECSMALL`)  $[M, j, d]$ , in the same order.

GEN `MFnew_get_vj`(GEN `mf`) valid only for a *new* space. Then the functions in `MF_get_S` are of the form  $T_j Tr_N^{new}$ . This returns a `t_VECSMALL` of the Hecke indices  $j$ , in the same order.

### 17.1.2 Accessors for individual modular forms.

GEN `mf_get_gN`(GEN `F`) return the level of  $F$ , which may be a multiple of the conductor, as a `t_INT`

`long mf_get_N`(GEN `F`) return the level as a `long`.

GEN `mf_get_gk`(GEN `F`) return the weight of  $F$  as a `t_INT` or a `t_FRAC` with denominator 2 (half-integral weight).

`long mf_get_k`(GEN `F`) return the weight as a `long`; if the weight is not integral, this raises an exception.

`long mf_get_r`(GEN `F`) assuming  $F$  is a modular form of half-integral weight  $k = (2r+1)/2$ , return  $r = k - (1/2)$ .

GEN `mf_get_CHI`(GEN `F`) return the nebentypus, which is a special form of character structure attached to Dirichlet characters (see next section). Its values are given as algebraic numbers: either  $\pm 1$  or `t_POLMOD` in  $t$ .

GEN `mf_get_field`(GEN `F`) return the polynomial (in variable  $y$ ) defining  $\mathbf{Q}(f)$  over  $\mathbf{Q}(\chi)$ .

GEN `mf_get_NK`(GEN `F`) return the tag attached to  $F$ : a vector containing `gN`, `gk`, `CHI`, `field`. Never use its component directly, use individual accessors as above.

`long mf_get_type`(GEN `F`) returns a symbolic name for the constructor used to create the form, e.g. `t_MF_EISEN` for a general Eisenstein series. A form has a recursive structure represented by a tree: its definition may involve other forms, e.g. the tree attached to  $T_n f$  contains  $f$  as a subtree. Such trees have *leaves*, forms which do not contain a strict subtree, e.g. `t_MF_DELTA` is a leaf, attached to Ramanujan’s  $\Delta$ .

Here is the current list of types; since the names are liable to change, they are not documented at this point. Use `mfdescribe` to visualize their mathematical structure.

```
/*leaves*/
  t_MF_CONST, t_MF_EISEN, t_MF_Ek, t_MF_DELTA, t_MF_ETAQUO, t_MF_ELL,
  t_MF_DIHEDRAL, t_MF_THETA, t_MF_TRACE, t_MF_NEWTRACE,
/*recursive*/
  t_MF_MUL, t_MF_POW, t_MF_DIV, t_MF_BRACKET, t_MF_LINEAR, t_MF_LINEAR_BHN,
  t_MF_SHIFT, t_MF_DERIV, t_MF_DERIVE2, t_MF_TWIST, t_MF_HECKE,
  t_MF_BD,
```

**17.1.3 Nebentypus.** The characters stored in modular forms and modular form spaces have a special structure. One can recover the parameters of an ordinary Dirichlet character by `G = gel(CHI,1)` (the underlying `znstar`) and `chi = gel(CHI,2)` (the underlying character in `znconreylog` form).

`long mfcharmodulus(GEN CHI)` the modulus of  $\chi$ .

`long mfcharorder(GEN CHI)` the order of  $\chi$ .

`GEN mfcharpol(GEN CHI)` the cyclotomic polynomial  $\Phi_n$  defining  $\mathbf{Q}(\chi)$ , always normalized so that  $n$  is not 2 mod 4.

#### 17.1.4 Miscellaneous functions.

`long mfnewdim(long N, long k, GEN CHI)` dimension of the new part of the cuspidal space.

`long mfcuspdim(long N, long k, GEN CHI)` dimension of the cuspidal space.

`long mfoldddim(long N, long k, GEN CHI)` dimension of the old part of the cuspidal space.

`long mfeisensteindim(long N, long k, GEN CHI)` dimension of the Eisenstein subspace.

`long mffulldim(long N, long k, GEN CHI)` dimension of the full space.

`GEN mfeisensteinspaceinit(GEN NK)`

`GEN mfdiv_val(GEN F, GEN G, long vG)`

`GEN mfembed(GEN E, GEN v)`

`GEN mfmatembed(GEN E, GEN v)`

`GEN mfvecembed(GEN E, GEN v)`

`long mfsturmNgk(long N, GEN k)`

`long mfsturmNk(long N, long k)`

`long mfsturm_mf(GEN mf)`

`long mfiscuspidal(GEN mf, GEN F)`

`GEN mftobasisES(GEN mf, GEN F)`

`GEN mftocol(GEN F, long lim, long d)`

`GEN mfvectomat(GEN vF, long lim, long d)`



## Chapter 18:

### Plots

A `PARI_plot` canvas is a record of dimensions, with the following fields:

```
long width; /* window width */
long height; /* window height */
long hunit; /* length of horizontal 'ticks' */
long vunit; /* length of vertical 'ticks' */
long fwidth; /* font width */
long fheight; /* font height */
void (*draw)(PARI_plot *T, GEN w, GEN x, GEN y);
```

The `draw` method performs the actual drawing of a `t_VECSMALL` `w` (rectwindow indices);  $x$  and  $y$  are `t_VECSMALL`s of the same length and rectwindow  $w[i]$  is drawn with its upper left corner at offset  $(x[i], y[i])$ . No plot engine is available in `libpari` by default, since this would introduce a dependency on extra graphical libraries. See the files `src/graph/plot*` for basic implementations of various plot engines: `plotsvg` is particularly simple (`draw` is a 1-liner).

`void pari_set_plot_engine(void (*T)(PARI_plot *))` installs the graphical engine  $T$  and initializes the graphical subsystem. No routine in this chapter will work without this initialization.

`void pari_kill_plot_engine(void)` closes the graphical subsystem and frees the resources it occupies.

### 18.1 Highlevel functions.

Those functions plot  $f(E, x)$  for  $x \in [a, b]$ , using  $n$  regularly spaced points (by default).

`GEN ploth(void *E, GEN(*f)(void*, GEN), GEN a, GEN b, long flags, long n, long prec)`  
draw physically.

`GEN plotrecth(void *E, GEN(*f)(void*, GEN), long w, GEN a, GEN b, ulong flags, long n, long prec)` draw in rectwindow  $w$ .

## 18.2 Function.

```
void plotbox(long ne, GEN gx2, GEN gy2)
void plotclip(long rect)
void plotcolor(long ne, long color)
void plotcopy(long source, long dest, GEN xoff, GEN yoff, long flag)
GEN plotcursor(long ne)
void plotdraw(GEN list, long flag)
GEN plothrow(GEN listx, GEN listy, long flag)
GEN plotsizes(long flag)
void plotinit(long ne, GEN x, GEN y, long flag)
void plotkill(long ne)
void plotline(long ne, GEN x2, GEN y2)
void plotlines(long ne, GEN listx, GEN listy, long flag)
void plotlinetype(long ne, long t)
void plotmove(long ne, GEN x, GEN y)
void plotpoints(long ne, GEN listx, GEN listy)
void plotpointsize(long ne, GEN size)
void plotpointtype(long ne, long t)
void plotrbox(long ne, GEN x2, GEN y2)
GEN plotrecthrow(long ne, GEN data, long flags)
void plotrline(long ne, GEN x2, GEN y2)
void plotrmove(long ne, GEN x, GEN y)
void plotrpoint(long ne, GEN x, GEN y)
void plotscale(long ne, GEN x1, GEN x2, GEN y1, GEN y2)
void plotstring(long ne, char*x, long dir)
```

**18.2.1 Obsolete functions.** These draw directly to a PostScript file specified by a global variable and should no longer be used. Use `plotexport` and friends instead.

```
void psdraw(GEN list, long flag)
GEN psplothrow(GEN listx, GEN listy, long flag)
GEN psplotth(void *E, GEN(*f)(void*, GEN), GEN a, GEN b, long flags, long n, long
prec) draw to a PostScript file.
```

### 18.3 Dump rectwindows to a PostScript or SVG file.

$w, x, y$  are three `t_VECSMALL`s indicating the rectwindows to dump, at which offsets. If  $T$  is `NULL`, rescale with respect to the installed graphic engine dimensions; else with respect to  $T$ .

`char* rect2ps(GEN w, GEN x, GEN y, PARI_plot *T)`

`char* rect2ps_i(GEN w, GEN x, GEN y, PARI_plot *T, int plotps)` if `plotps` is 0, as above; else private version used to implement the `plotps` graphic engine (do not rescale, rotate to portrait orientation).

`char* rect2svg(GEN w, GEN x, GEN y, PARI_plot *T)`

### 18.4 Technical functions exported for convenience.

`void pari_plot_by_file(const char *env, const char *suf, const char *img)` backend used by the `plotps` and `plotsvg` graphic engines.

`void colorname_to_rgb(const char *s, int *r, int *g, int *b)` convert an X11 colorname to RGB values.

`void color_to_rgb(GEN c, int *r, int *g, int *b)` convert a pari color (`t_VECSMALL` RGB triple or `t_STR` name) to RGB values.

`void long_to_rgb(long c, int *r, int *g, int *b)` split a standard hexadecimal color value `0xfdf5e6` to its rgb components (`0xfd`, `0xf5`, `0xe6`).





## Appendix A:

### A Sample program and Makefile

We assume that you have installed the PARI library and include files as explained in Appendix A or in the installation guide. If you chose differently any of the directory names, change them accordingly in the Makefiles.

If the program example that we have given is in the file `extgcd.c`, then a sample Makefile might look as follows. Note that the actual file `examples/Makefile` is more elaborate and you should have a look at it if you intend to use `install()` on custom made functions.

```
CC = cc
INCDIR = /usr/pkg/include
LIBDIR = /usr/pkg/lib
CFLAGS = -O -I$(INCDIR) -L$(LIBDIR)

all: extgcd

extgcd: extgcd.c
        $(CC) $(CFLAGS) -o extgcd extgcd.c -lpari -lm
```

We then give the listing of the program `examples/extgcd.c` seen in detail in Section 4.10.

```
#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/

/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
    pari_sp av = avma;
    GEN ux = gen_1, vx = gen_0, a = A, b = B;
    if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
    if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
    if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
    while (!gequal0(b))
    {
        GEN r, q = dvmdii(a, b, &r), v = vx;
        vx = subii(ux, mulii(q, vx));
        ux = v; a = b; b = r;
    }
    *U = ux;
    *V = diviixact( subii(a, mulii(A,ux)), B );
    gerepileall(av, 3, &a, U, V); return a;
}

int
```

```

main()
{
    GEN x, y, d, u, v;
    pari_init(1000000,2);
    printf("x = "); x = gp_read_stream(stdin);
    printf("y = "); y = gp_read_stream(stdin);
    d = extgcd(x, y, &u, &v);
    pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
    pari_close();
    return 0;
}

```

## Appendix B:

### PARI and threads

To use PARI in multi-threaded programs, you must configure it using `Configure --enable-tls`. Your system must implement the `_thread` storage class. As a major side effect, this breaks the `libpari` ABI: the resulting library is not compatible with the old one, and `-tls` is appended to the PARI library `soname`. On the other hand, this library is now thread-safe.

PARI provides some functions to set up PARI subthreads. In our model, each concurrent thread needs its own PARI stack. The following scheme is used:

Child thread:

```
void *child_thread(void *arg)
{
    GEN data = pari_thread_start((struct pari_thread*)arg);
    GEN result = ...; /* Compute result from data */
    pari_thread_close();
    return (void*)result;
}
```

Parent thread:

```
pthread_t th;
struct pari_thread pth;
GEN data, result;

pari_thread_alloc(&pth, s, data);
pthread_create(&th, NULL, &child_thread, (void*)&pth); /* start child */
... /* do stuff in parent */
pthread_join(th, (void*)&result); /* wait until child terminates */
result = gcopy(result); /* copy result from thread stack to main stack */
pari_thread_free(&pth); /* ... and clean up */
```

`void pari_thread_valloc(struct pari_thread *pth, size_t s, size_t v, GEN arg)` Allocate a PARI stack of size `s` which can grow to at most `v` (as with `parisize` and `parisizemax`) and associate it, together with the argument `arg`, with the PARI thread data `pth`.

`void pari_thread_alloc(struct pari_thread *pth, size_t s, GEN arg)` As above but the stack cannot grow beyond `s`.

`void pari_thread_free(struct pari_thread *pth)` Free the PARI stack attached to the PARI thread data `pth`. This is called after the child thread terminates, i.e. after `pthread_join` in the parent. Any GEN objects returned by the child in the thread stack need to be saved before running this command.

`void pari_thread_init(void)` Initialize the thread-local PARI data structures. This function is called by `pari_thread_start`.

GEN `pari_thread_start(struct pari_thread *t)` Initialize the thread-local PARI data structures and set up the thread stack using the PARI thread data `pth`. This function returns the thread argument `arg` that was given to `pari_thread_alloc`.

void `pari_thread_close(void)` Free the thread-local PARI data structures, but keeping the thread stack, so that a GEN returned by the thread remains valid.

Under this model, some PARI states are reset in new threads. In particular

- the random number generator is reset to the starting seed;
- the system stack exhaustion checking code, meant to catch infinite recursions, is disabled (use `pari_stackcheck_init()` to reenale it);
- cached real constants (returned by `mppi`, `mpeuler` and `mplog2`) are not shared between threads and will be recomputed as needed;

The following sample program can be compiled using

```
cc thread.c -o thread.o -lpari -lpthread
```

(Add `-I/-L` paths as necessary.)

```
#include <pari/pari.h> /* Include PARI headers */
#include <pthread.h>    /* Include POSIX threads headers */

void *
mydet(void *arg)
{
    GEN F, M;
    /* Set up thread stack and get thread parameter */
    M = pari_thread_start((struct pari_thread*) arg);
    F = QM_det(M);
    /* Free memory used by the thread */
    pari_thread_close();
    return (void*)F;
}

void *
myfactor(void *arg) /* same principle */
{
    GEN F, N;
    N = pari_thread_start((struct pari_thread*) arg);
    F = factor(N);
    pari_thread_close();
    return (void*)F;
}

int
main(void)
{
    long prec = DEFAULTPREC;
    GEN M1,M2, N1,N2, F1,F2, D1,D2;
    pthread_t th1, th2, th3, th4; /* POSIX-thread variables */
    struct pari_thread pth1, pth2, pth3, pth4; /* pari thread variables */
```

```

/* Initialise the main PARI stack and global objects (gen_0, etc.) */
pari_init(32000000,500000);
/* Compute in the main PARI stack */
N1 = addis(int2n(256), 1); /*  $2^{256} + 1$  */
N2 = subis(int2n(193), 1); /*  $2^{193} - 1$  */
M1 = mathilbert(149);
M2 = mathilbert(150);
/* Allocate pari thread structures */
pari_thread_alloc(&pth1,8000000,N1);
pari_thread_alloc(&pth2,8000000,N2);
pari_thread_alloc(&pth3,32000000,M1);
pari_thread_alloc(&pth4,32000000,M2);
/* pthread_create() and pthread_join() are standard POSIX-thread
   * functions to start and get the result of threads. */
pthread_create(&th1,NULL, &myfactor, (void*)&pth1);
pthread_create(&th2,NULL, &myfactor, (void*)&pth2);
pthread_create(&th3,NULL, &mydet, (void*)&pth3);
pthread_create(&th4,NULL, &mydet, (void*)&pth4); /* Start 4 threads */
pthread_join(th1,(void*)&F1);
pthread_join(th2,(void*)&F2);
pthread_join(th3,(void*)&D1);
pthread_join(th4,(void*)&D2); /* Wait for termination, get the results */
pari_printf("F1=%Ps\nF2=%Ps\nlog(D1)=%Ps\nlog(D2)=%Ps\n",
            F1,F2, glog(D1,prec),glog(D2,prec));
pari_thread_free(&pth1);
pari_thread_free(&pth2);
pari_thread_free(&pth3);
pari_thread_free(&pth4); /* clean up */
return 0;
}

```

## Index

*SomeWord* refers to PARI-GP concepts.  
*SomeWord* is a PARI-GP keyword.  
*SomeWord* is a generic index entry.

### A

ABC_to_bnr . . . . .	317	addsi_sign . . . . .	98
abelian_group . . . . .	257	addui . . . . .	97
abgrp_get_cyc . . . . .	290	addui_sign . . . . .	98
abgrp_get_gen . . . . .	290	addumului . . . . .	97
abgrp_get_no . . . . .	290	adduu . . . . .	97
abmap_kernel . . . . .	318	affc_fixlg . . . . .	253
abmap_subgroup_image . . . . .	318	affects_sign . . . . .	64
abscmpii . . . . .	95	affects_sign_safe . . . . .	64
abscmpiu . . . . .	95	affgr . . . . .	89
abscmpr . . . . .	95	affii . . . . .	89
abscmpui . . . . .	95	affir . . . . .	89
absdiviu_rem . . . . .	99	affiz . . . . .	89
absequalii . . . . .	95	affrr . . . . .	89
absequaliu . . . . .	95	affrr_fixlg . . . . .	89, 253
absequalui . . . . .	95	affsi . . . . .	89
absfrac . . . . .	243	affsr . . . . .	89
absfrac_shallow . . . . .	243	affsz . . . . .	89
absi . . . . .	94	affui . . . . .	89
absi_shallow . . . . .	94	affur . . . . .	89
absr . . . . .	94	alarm . . . . .	269
absrnz_equal1 . . . . .	95	algininit . . . . .	337
absrnz_equal2n . . . . .	95	alglat_get_primbasis . . . . .	338
abstorel . . . . .	318	alglat_get_scalar . . . . .	338
absZ_factor . . . . .	173	algsimpledec_ss . . . . .	338
absZ_factor_limit . . . . .	173	algtype . . . . .	337
absZ_factor_limit_strict . . . . .	173	alg_changeorder . . . . .	336
addhelp . . . . .	79	alg_complete . . . . .	336
addii . . . . .	15	alg_csa_table . . . . .	336
addii_sign . . . . .	98	alg_cyclic . . . . .	336
addir . . . . .	15	alg_get_absdim . . . . .	337
addir_sign . . . . .	98	alg_get_abssplitting . . . . .	338
addis . . . . .	15	alg_get_aut . . . . .	337
addiu . . . . .	97	alg_get_auts . . . . .	337
addll . . . . .	83	alg_get_b . . . . .	337
addllx . . . . .	83	alg_get_basis . . . . .	337
addmul . . . . .	83	alg_get_center . . . . .	337
addmulii . . . . .	97	alg_get_char . . . . .	337
addmulii_inplace . . . . .	97	alg_get_degree . . . . .	337
addmuliu . . . . .	97	alg_get_dim . . . . .	337
addmuliu_inplace . . . . .	97	alg_get_hasse_f . . . . .	337
addri . . . . .	15	alg_get_hasse_i . . . . .	337
addr . . . . .	15	alg_get_invbasis . . . . .	337
addr_sign . . . . .	98	alg_get_multable . . . . .	337
		alg_get_relmultable . . . . .	337
		alg_get_splitpol . . . . .	338
		alg_get_splittingbasis . . . . .	338
		alg_get_splittingbasisinv . . . . .	338
		alg_get_splittingdata . . . . .	338
		alg_get_splittingfield . . . . .	337

alg_get_tracebasis	338
alg_hasse	336
alg_hilbert	336
alg_matrix	336
alg_model	337
alg_type	337
assignment	26
atanhui	252
atanhuu	252
avma	17, 26

## B

bb_algebra	214
bb_field	212
bb_group	210
bb_ring	215
bernfrac	254, 255
Bernoulli	254, 255
bernreal	254, 255
bezout	48, 102
bfffo	83
bid_get_arch	294
bid_get_archp	294
bid_get_cyc	294
bid_get_fact	294
bid_get_fact2	294
bid_get_gen	294
bid_get_gen_nocheck	294
bid_get_grp	294
bid_get_ideal	294
bid_get_mod	294
bid_get_no	294
bid_get_sarch	295
bid_get_sprk	294
bid_get_U	295
BIGDEFAULTPREC	16, 66
bigomegau	106
BIL	53
binary quadratic form	34
binary_2k	92
binary_2k_nv	92
binary_zv	92
bincopy_relink	70
binomial	243
binomialuu	243
bin_copy	69
bitprecision0	219
BITS_IN_HALFULONG	66

BITS_IN_LONG	16, 53, 66, 92
bits_to_int	92
bits_to_u	92
bit_accuracy	16, 61
bit_accuracy_mul	61
bit_prec	61
bl_base	74
bl_next	74
bl_num	74
bl_prev	74
bl_refc	74
bnfgwgeneric	318
bnfisprincipal0	296, 312, 315
bnfisunit	302
bnfnewprec	296, 312, 313
bnfnewprec_shallow	296
bnftestprimes	313
bnf_build_cheapfu	295
bnf_build_cycgen	295
bnf_build_matalpha	295
bnf_build_units	295
bnf_compactfu	293
bnf_compactfu_mat	293
bnf_get_clgp	292
bnf_get_cyc	292
bnf_get_fu	292
bnf_get_fu_nocheck	292
bnf_get_gen	292
bnf_get_logfu	292
bnf_get_nf	291
bnf_get_no	292
bnf_get_reg	292
bnf_get_sunits	292
bnf_get_tuN	292
bnf_get_tuU	292
bnf_has_fu	292
bnrchar_primitive	317
bnrchar_primitive_raw	317
bnrclassno	316
bnrconductor	317
bnrconductorofchar	321
bnrconductor_factored	317
bnrconductor_i	321
bnrconductor_raw	317
bnrdisc	317
bnrdisc0	321
bnrinit0	320
bnrisconductor	317
bnrisprincipal	321

bnrnewprec . . . . .	296	centermodii . . . . .	98
bnrnewprec_shallow . . . . .	296	centermod_i . . . . .	234
bnrsurjection . . . . .	317	cgcd . . . . .	102
bnr_char_sanitize . . . . .	317	cgetalloc . . . . .	69
bnr_get_bid . . . . .	293	cgetc . . . . .	24, 59, 68, 88, 253
bnr_get_bnf . . . . .	293	cgetg . . . . .	24, 25, 59, 68
bnr_get_clgp . . . . .	293	cgetg_block . . . . .	74
bnr_get_cyc . . . . .	293, 317	cgetg_copy . . . . .	60
bnr_get_gen . . . . .	293	cgeti . . . . .	24, 59, 68, 88
bnr_get_gen_nocheck . . . . .	293	cgetineg . . . . .	88
bnr_get_mod . . . . .	293	cgetipos . . . . .	88
bnr_get_nf . . . . .	293	cgetp . . . . .	68
bnr_get_no . . . . .	293	cgetr . . . . .	24, 59, 68, 88
bnr_subgroup_check . . . . .	316	cgetr_block . . . . .	74
bnr_subgroup_sanitize . . . . .	316	cgiv . . . . .	18, 70
both_odd . . . . .	84	character string . . . . .	34
boundfact . . . . .	174	<i>character</i> . . . . .	334
BPSW_isprime . . . . .	178	characteristic . . . . .	245
BPSW_psp . . . . .	178	charconj . . . . .	334
brent_kung_optpow . . . . .	214	chardiv . . . . .	334
brute . . . . .	264	charmulo . . . . .	334
buchimag . . . . .	320	chartoGENstr . . . . .	261
Buchray . . . . .	316	char_check . . . . .	334
buchreal . . . . .	320	char_denormalize . . . . .	334

## C

CATCH_ALL . . . . .	47	char_normalize . . . . .	317, 334
cbezout . . . . .	102	char_simplify . . . . .	334
cbrtr . . . . .	251	checkabgrp . . . . .	290
cbrtr_abs . . . . .	252	checkalg . . . . .	336
cb_pari_ask_confirm . . . . .	57, 58	checkbid . . . . .	289
cb_pari_break_loop . . . . .	57	checkbid_i . . . . .	289
cb_pari_err_handle . . . . .	57	checkbnf . . . . .	289
cb_pari_err_recover . . . . .	57	checkbnf_i . . . . .	289
cb_pari_handle_exception . . . . .	57	checkbnr . . . . .	289
cb_pari_init_histfile . . . . .	57	checkbnr_i . . . . .	289
cb_pari_is_interactive . . . . .	57, 59	checkell . . . . .	339
cb_pari_pre_recover . . . . .	57	checkell5 . . . . .	339
cb_pari_quit . . . . .	57	checkellisog . . . . .	340
cb_pari_sigint . . . . .	57	checkellpt . . . . .	339
cb_pari_start_output . . . . .	57	checkell_Fq . . . . .	340
cb_pari_whatnow . . . . .	57	checkell_i . . . . .	339
ceildivuu . . . . .	100	checkell_Q . . . . .	340
ceilr . . . . .	90	checkell_Qp . . . . .	340
ceil_safe . . . . .	91	checkfarey_i . . . . .	361
centerlift . . . . .	219	checkgal . . . . .	289
centerlift0 . . . . .	219	checkgroup . . . . .	256
centermod . . . . .	234	checkgroupelts . . . . .	256
		checkhasse . . . . .	337
		checklat . . . . .	337
		checkMF . . . . .	361



checkMF_i	361	closure_context	283
checkmf_i	361	closure_deriv	282
checkmodpr	290	closure_derivn	282
checkms	359	closure_disassemble	281
checkmspadic	359	closure_err	283
checknf	289	closure_evalbrk	282
checknfelt_mod	290	closure_evalgen	77, 281
checknf_i	289	closure_evalnobrk	281
checkprid	290	closure_evalres	282
checkprid_i	290	closure_evalvoid	77, 281
checkrnf	289	closure_func_err	57
checkrnf_i	289	closure_is_variadic	34
checksqmat	289	closure_trapgen	282
checkznstar_i	289	cmpii	94
check_arith_all	176	cmpir	94
check_arith_non0	176	cmpis	94
check_arith_pos	176	cmpiu	94
check_ecppcert	179	cmpri	95
check_modinv	352	cmprr	95
check_quaddisc	322	cmprrs	95
check_quaddisc_imag	322	cmpsi	95
check_quaddisc_real	322	cmpsr	95
check_ZKmodule	290	cmpss	94
check_ZKmodule_i	290	cmpui	95
chinese1	160	cmpuu	94
chinese1_coprime_Z	160	cmp_Flx	233
chk_gerepileupto	73	cmp_nodata	232
classno	323	cmp_padic	233
classno2	323	cmp_prime_ideal	233
clcm	102	cmp_prime_over_p	233
cleanroots	199, 245	cmp_RgX	233
clean_Z_factor	176	cmp_universal	191, 228, 232
clone	73	colorname_to_rgb	367
clone	14, 27	colors	263, 264
CLONEBIT	66	color_to_rgb	367
closemodinvertible	334	coltrunc_init	60
closure	77	column vector	34
closure	34	col_ei	222
closure_arity	34	compile_str	58, 282
closure_callgen0prec	281	complex number	31
closure_callgen1	77, 281	compo	65
closure_callgen1prec	281	conjclasses_algcenter	338
closure_callgen2	281	conjclasses_repr	257
closure_callgenall	281	conjvec	245, 253
closure_callgenvec	281	conj_i	243
closure_callgenvecdef	281	constant_coeff	32, 65
closure_callgenvecdefprec	281	constbern	254
closure_callgenvecprec	281	constcatalan	254
closure_callvoid1	281	consteuler	254



diviiround . . . . .	91	ec_dmFdy_evalQ . . . . .	344
divisors . . . . .	43	ec_f_evalx . . . . .	344
divisorsu . . . . .	107	ec_half_deriv_2divpol_evalx . . . . .	344
divisorsu_fact . . . . .	107	ec_h_evalx . . . . .	344
divisorsu_fact_factored . . . . .	107	ec_phi2 . . . . .	344
divisorsu_moebius . . . . .	107	effective length . . . . .	29
divis_rem . . . . .	99	ei_multable . . . . .	299
diviuexact . . . . .	98	ellff_get_o . . . . .	343
diviuuexact . . . . .	98	ell . . . . .	339
divll . . . . .	83	ellanalyticrank . . . . .	357
divll_pre . . . . .	85	ellanal_globalred . . . . .	341
divrunextu . . . . .	233	ellanal_globalred_all . . . . .	341
divsBIL . . . . .	66	ellap . . . . .	344
divsi_rem . . . . .	100	ellap_CM_fast . . . . .	340
divss_rem . . . . .	100	ellbasechar . . . . .	344
div_content . . . . .	236	ellchangeinvert . . . . .	344
dvdii . . . . .	98	elldatagenerators . . . . .	341
dvdiiiz . . . . .	98	elleulerf . . . . .	342
dvdis . . . . .	98	ellff_get_a4a6 . . . . .	343
dvdisz . . . . .	98	ellff_get_card . . . . .	343
dvdiu . . . . .	98	ellff_get_D . . . . .	343
dvdiuz . . . . .	98	ellff_get_field . . . . .	343
dvdsi . . . . .	98	ellff_get_gens . . . . .	343
dvdui . . . . .	98	ellff_get_group . . . . .	343
dvmdii . . . . .	99	ellff_get_m . . . . .	343
dvmdiiiz . . . . .	99	ellff_get_o . . . . .	343
dvmdis . . . . .	99	ellff_get_p . . . . .	342
dvmdsBIL . . . . .	66	ellheightoo . . . . .	341
dvmdsi . . . . .	99	ellinf . . . . .	344
dvmdss . . . . .	99	ellinit . . . . .	339
dvmduBIL . . . . .	66	ellintegralmodel . . . . .	341
dynamic array . . . . .	273	ellintegralmodel_i . . . . .	341
d_ACKNOWLEDGE . . . . .	284, 286, 287	ellisogeny . . . . .	340
d_INITRC . . . . .	284, 286	ellL1 . . . . .	357
d_RETURN . . . . .	284, 286, 287	ellnf_get_bnf . . . . .	342
d_SILENT . . . . .	284	ellnf_get_nf . . . . .	342
<b>E</b>			
echo . . . . .	59	ellnf_vecarea . . . . .	342
ecpp . . . . .	179	ellnf_veceta . . . . .	342
ecpp0 . . . . .	179	ellnf_vecomega . . . . .	342
ecppexport . . . . .	179	ellorder . . . . .	341
ecppisvalid . . . . .	179	ellorder_Q . . . . .	341
ec_2divpol_evalx . . . . .	344	ellQp_ab . . . . .	342
ec_3divpol_evalx . . . . .	344	ellQp_AGM . . . . .	342
ec_bmodel . . . . .	344	ellQp_Ei . . . . .	254
ec_dFdx_evalQ . . . . .	344	ellQp_get_p . . . . .	342
ec_dFdy_evalQ . . . . .	344	ellQp_get_prec . . . . .	342
		ellQp_get_zero . . . . .	342
		ellQp_L . . . . .	342
		ellQp_q . . . . .	342



F2m_coeff	118	F2xn_red	154
F2m_copy	119	F2xE_add	349
F2m_deplin	120	F2xE_changepoint	349
F2m_det	120	F2xE_changepointinv	349
F2m_det_sp	120	F2xE_dbl	349
F2m_F2c_gauss	119	F2xE_log	349
F2m_F2c_invimage	119	F2xE_mul	349
F2m_F2c_mul	119	F2xE_neg	349
F2m_flip	119	F2xE_order	349
F2m_gauss	119	F2xE_sub	349
F2m_image	119	F2xE_tatepairing	349
F2m_indexrank	119	F2xE_weilpairing	349
F2m_inv	120	F2xM_deplin	155
F2m_invimage	119	F2xM_det	155
F2m_ker	120	F2xM_F2xC_gauss	155
F2m_ker_sp	120	F2xM_F2xC_invimage	155
F2m_mul	119	F2xM_F2xC_mul	155
F2m_powu	119	F2xM_gauss	155
F2m_rank	119	F2xM_image	155
F2m_row	119	F2xM_indexrank	155
F2m_rowslice	119	F2xM_inv	155
F2m_set	119	F2xM_invimage	155
F2m_suppl	119	F2xM_ker	155
F2m_to_F2Ms	192	F2xM_mul	155
F2m_to_Flm	120	F2xM_rank	155
F2m_to_mod	158	F2xM_suppl	155
F2m_to_ZM	120	F2xXQV_red	157
F2m_transpose	119	F2xXQ_autpow	157
F2v_add_inplace	120	F2xXQ_auttrace	157
F2v_and_inplace	120	F2xXQ_inv	156
F2v_clear	118	F2xXQ_invsafe	156
F2v_coeff	118	F2xXQ_mul	157
F2v_copy	118	F2xXQ_pow	157
F2v_dotproduct	120	F2xXQ_powers	157
F2v_ei	119	F2xXQ_sqr	157
F2v_equal0	118	F2xX_ddf	157
F2v_flip	118	F2xX_degfact	157
F2v_hamming	120	F2xX_disc	157
F2v_negimply_inplace	120	F2xX_div	156
F2v_or_inplace	120	F2xX_divrem	156
F2v_set	118	F2xX_extgcd	156
F2v_slice	118	F2xX_F2xXQV_eval	157
F2v_subset	120	F2xX_F2xXQ_eval	157
F2v_to_F2x	152	F2xX_F2xq_mul	156
F2v_to_Flv	120	F2xX_F2xq_mul_to_monic	156
F2xC_to_FlxC	172	F2xX_factor	157
F2xC_to_ZXC	172	F2xX_factor_squarefree	157
F2xn_div	154	F2xX_gcd	156
F2xn_inv	154	F2xX_get_red	156

F2xqX_halfgcd . . . . .	157	F2xY_degreeex . . . . .	155
F2xqX_invBarrett . . . . .	156	F2xY_F2xqV_evalx . . . . .	156
F2xqX_isplayer . . . . .	157	F2xY_F2xq_evalx . . . . .	156
F2xqX_mul . . . . .	156	F2x_1_add . . . . .	153
F2xqX_normalize . . . . .	156	F2x_add . . . . .	153
F2xqX_powu . . . . .	156	F2x_clear . . . . .	152
F2xqX_red . . . . .	156	F2x_coeff . . . . .	152
F2xqX_rem . . . . .	156	F2x_copy . . . . .	152
F2xqX_resultant . . . . .	157	F2x_ddf . . . . .	154
F2xqX_roots . . . . .	157	F2x_deflate . . . . .	153
F2xqX_sqr . . . . .	156	F2x_degfact . . . . .	153
F2xq_Artin_Schreier . . . . .	154	F2x_degree . . . . .	153
F2xq_autpow . . . . .	154	F2x_deriv . . . . .	153
F2xq_conjvec . . . . .	154	F2x_div . . . . .	153
F2xq_div . . . . .	154	F2x_divrem . . . . .	153
F2xq_ellcard . . . . .	349	F2x_equal . . . . .	153
F2xq_ellgens . . . . .	349	F2x_equal1 . . . . .	153
F2xq_ellgroup . . . . .	349	F2x_eval . . . . .	153
F2xq_elltwist . . . . .	349	F2x_even_odd . . . . .	153
F2xq_inv . . . . .	154	F2x_extgcd . . . . .	153
F2xq_invsafe . . . . .	154	F2x_F2xqV_eval . . . . .	154
F2xq_log . . . . .	154	F2x_F2xq_eval . . . . .	154
F2xq_matrix_pow . . . . .	154	F2x_factor . . . . .	153
F2xq_mul . . . . .	154	F2x_factor_squarefree . . . . .	153
F2xq_order . . . . .	154	F2x_flip . . . . .	152
F2xq_pow . . . . .	154	F2x_Frobenius . . . . .	153
F2xq_powers . . . . .	154	F2x_gcd . . . . .	153
F2xq_powu . . . . .	154	F2x_get_red . . . . .	152
F2xq_pow_init . . . . .	154	F2x_halfgcd . . . . .	153
F2xq_pow_table . . . . .	154	F2x_issquare . . . . .	153
F2xq_sqr . . . . .	154	F2x_is_irred . . . . .	153
F2xq_sqrt . . . . .	154	F2x_matFrobenius . . . . .	153
F2xq_sqrtn . . . . .	154	F2x_mul . . . . .	153
F2xq_sqrt_fast . . . . .	154	F2x_recip . . . . .	153
F2xq_trace . . . . .	154	F2x_rem . . . . .	153
F2xV_to_F2m . . . . .	172	F2x_renormalize . . . . .	153
F2xV_to_FlxV_inplace . . . . .	170	F2x_set . . . . .	152
F2xV_to_ZXV_inplace . . . . .	170	F2x_shift . . . . .	153
F2xXC_to_ZXXC . . . . .	156	F2x_sqr . . . . .	153
F2xXV_to_F2xM . . . . .	155	F2x_sqrt . . . . .	153
F2xX_add . . . . .	155	F2x_Teichmuller . . . . .	154
F2xX_deriv . . . . .	155	F2x_to_F2v . . . . .	172
F2xX_F2x_add . . . . .	155	F2x_to_F2xX . . . . .	152
F2xX_F2x_mul . . . . .	155	F2x_to_Flx . . . . .	152
F2xX_renormalize . . . . .	155	F2x_to_ZX . . . . .	152
F2xX_to_F2xC . . . . .	155	F2x_valrem . . . . .	153
F2xX_to_FlxX . . . . .	155	F3c_to_mod . . . . .	158
F2xX_to_Kronecker . . . . .	155	F3c_to_ZC . . . . .	121
F2xX_to_ZXX . . . . .	155	F3m_coeff . . . . .	121

F3m_copy	121	famat_reduce	300, 302
F3m_ker	121	famat_remove_trivial	302
F3m_ker_sp	121	famat_sqr	301
F3m_mul	121	famat_to_nf	302
F3m_row	121	famat_to_nf_moddivisor	315
F3m_set	121	famat_to_nf_modideal_coprime	315, 316
F3m_to_Flm	121	famat_Z_gcd	301
F3m_to_mod	158	fetch_user_var	36, 75
F3m_to_ZM	121	fetch_var	36, 75
F3m_transpose	121	fetch_var_higher	36
F3v_clear	121	fetch_var_value	36, 75
F3v_coeff	121	FFM_deplin	250
F3v_set	121	FFM_det	250
F3v_to_Flv	121	FFM_FFC_gauss	250
factmod	159	FFM_FFC_invimage	250
factor	339	FFM_FFC_mul	250
factorback	239	FFM_gauss	250
factoredpolred	321	FFM_image	250
factoredpolred2	321	FFM_indexrank	250
factorial_Fl	85	FFM_inv	250
factorial_Fp	105	FFM_invimage	250
factorial_lval	93	FFM_ker	250
factorint	176	FFM_mul	250
factoru	174	FFM_rank	250
factoru_pow	174	FFM_suppl	250
factor_Aurifeuille	174	FFXQ_inv	251
factor_Aurifeuille_prime	174	FFXQ_minpoly	251
factor_pn_1	174	FFXQ_mul	251
factor_pn_1_limit	174	FFXQ_sqr	251
factor_proven	177	FFX_add	249
famat	300	FFX_ddf	249
famat_small_reduce	302	FFX_degfact	249
famatV_factorback	301	FFX_disc	249
famatV_zv_factorback	301	FFX_extgcd	249
famat_div	301	FFX_factor	249
famat_div_shallow	301	FFX_factor_squarefree	249
famat_idealfactor	302	FFX_gcd	249
famat_inv	301	FFX_halfgcd	249
famat_inv_shallow	301	FFX_ispower	249
famat_makecoprime	316	FFX_mul	249
famat_mul	301	FFX_preimage	250
famat_mulpow	301	FFX_preimagerel	250
famat_mulpows_shallow	301	FFX_rem	249
famat_mulpow_shallow	301	FFX_resultant	249
famat_mul_shallow	301	FFX_roots	250
famat_nfvalrem	302	FFX_sqr	249
famat_pow	301	FF_1	247
famat_pows_shallow	301	FF_add	247
famat_pow_shallow	301	FF_charpoly	248

FF_conjvec	248	FF_to_Flxq_i	246
FF_div	248	FF_to_FpXQ	246
FF_ellcard	345	FF_to_FpXQ_i	246
FF_ellcard_SEA	345	FF_trace	248
FF_ellgens	345	FF_var	246
FF_ellgroup	345	FF_zero	247
FF_elllog	345	FF_Z_add	247
FF_ellmul	345	FF_Z_mul	247
FF_ellorder	345	FF_Z_Z_muldiv	248
FF_ellrandom	345	file_is_binary	264
FF_elltatepairing	345	finite field element	31
FF_elltwist	345	fixlg	72, 89
FF_ellweilpairing	345	Flc_Flv_mul	116
FF_equal	247	Flc_FpV_mul	117
FF_equal0	247	Flc_lincomb1_inplace	116
FF_equal1	247	Flc_to_mod	158
FF_equality1	247	Flc_to_ZC	170
FF_f	246	Flc_to_ZC_inplace	170
FF_Frobenius	248	Fle_add	347
FF_gen	246	Fle_changepoint	348
FF_inv	248	Fle_changepointinv	348
FF_ispower	248	Fle_dbl	347
FF_issquare	248	Fle_log	348
FF_issquareall	248	Fle_mul	347
FF_log	248	Fle_mulu	347
FF_map	249	Fle_order	347
FF_minpoly	248	Fle_sub	347
FF_mod	246	Fle_tatepairing	348
FF_mul	247	Fle_to_Flj	348
FF_mul2n	248	Fle_weilpairing	348
FF_neg	248	FljV_factorback_pre	348
FF_neg_i	248	Flj_add_pre	348
FF_norm	248	Flj_changepointinv_pre	348
FF_order	248	Flj_dbl_pre	348
FF_p	246	Flj_mulu_pre	348
ff_parse_Tp	127	Flj_neg	348
FF_pow	248	Flj_to_Fle	348
FF_primroot	248	Flj_to_Fle_pre	348
FF_p_i	246	Flm_add	117
FF_q	246	Flm_adjoint	118
FF_Q_add	247	Flm_center	116
FF_samefield	247	Flm_charpoly	117
FF_sqr	248	Flm_copy	116
FF_sqrt	248	Flm_deplin	118
FF_sqrtn	248	Flm_det	118
FF_sub	247	Flm_det_sp	118
FF_to_F2xq	246	Flm_Flc_gauss	118
FF_to_F2xq_i	246	Flm_Flc_invimage	118
FF_to_Flxq	246	Flm_Flc_mul	116



Flm_Flc_mul_pre . . . . .	116	Flv_Fl_mul_part_inplace . . . . .	116
Flm_Flc_mul_pre_Flx . . . . .	116	Flv_inv . . . . .	117
Flm_Fl_add . . . . .	116	Flv_invVandermonde . . . . .	143
Flm_Fl_mul . . . . .	116	Flv_inv_inplace . . . . .	117
Flm_Fl_mul_inplace . . . . .	116	Flv_inv_pre . . . . .	117
Flm_Fl_mul_pre . . . . .	116	Flv_inv_pre_inplace . . . . .	117
Flm_Fl_sub . . . . .	116	Flv_neg . . . . .	116
Flm_gauss . . . . .	118	Flv_neg_inplace . . . . .	116
Flm_hess . . . . .	118	Flv_polint . . . . .	143
Flm_image . . . . .	118	Flv_prod . . . . .	117
Flm_indexrank . . . . .	118	Flv_prod_pre . . . . .	117
Flm_intersect . . . . .	118	Flv_roots_to_pol . . . . .	143
Flm_intersect_i . . . . .	118	Flv_sub . . . . .	117
Flm_inv . . . . .	118	Flv_sub_inplace . . . . .	117
Flm_invimage . . . . .	118	Flv_sum . . . . .	117
Flm_ker . . . . .	118	Flv_to_F2v . . . . .	120
Flm_ker_sp . . . . .	118	Flv_to_F3v . . . . .	121
Flm_mul . . . . .	117	Flv_to_Flx . . . . .	171
Flm_mul_pre . . . . .	117	Flv_to_ZV . . . . .	170
Flm_neg . . . . .	116	FlxC_eval_powers_pre . . . . .	143
Flm_powers . . . . .	117	FlxC_FlxqV_eval . . . . .	145
Flm_powu . . . . .	117	FlxC_FlxqV_eval_pre . . . . .	145
Flm_rank . . . . .	118	FlxC_Flxq_eval . . . . .	145
Flm_row . . . . .	117	FlxC_Flxq_eval_pre . . . . .	145
Flm_sub . . . . .	117	FlxC_neg . . . . .	143
Flm_suppl . . . . .	118	FlxC_sub . . . . .	143
Flm_to_F2m . . . . .	120	FlxC_to_F2xC . . . . .	172
Flm_to_F3m . . . . .	121	FlxC_to_ZXC . . . . .	170
Flm_to_FlxV . . . . .	171	FlxM_eval_powers_pre . . . . .	143
Flm_to_FlxX . . . . .	171	FlxM_Flx_add_shallow . . . . .	121
Flm_to_mod . . . . .	158	FlxM_neg . . . . .	143
Flm_to_ZM . . . . .	170	FlxM_sub . . . . .	143
Flm_to_ZM_inplace . . . . .	170	FlxM_to_FlxXV . . . . .	171
Flm_transpose . . . . .	118	FlxM_to_ZXM . . . . .	170
floorr . . . . .	90	Flxn_div . . . . .	144
floor_safe . . . . .	91	Flxn_div_pre . . . . .	144
flush . . . . .	262	Flxn_exp . . . . .	144
Flv_add . . . . .	116	Flxn_expint . . . . .	144
Flv_add_inplace . . . . .	116, 309	Flxn_inv . . . . .	144
Flv_center . . . . .	116	Flxn_mul . . . . .	143
Flv_copy . . . . .	116	Flxn_mul_pre . . . . .	143
Flv_dotproduct . . . . .	117	Flxn_red . . . . .	144
Flv_dotproduct_pre . . . . .	117	Flxn_sqr . . . . .	143
Flv_factorback . . . . .	117	Flxn_sqr_pre . . . . .	143
Flv_Flm_polint . . . . .	143	FlxqC_Flxq_mul . . . . .	121
Flv_Fl_div . . . . .	116	FlxqE_add . . . . .	350
Flv_Fl_div_inplace . . . . .	116	FlxqE_changepoint . . . . .	350
Flv_Fl_mul . . . . .	116	FlxqE_changepointinv . . . . .	350
Flv_Fl_mul_inplace . . . . .	116	FlxqE_dbl . . . . .	350

FlxqE_log	350	FlxqXQ_invsafe	150, 157
FlxqE_mul	350	FlxqXQ_invsafe_pre	151
FlxqE_neg	350	FlxqXQ_inv_pre	150
FlxqE_order	350	FlxqXQ_matrix_pow	151
FlxqE_sub	350	FlxqXQ_minpoly	151
FlxqE_tatepairing	350	FlxqXQ_minpoly_pre	151
FlxqE_weilpairing	350	FlxqXQ_mul	150
FlxqE_weilpairing_pre	350	FlxqXQ_mul_pre	150
FlxqM_deplin	121	FlxqXQ_pow	151
FlxqM_det	122	FlxqXQ_powers	151
FlxqM_FlxqC_gauss	121	FlxqXQ_powers_pre	151
FlxqM_FlxqC_invimage	121	FlxqXQ_powu	151
FlxqM_FlxqC_mul	121	FlxqXQ_powu_pre	151
FlxqM_Flxq_mul	121	FlxqXQ_pow_pre	151
FlxqM_gauss	122	FlxqXQ_sqr	150
FlxqM_image	122	FlxqXQ_sqr_pre	150
FlxqM_indexrank	122	FlxqXV_prod	149
FlxqM_inv	122	FlxqX_ddf	150
FlxqM_invimage	122	FlxqX_ddf_degree	150
FlxqM_ker	122	FlxqX_degfact	150
FlxqM_mul	122	FlxqX_disc	149
FlxqM_rank	122	FlxqX_div	149
FlxqM_suppl	122	FlxqX_divrem	148
FlxqV_dotproduct	121	FlxqX_divrem_pre	149
FlxqV_dotproduct_pre	121	FlxqX_div_pre	149
FlxqV_factorback	144	FlxqX_dotproduct	149
FlxqV_roots_to_pol	145	FlxqX_extgcd	149
FlxqXC_FlxqXQV_eval	150	FlxqX_extgcd_pre	149
FlxqXC_FlxqXQV_eval_pre	150	FlxqX_factor	150
FlxqXC_FlxqXQ_eval	150	FlxqX_factor_squarefree	150
FlxqXC_FlxqXQ_eval_pre	150	FlxqX_factor_squarefree_pre	150
FlxqXn_expint	151	FlxqX_FlxqXQV_eval	150
FlxqXn_expint_pre	152	FlxqX_FlxqXQV_eval_pre	150
FlxqXn_inv	151	FlxqX_FlxqXQ_eval	150
FlxqXn_inv_pre	151	FlxqX_FlxqXQ_eval_pre	150
FlxqXn_mul	151	FlxqX_Flxq_mul	148
FlxqXn_mul_pre	151	FlxqX_Flxq_mul_pre	148
FlxqXn_sqr	151	FlxqX_Flxq_mul_to_monic	148
FlxqXn_sqr_pre	151	FlxqX_Flxq_mul_to_monic_pre	148
FlxqXQ_autpow	151	FlxqX_Frobenius	149
FlxqXQ_autpow_pre	151	FlxqX_Frobenius_pre	150
FlxqXQ_autsum	151	FlxqX_fromNewton	149
FlxqXQ_autsum_pre	151	FlxqX_fromNewton_pre	149
FlxqXQ_auttrace	151	FlxqX_gcd	149
FlxqXQ_auttrace_pre	151	FlxqX_gcd_pre	149
FlxqXQ_div	151	FlxqX_get_red	147
FlxqXQ_div_pre	151	FlxqX_get_red_pre	147
FlxqXQ_halfFrobenius	151	FlxqX_halfgcd	149
FlxqXQ_inv	150, 157	FlxqX_halfgcd_pre	149

FlxqX_invBarrett . . . . .	149	Flxq_issquare . . . . .	145
FlxqX_invBarrett_pre . . . . .	149	Flxq_log . . . . .	145
FlxqX_ispower . . . . .	149	Flxq_lroot . . . . .	145
FlxqX_is_squarefree . . . . .	149	Flxq_lroot_fast . . . . .	145
FlxqX_mul . . . . .	148	Flxq_lroot_fast_pre . . . . .	146
FlxqX_mul_pre . . . . .	148	Flxq_lroot_pre . . . . .	145
FlxqX_nbfact . . . . .	150	Flxq_matrix_pow . . . . .	145
FlxqX_nbfact_by_degree . . . . .	150	Flxq_matrix_pow_pre . . . . .	145
FlxqX_nbfact_Frobenius . . . . .	150	Flxq_minpoly . . . . .	146
FlxqX_nbroots . . . . .	150	Flxq_minpoly_pre . . . . .	146
FlxqX_Newton . . . . .	149	Flxq_mul . . . . .	144
FlxqX_Newton_pre . . . . .	149	Flxq_mul_pre . . . . .	144
FlxqX_normalize . . . . .	148	Flxq_norm . . . . .	146
FlxqX_normalize_pre . . . . .	148	Flxq_order . . . . .	145
FlxqX_powu . . . . .	148	Flxq_pow . . . . .	144
FlxqX_powu_pre . . . . .	148	Flxq_powers . . . . .	144
FlxqX_red . . . . .	148	Flxq_powers_pre . . . . .	145
FlxqX_red_pre . . . . .	148	Flxq_powu . . . . .	144
FlxqX_rem . . . . .	149	Flxq_powu_pre . . . . .	144
FlxqX_rem_pre . . . . .	149	Flxq_pow_init . . . . .	144
FlxqX_resultant . . . . .	149	Flxq_pow_init_pre . . . . .	144
FlxqX_roots . . . . .	150	Flxq_pow_pre . . . . .	144
FlxqX_safegcd . . . . .	149	Flxq_pow_table . . . . .	144
FlxqX_saferes resultant . . . . .	149	Flxq_pow_table_pre . . . . .	144
FlxqX_sqr . . . . .	148	Flxq_sqr . . . . .	144
FlxqX_sqr_pre . . . . .	148	Flxq_sqrt . . . . .	145
Flxq_add . . . . .	144	Flxq_sqrtn . . . . .	145
Flxq_autpow . . . . .	145	Flxq_sqr_pre . . . . .	144
Flxq_autpowers . . . . .	145	Flxq_sub . . . . .	144
Flxq_autpow_pre . . . . .	145	Flxq_trace . . . . .	146
Flxq_autsum . . . . .	145	FlxT_red . . . . .	143
Flxq_auttrace . . . . .	145	FlxV_Flc_mul . . . . .	143
Flxq_auttrace_pre . . . . .	145	FlxV_Flv_multieval . . . . .	143
Flxq_charpoly . . . . .	146	FlxV_Flx_fromdigits . . . . .	139
Flxq_conjvec . . . . .	146	FlxV_prod . . . . .	143
Flxq_div . . . . .	144	FlxV_red . . . . .	143
Flxq_div_pre . . . . .	144	FlxV_to_Flm . . . . .	171
Flxq_ellcard . . . . .	350	FlxV_to_FlxX . . . . .	172
Flxq_ellgens . . . . .	350	FlxV_to_ZXV . . . . .	170
Flxq_ellgroup . . . . .	350	FlxV_to_ZXV_inplace . . . . .	170
Flxq_ellj . . . . .	349	FlxXC_sub . . . . .	147
Flxq_ellj_to_a4a6 . . . . .	349	FlxXC_to_F2xXC . . . . .	156
Flxq_elltwist . . . . .	350	FlxXC_to_ZXXC . . . . .	170
Flxq_ffisom_inv . . . . .	145	FlxXM_to_ZXXM . . . . .	170
Flxq_inv . . . . .	144	FlxXn_red . . . . .	151
Flxq_invsafe . . . . .	144	FlxXV_to_FlxM . . . . .	172
Flxq_invsafe_pre . . . . .	144	FlxX_add . . . . .	146
Flxq_inv_pre . . . . .	144	FlxX_blocks . . . . .	147
Flxq_is2npower . . . . .	145	FlxX_deriv . . . . .	146

FlxX_double . . . . .	146	Flx_dotproduct_pre . . . . .	142
FlxX_Flx_add . . . . .	146	Flx_double . . . . .	139
FlxX_Flx_mul . . . . .	146	Flx_equal . . . . .	138
FlxX_Flx_sub . . . . .	146	Flx_equal1 . . . . .	138
FlxX_Fl_mul . . . . .	146	Flx_eval . . . . .	141
FlxX_invLaplace . . . . .	146	Flx_eval_powers_pre . . . . .	141
FlxX_Laplace . . . . .	146	Flx_eval_pre . . . . .	141
FlxX_neg . . . . .	146	Flx_extgcd . . . . .	140
FlxX_renormalize . . . . .	147	Flx_extgcd_pre . . . . .	140
FlxX_resultant . . . . .	147	Flx_extresultant . . . . .	141
FlxX_shift . . . . .	147	Flx_factcyclo . . . . .	140
FlxX_sub . . . . .	146	Flx_factor . . . . .	140
FlxX_swap . . . . .	147	Flx_factorff_irred . . . . .	140
FlxX_to_F2xX . . . . .	155	Flx_factor_squarefree . . . . .	140
FlxX_to_Flm . . . . .	171	Flx_factor_squarefree_pre . . . . .	140
FlxX_to_Flx . . . . .	171	Flx_ffintersect . . . . .	142
FlxX_to_FlxC . . . . .	171	Flx_ffisom . . . . .	141
FlxX_to_ZXX . . . . .	170	Flx_Flv_multieval . . . . .	142
FlxX_translate1 . . . . .	146	Flx_FlxqV_eval . . . . .	145
FlxX_triple . . . . .	146	Flx_FlxqV_eval_pre . . . . .	145
FlxYqq_pow . . . . .	147	Flx_Flxq_eval . . . . .	145
FlxY_degreeex . . . . .	146	Flx_Flxq_eval_pre . . . . .	145
FlxY_evalx . . . . .	146	Flx_FlxY_resultant . . . . .	147
FlxY_evalx_powers_pre . . . . .	147	Flx_Fl_add . . . . .	138
FlxY_evalx_pre . . . . .	146	Flx_Fl_mul . . . . .	139
FlxY_eval_powers_pre . . . . .	147	Flx_Fl_mul_to_monic . . . . .	139
FlxY_FlxqV_evalx . . . . .	147	Flx_Fl_sub . . . . .	139
FlxY_FlxqV_evalx_pre . . . . .	147	Flx_Frobenius . . . . .	139
FlxY_Flxq_evalx . . . . .	147	Flx_Frobenius_pre . . . . .	139
FlxY_Flxq_evalx_pre . . . . .	147	Flx_fromNewton . . . . .	142
FlxY_Flx_div . . . . .	146	Flx_gcd . . . . .	140
FlxY_Flx_translate . . . . .	147	Flx_gcd_pre . . . . .	140
Flx_add . . . . .	138	Flx_get_red . . . . .	138
Flx_blocks . . . . .	142	Flx_get_red_pre . . . . .	138
Flx_constant . . . . .	138	Flx_halfgcd . . . . .	140
Flx_copy . . . . .	138	Flx_halfgcd_pre . . . . .	140
Flx_ddf . . . . .	140	Flx_halve . . . . .	139
Flx_ddf_pre . . . . .	140	Flx_inflate . . . . .	142
Flx_deflate . . . . .	142	Flx_integ . . . . .	139
Flx_degfact . . . . .	140, 142	Flx_invBarrett . . . . .	141
Flx_deriv . . . . .	139	Flx_invLaplace . . . . .	142
Flx_diff1 . . . . .	139	Flx_isplayer . . . . .	140
Flx_digits . . . . .	139	Flx_is_irred . . . . .	142
Flx_div . . . . .	139	Flx_is_smooth . . . . .	142
Flx_divrem . . . . .	139	Flx_is_smooth_pre . . . . .	142
Flx_divrem_pre . . . . .	139	Flx_is_squarefree . . . . .	142
Flx_div_by_X_x . . . . .	141	Flx_is_totally_split . . . . .	142
Flx_div_pre . . . . .	139	Flx_Laplace . . . . .	142
Flx_dotproduct . . . . .	142	Flx_lead . . . . .	138

Flx_matFrobenius . . . . .	140	Flx_valrem . . . . .	141
Flx_matFrobenius_pre . . . . .	140	Fly_to_FlxY . . . . .	172
Flx_mod_Xn1 . . . . .	140	Fl_2gener_pre . . . . .	86
Flx_mod_Xnm1 . . . . .	140	Fl_add . . . . .	84
Flx_mul . . . . .	139	Fl_addmulmul_pre . . . . .	86
Flx_mulu . . . . .	139	Fl_addmul_pre . . . . .	86
Flx_mul_pre . . . . .	139	Fl_center . . . . .	84
Flx_nbfact . . . . .	142	Fl_div . . . . .	85
Flx_nbfact_by_degree . . . . .	142	Fl_double . . . . .	84
Flx_nbfact_Frobenius . . . . .	142	Fl_elldisc . . . . .	347
Flx_nbfact_Frobenius_pre . . . . .	142	Fl_elldisc_pre . . . . .	347
Flx_nbfact_pre . . . . .	142	Fl_ellj . . . . .	347
Flx_nbroots . . . . .	142	Fl_ellj_pre . . . . .	347
Flx_neg . . . . .	138	Fl_ellj_to_a4a6 . . . . .	347
Flx_neg_inplace . . . . .	138	Fl_ellptors . . . . .	347
Flx_Newton . . . . .	142	Fl_elltrace . . . . .	347
Flx_normalize . . . . .	141	Fl_elltrace_CM . . . . .	347
Flx_oneroot . . . . .	140	Fl_elltwist . . . . .	347
Flx_oneroot_pre . . . . .	140	Fl_elltwist_disc . . . . .	347
Flx_oneroot_split . . . . .	140	Fl_half . . . . .	84
Flx_oneroot_split_pre . . . . .	140	Fl_inv . . . . .	84
Flx_powu . . . . .	139	Fl_invgen . . . . .	84
Flx_powu_pre . . . . .	139	Fl_invsafe . . . . .	84
Flx_recip . . . . .	141	Fl_log . . . . .	85
Flx_red . . . . .	138	Fl_log_pre . . . . .	86
Flx_rem . . . . .	139	Fl_mul . . . . .	84
Flx_rem_pre . . . . .	139	Fl_mul_pre . . . . .	86
Flx_renormalize . . . . .	141	Fl_neg . . . . .	84
Flx_rescale . . . . .	141	Fl_order . . . . .	85
Flx_resultant . . . . .	141	Fl_powers . . . . .	85
Flx_resultant_pre . . . . .	141	Fl_powers_pre . . . . .	86
Flx_roots . . . . .	140	Fl_powu . . . . .	85
Flx_rootsff . . . . .	140	Fl_powu_pre . . . . .	86
Flx_roots_pre . . . . .	140	Fl_sqr . . . . .	84
Flx_shift . . . . .	141	Fl_sqrt . . . . .	85
Flx_splitting . . . . .	142	Fl_sqrt1 . . . . .	85
Flx_sqr . . . . .	139	Fl_sqrt1_pre . . . . .	86
Flx_sqr_pre . . . . .	139	Fl_sqrttn . . . . .	85
Flx_sub . . . . .	138	Fl_sqrttn_pre . . . . .	86
Flx_Teichmuller . . . . .	142	Fl_sqrt_pre . . . . .	86
Flx_to_F2x . . . . .	152	Fl_sqrt_pre_i . . . . .	86
Flx_to_Flv . . . . .	171	Fl_sqr_pre . . . . .	86
Flx_to_FlxX . . . . .	170	Fl_sub . . . . .	84
Flx_to_ZX . . . . .	170	Fl_to_Flx . . . . .	171
Flx_to_ZX_inplace . . . . .	170	Fl_triple . . . . .	84
Flx_translate1 . . . . .	139	forallsubset_init . . . . .	44
Flx_translate1_basecase . . . . .	139	forcomposite . . . . .	43
Flx_triple . . . . .	139	forcomposite_init . . . . .	43
Flx_val . . . . .	141	forcomposite_next . . . . .	43

fordiv	43	FpE_to_mod	347
forell	43	FpE_weilpairing	347
forell(ell,a,b,,flag)	43	FpJ_add	348
forksubset_init	44	FpJ_dbl	348
format	41	FpJ_mul	348
forpart	43	FpJ_neg	348
forpart_init	43	FpJ_to_FpE	348
forpart_next	44	FpMs_FpCs_solve	193
forpart_prev	44	FpMs_FpCs_solve_safe	193
forpart_t	44	FpMs_FpC_mul	192
forperm	44	FpMs_leftkernel_elt	193
forperm_init	44	FpM_add	113
forperm_next	44	FpM_center	113
forprime	43	FpM_center_inplace	113
forprimestep	43	FpM_charpoly	115
forprimestep_init	44, 179	FpM_deplin	114
forprime_init	44, 45, 179	FpM_det	114
forprime_next	45, 179	FpM_FpC_gauss	114
forprime_t	44, 45	FpM_FpC_invimage	115
forqfvec	43	FpM_FpC_mul	114
forqfvec1	43	FpM_FpC_mul_FpX	114
forsubgroup	43	FpM_Fp_mul	114
forsubgroup(H = G, B,)	43	FpM_gauss	114
forsubset	44	FpM_hess	115
forsubset_init	44	FpM_image	114
forsubset_next	44	FpM_indexrank	115
forvec	43	FpM_intersect	114
forvec_init	43	FpM_intersect_i	114
forvec_next	43	FpM_inv	114
FpC_add	113	FpM_invimage	115
FpC_center	113	FpM_ker	115
FpC_center_inplace	113	FpM_mul	114
FpC_FpV_mul	114	FpM_powu	114
FpC_Fp_mul	114	FpM_rank	115
FpC_ratlift	163	FpM_ratlift	163
FpC_red	113	FpM_red	113
FpC_sub	113	FpM_sub	114
FpC_to_mod	157	FpM_suppl	115
FpE_add	346	FpM_to_mod	158
FpE_changepoint	346	FpVV_to_mod	158
FpE_changepointinv	346	FpV_add	113
FpE_dbl	346	FpV_dotproduct	114
FpE_log	347	FpV_dotsquare	114
FpE_mul	346	FpV_factorback	114
FpE_neg	346	FpV_FpC_mul	114
FpE_order	346	FpV_FpMs_mul	192
FpE_sub	346	FpV_FpM_polint	125
FpE_tatepairing	347	FpV_inv	104
FpE_to_FpJ	348	FpV_invVandermonde	125

FpV_polint . . . . .	125	FpXQXQ_powers . . . . .	135
FpV_prod . . . . .	104	FpXQXQ_sqr . . . . .	135
FpV_red . . . . .	113	FpXQXT_red . . . . .	133
FpV_roots_to_pol . . . . .	125	FpXQXV_FpXQX_fromdigits . . . . .	134
FpV_sub . . . . .	114	FpXQXV_prod . . . . .	134
FpV_to_mod . . . . .	158	FpXQXV_red . . . . .	133
FpXC_center . . . . .	131	FpXQX_ddf . . . . .	136
FpXC_FpXQV_eval . . . . .	130	FpXQX_ddf_degree . . . . .	137
FpXC_FpXQ_eval . . . . .	130	FpXQX_degfact . . . . .	137
FpXC_to_mod . . . . .	158	FpXQX_digits . . . . .	133
FpXM_center . . . . .	131	FpXQX_disc . . . . .	134
FpXM_FpXQV_eval . . . . .	130	FpXQX_div . . . . .	133
FpXM_to_mod . . . . .	158	FpXQX_divrem . . . . .	133
FpXn_div . . . . .	131	FpXQX_div_by_X_x . . . . .	133
FpXn_exp . . . . .	131	FpXQX_dotproduct . . . . .	134
FpXn_expint . . . . .	131	FpXQX_extgcd . . . . .	134
FpXn_inv . . . . .	131	FpXQX_factor . . . . .	136
FpXn_mul . . . . .	131	FpXQX_factor_squarefree . . . . .	136
FpXn_sqr . . . . .	131	FpXQX_FpXQXQV_eval . . . . .	135
FpXQC_to_mod . . . . .	158	FpXQX_FpXQXQ_eval . . . . .	134
FpXQE_add . . . . .	351	FpXQX_FpXQ_mul . . . . .	133
FpXQE_changepoint . . . . .	351	FpXQX_Frobenius . . . . .	138
FpXQE_changepointinv . . . . .	351	FpXQX_gcd . . . . .	134
FpXQE_dbl . . . . .	351	FpXQX_get_red . . . . .	134
FpXQE_log . . . . .	351	FpXQX_halfgcd . . . . .	134
FpXQE_mul . . . . .	351	FpXQX_invBarrett . . . . .	134
FpXQE_neg . . . . .	351	FpXQX_isplayer . . . . .	137
FpXQE_order . . . . .	351	FpXQX_mul . . . . .	133
FpXQE_sub . . . . .	351	FpXQX_nbfact . . . . .	137
FpXQE_tatepairing . . . . .	352	FpXQX_nbfact_Frobenius . . . . .	137
FpXQE_weilpairing . . . . .	352	FpXQX_nbroots . . . . .	137
FpXQM_autsum . . . . .	130	FpXQX_normalize . . . . .	132
FpXQXn_div . . . . .	134	FpXQX_powu . . . . .	133
FpXQXn_exp . . . . .	134	FpXQX_red . . . . .	133
FpXQXn_expint . . . . .	134	FpXQX_rem . . . . .	133
FpXQXn_inv . . . . .	134	FpXQX_renormalize . . . . .	133
FpXQXn_mul . . . . .	134	FpXQX_resultant . . . . .	134
FpXQXn_sqr . . . . .	134	FpXQX_roots . . . . .	136
FpXQXQ_autpow . . . . .	135	FpXQX_split_part . . . . .	137
FpXQXQ_autsum . . . . .	135	FpXQX_sqr . . . . .	133
FpXQXQ_auttrace . . . . .	136	FpXQX_to_mod . . . . .	158
FpXQXQ_div . . . . .	135	FpXQ_add . . . . .	128
FpXQXQ_halfFrobenius . . . . .	135	FpXQ_autpow . . . . .	130
FpXQXQ_inv . . . . .	135	FpXQ_autpowers . . . . .	130
FpXQXQ_invsafe . . . . .	135	FpXQ_autsum . . . . .	130
FpXQXQ_matrix_pow . . . . .	135	FpXQ_auttrace . . . . .	130
FpXQXQ_minpoly . . . . .	135	FpXQ_charpoly . . . . .	129
FpXQXQ_mul . . . . .	135	FpXQ_conjvec . . . . .	130
FpXQXQ_pow . . . . .	135	FpXQ_div . . . . .	128

FpXQ_ellcard	351	FpXY_FpXQ_evalx	132
FpXQ_elldivpol	351	FpXY_FpXQ_evaly	131
FpXQ_ellgens	351	FpXY_Fq_evaly	131
FpXQ_ellgroup	351	FpX_add	123
FpXQ_ellj	351	FpX_center	123, 124
FpXQ_elljissupersingular	351	FpX_center_i	124
FpXQ_elltwist	351	FpX_chinese_coprime	125
FpXQ_ffisom_inv	137	FpX_convolution	123
FpXQ_inv	128	FpX_ddf	126
FpXQ_invsafe	128	FpX_ddf_degree	126
FpXQ_issquare	128	FpX_degfact	126, 140, 153
FpXQ_log	128, 129, 145	FpX_deriv	123
FpXQ_matrix_pow	130	FpX_digits	123
FpXQ_minpoly	129	FpX_disc	126
FpXQ_mul	128	FpX_div	123
FpXQ_norm	129	FpX_divrem	123
FpXQ_order	128, 145	FpX_divu	124
FpXQ_pow	128	FpX_div_by_X_x	123
FpXQ_powers	130	FpX_dotproduct	124
FpXQ_powu	128	FpX_eval	124
FpXQ_red	128	FpX_extgcd	123
FpXQ_sqr	128	FpX_factcyclo	126
FpXQ_sqrt	128	FpX_factor	125
FpXQ_sqrtn	129, 145	FpX_factorff	137
FpXQ_sub	128	FpX_factorff_irred	137, 140
FpXQ_trace	129	FpX_factor_squarefree	125
FpXT_red	122	FpX_ffintersect	137
FpXV_chinese	125	FpX_ffisom	137, 141
FpXV_factorback	125	FpX_FpC_nfpoleval	297
FpXV_FpC_mul	124	FpX_FpV_multieval	124
FpXV_FpX_fromdigits	123	FpX_FpXQV_eval	130
FpXV_prod	125	FpX_FpXQ_eval	130
FpXV_red	122	FpX_FpXV_multirem	125
FpXX_add	131	FpX_FpXY_resultant	126
FpXX_deriv	131	FpX_Fp_add	124
FpXX_FpX_mul	131	FpX_Fp_add_shallow	124
FpXX_Fp_mul	131	FpX_Fp_div	124
FpXX_halve	131	FpX_Fp_mul	124
FpXX_integ	131	FpX_Fp_mulspec	124
FpXX_mulu	131	FpX_Fp_mul_to_monic	124
FpXX_neg	131	FpX_Fp_sub	124
FpXX_red	131	FpX_Fp_sub_shallow	124
FpXX_renormalize	131	FpX_Frobenius	124
FpXX_sub	131	FpX_fromNewton	126
FpXYQQ_pow	132	FpX_gcd	123
FpXY_eval	131	FpX_gcd_check	312
FpXY_evalx	131	FpX_get_red	127
FpXY_evaly	131	FpX_halfgcd	123
FpXY_FpXQV_evalx	132	FpX_halve	123



FpX_integ	123	Fp_ellj	345
FpX_invBarret	127	Fp_elljissupersingular	345
FpX_invBarrett	124	Fp_elltwist	346
FpX_invLaplace	126	Fp_factored_order	104
FpX_isplayer	125	Fp_ffellcard	346
FpX_is_irred	125, 153	Fp_FpXQ_log	128
FpX_is_squarefree	125	Fp_FpX_sub	124
FpX_is_totally_split	125	Fp_half	103
FpX_Laplace	126	Fp_inv	104
FpX_matFrobenius	124	Fp_invgen	104
FpX_mul	123	Fp_invsafe	104
FpX_mulspec	123	Fp_isplayer	104
FpX_mulu	124	Fp_issquare	104
FpX_nbfact	126	Fp_log	104, 129
FpX_nbfact_Frobenius	126	Fp_modinv_to_j	352
FpX_nbroots	126	Fp_mul	103
FpX_neg	123	Fp_muls	103
FpX_Newton	126	Fp_mulu	103
FpX_normalize	124	Fp_neg	103
FpX_oneroot	126	Fp_order	104
FpX_oneroot_split	126	Fp_polmodular_evalx	352
FpX_powu	123	Fp_pow	104
FpX_ratlift	163	Fp_powers	104
FpX_red	122	Fp_pows	104
FpX_rem	123	Fp_powu	103
FpX_renormalize	123	Fp_pow_init	104
FpX_rescale	124	Fp_pow_table	104
FpX_resultant	126	Fp_ratlift	163
FpX_roots	126	Fp_red	103
FpX_rootsff	137, 140	Fp_sqr	103
FpX_roots_mult	126	Fp_sqrt	104
FpX_split_part	126	Fp_sqrtn	105
FpX_sqr	123	Fp_sqrt_i	105
FpX_sub	123	Fp_sub	103
FpX_to_mod	157	Fp_to_mod	157
FpX_translate	123	FqC_add	115
FpX_valrem	123	FqC_FqV_mul	115
Fp_2gener	105	FqC_Fq_mul	115
Fp_add	15, 103	FqC_sub	115
Fp_addmul	103	FqC_to_mod	158
Fp_center	103	FqM_deplin	115
Fp_center_i	103	FqM_det	115
Fp_div	104	FqM_FqC_gauss	115
Fp_divu	104	FqM_FqC_invimage	115
Fp_ellcard	346	FqM_FqC_mul	115
Fp_ellcard_SEA	346	FqM_gauss	115
Fp_elldivpol	346	FqM_image	115
Fp_ellgens	346	FqM_indexrank	115
Fp_ellgroup	346	FqM_inv	115

FqM_invimage	115	FqX_halfgcd	133
FqM_ker	115	FqX_halve	132
FqM_mul	115	FqX_integ	132
FqM_rank	115	FqX_isplayer	137
FqM_suppl	115	FqX_is_squarefree	136
FqM_to_mod	158	FqX_mul	132
FqM_to_nfM	308	FqX_mulu	132
FqV_factorback	129	FqX_nbfact	137
FqV_inv	129	FqX_nbroots	137
FqV_red	128	FqX_neg	132
FqV_roots_to_pol	136	FqX_normalize	132
FqV_to_nfV	308	FqX_powu	132
FqXC_to_mod	158	FqX_red	128
FqXM_to_mod	158	FqX_rem	132
FqXn_exp	134	FqX_roots	136
FqXn_expint	134	FqX_sqr	132
FqXn_inv	134	FqX_sub	132
FqXn_mul	134	FqX_to_FFX	247
FqXn_sqr	134	FqX_to_mod	158
FqXQ_add	136	FqX_to_nfX	308
FqXQ_div	136	FqX_translate	133
FqXQ_inv	136	Fq_add	129
FqXQ_invsafe	136	Fq_div	129
FqXQ_matrix_pow	136	Fq_ellcard_SEA	351
FqXQ_mul	136	Fq_elldivpolmod	351
FqXQ_pow	136	Fq_Fp_mul	129
FqXQ_powers	136	Fq_halve	129
FqXQ_sqr	136	Fq_inv	129
FqXQ_sub	136	Fq_invsafe	129
FqXY_eval	133	Fq_isplayer	129
FqXY_evalx	133	Fq_issquare	129
FqX_add	132	Fq_log	129
FqX_ddf	137	Fq_mul	129
FqX_degfact	137	Fq_mulu	129
FqX_deriv	132	Fq_neg	129
FqX_div	132	Fq_neg_inv	129
FqX_divrem	132	Fq_pow	129
FqX_div_by_X_x	132	Fq_powu	129
FqX_eval	133	Fq_red	128
FqX_extgcd	133	Fq_sqr	129
FqX_factor	136	Fq_sqrt	129
FqX_factor_squarefree	136	Fq_sqrtn	129
FqX_Fp_mul	132	Fq_sub	129
FqX_Fq_add	132	Fq_to_FF	247
FqX_Fq_mul	132	Fq_to_FpXQ	128
FqX_Fq_mul_to_monic	132	Fq_to_nf	308
FqX_Fq_sub	132	fractor	217
FqX_gcd	133	Frobeniusform	190
FqX_get_red	135	fromdigitsu	92

fromdigits_2k . . . . .	92
fujiwara_bound . . . . .	245
fujiwara_bound_real . . . . .	245
fun(E, ell) . . . . .	43
fun(E, H) . . . . .	43
functions_basic . . . . .	56
functions_default . . . . .	56
functions_gp . . . . .	56
fuse_Z_factor . . . . .	176
f_PRETTYMAT . . . . .	261
f_RAW . . . . .	261, 262
f_TEX . . . . .	261, 262

## G

gabs[z] . . . . .	236
gadd . . . . .	87, 237
gaddgs . . . . .	15, 237
gaddsg . . . . .	15, 237
gaddz . . . . .	15, 26, 88, 238
gadd[z] . . . . .	87
gaffect . . . . .	26, 27, 217
gaffsg . . . . .	27, 217
galoisexport . . . . .	259
galoisidentify . . . . .	259
galoisinit . . . . .	256, 321
galois_group . . . . .	256
gal_get_den . . . . .	322
gal_get_e . . . . .	321
gal_get_gen . . . . .	322
gal_get_group . . . . .	322
gal_get_invvdm . . . . .	322
gal_get_mod . . . . .	321
gal_get_orders . . . . .	322
gal_get_p . . . . .	321
gal_get_pol . . . . .	321
gal_get_roots . . . . .	322
gammamellininv . . . . .	357
gammamellinininit . . . . .	357
gammamellininvrt . . . . .	357
gand . . . . .	231
garbage collecting . . . . .	17
gassoc_proto . . . . .	106
gaussred_from_QR . . . . .	190
gbezout . . . . .	235
gboundcf . . . . .	102
gcdii . . . . .	102
gceil . . . . .	227
gchari_lfun . . . . .	336

gclone . . . . .	27, 72, 73
gcloneref . . . . .	73
gclone_refc . . . . .	74
gcmp . . . . .	228
gcmpgs . . . . .	230
gcmpsg . . . . .	230
gcoeff . . . . .	15, 65, 274
gconj . . . . .	243
gcopy . . . . .	27, 73
gcopy_avma . . . . .	72
gcopy_lg . . . . .	73
gcvtoi . . . . .	228
gcvtop . . . . .	218
gc_all . . . . .	70
gc_bool . . . . .	70
gc_const . . . . .	70
gc_double . . . . .	70
gc_int . . . . .	70
gc_long . . . . .	70
gc_needed . . . . .	23
gc_NULL . . . . .	70
gc_ulong . . . . .	70
gdeuc . . . . .	234
gdiv . . . . .	237
gdiventgs[z] . . . . .	233
gdiventres . . . . .	233
gdiventsg . . . . .	233
gdivent[z] . . . . .	233
gdivexact . . . . .	233
gdivgs . . . . .	237
gdivgu . . . . .	233, 237
gdivgunextu . . . . .	233
gdivmod . . . . .	234
gdivround . . . . .	234
gdivsg . . . . .	237
gdivz . . . . .	238
gdvd . . . . .	233
gel . . . . .	14, 15, 65, 274
GEN . . . . .	13
GENbinbase . . . . .	69
gener_F2xq . . . . .	154
gener_Flxq . . . . .	146
gener_FpXQ . . . . .	130
gener_FpXQ_local . . . . .	130
gener_Fq_local . . . . .	130
GENtoGENstr . . . . .	261
GENtoGENstr_nospace . . . . .	261
GENtostr . . . . .	40, 261
GENtostr_raw . . . . .	261

GENtostr_unquoted . . . . .	261	gen_Shanks . . . . .	211
GENtoTeXstr . . . . .	40, 261	gen_Shanks_init . . . . .	210, 211
gen_0 . . . . .	13, 33	gen_Shanks_log . . . . .	210
gen_1 . . . . .	13	gen_Shanks_sqrt . . . . .	211
gen_2 . . . . .	13	gen_sort . . . . .	231
gen_bkeval . . . . .	214	gen_sort_inplace . . . . .	232
gen_bkeval_powers . . . . .	214	gen_sort_shallow . . . . .	232
gen_cmp_RgX . . . . .	233	gen_sort_uniq . . . . .	232
gen_det . . . . .	213	gen_ZpM_Dixon_Wiedemann . . . . .	193
gen_digits . . . . .	215	gen_ZpM_Newton . . . . .	216
gen_ellgens . . . . .	212, 343	gen_ZpX_Dixon . . . . .	216
gen_ellgroup . . . . .	212	gen_ZpX_Newton . . . . .	216
gen_factorback . . . . .	239	geq . . . . .	231
gen_factored_order . . . . .	211	gequal . . . . .	201, 229
gen_FpM_Wiedemann . . . . .	193	gequal0 . . . . .	230
gen_fromdigits . . . . .	215	gequal1 . . . . .	230
gen_Gauss . . . . .	213	gequalgs . . . . .	230
gen_Gauss_pivot . . . . .	213	gequalm1 . . . . .	230
gen_gener . . . . .	211	gequalsg . . . . .	230
gen_indexsort . . . . .	232	gequalX . . . . .	228
gen_indexsort_uniq . . . . .	232	gerepile . . . . .	18, 20, 26, 27, 70, 97
gen_ker . . . . .	213	gerepileall . . . . .	23
gen_m1 . . . . .	13	gerepileall . . . . .	20, 23, 70
gen_m2 . . . . .	13	gerepileallsp . . . . .	20, 71
gen_matcolinvimage . . . . .	213	gerepilecoeffs . . . . .	71
gen_matcolmul . . . . .	213	gerepilecoeffssp . . . . .	71
gen_matid . . . . .	213	gerepilecopy . . . . .	20, 23, 71
gen_matinvimage . . . . .	213	gerepilemany . . . . .	71
gen_matmul . . . . .	213	gerepilemanysp . . . . .	71
gen_order . . . . .	211	gerepileupto . . . . .	20, 25, 27, 71, 97, 174, 224, 225, 274, 304
gen_PH_log . . . . .	211	gerepileuptoint . . . . .	71
gen_plog . . . . .	211	gerepileuptoleaf . . . . .	71
gen_Pollard_log . . . . .	211	getheap . . . . .	74
gen_pow . . . . .	238, 239	getrand . . . . .	103
gen_powers . . . . .	214, 239	getrealprecision . . . . .	251
gen_powu . . . . .	238, 239	gettime . . . . .	42
gen_powu_fold . . . . .	239	get_arith_Z . . . . .	211
gen_powu_fold_i . . . . .	239	get_arith_ZZM . . . . .	211
gen_powu_i . . . . .	239	get_avma . . . . .	70
gen_pow_fold . . . . .	239	get_bnf . . . . .	289
gen_pow_fold_i . . . . .	239	get_bnfpol . . . . .	289
gen_pow_i . . . . .	238	get_F2xqE_group . . . . .	212
gen_pow_init . . . . .	239	get_F2xqX_degree . . . . .	156
gen_pow_table . . . . .	239	get_F2xqX_mod . . . . .	156
gen_product . . . . .	238	get_F2xqX_var . . . . .	156
gen_RgX_bkeval . . . . .	214	get_F2xq_field . . . . .	213
gen_search . . . . .	232	get_F2x_degree . . . . .	152
gen_select_order . . . . .	211	get_F2x_mod . . . . .	152
gen_setminus . . . . .	232		

get_F2x_var . . . . .	152	glcm . . . . .	235
get_FlxqE_group . . . . .	212	gle . . . . .	230
get_FlxqXQ_algebra . . . . .	215	glt . . . . .	230
get_FlxqX_degree . . . . .	148	gmael . . . . .	15, 65
get_FlxqX_mod . . . . .	148	gmael1 . . . . .	15
get_FlxqX_var . . . . .	148	gmael2 . . . . .	65
get_Flxq_field . . . . .	213	gmael3 . . . . .	65
get_Flxq_star . . . . .	212	gmael4 . . . . .	65
get_Flx_degree . . . . .	138	gmael5 . . . . .	65
get_Flx_mod . . . . .	138	gmax . . . . .	229
get_Flx_var . . . . .	138	gmaxgs . . . . .	230
get_Fl_red . . . . .	85	gmaxsg . . . . .	230
get_FpE_group . . . . .	212	gmax_shallow . . . . .	229
get_FpXQE_group . . . . .	212	gmin . . . . .	229
get_FpXQXQ_algebra . . . . .	215	gmings . . . . .	230
get_FpXQX_algebra . . . . .	215	gminsg . . . . .	230
get_FpXQX_degree . . . . .	135	gmin_shallow . . . . .	229
get_FpXQX_mod . . . . .	135	gmodgs . . . . .	234
get_FpXQX_var . . . . .	135	gmodsg . . . . .	234
get_FpXQ_algebra . . . . .	215	gmodulgs . . . . .	219
get_FpXQ_star . . . . .	212	gmodulo . . . . .	219
get_FpX_algebra . . . . .	215	gmodulsg . . . . .	219
get_FpX_degree . . . . .	127	gmodulss . . . . .	219
get_FpX_mod . . . . .	127	gmod[z] . . . . .	234
get_FpX_var . . . . .	127	gmul . . . . .	237
get_Fp_field . . . . .	213	gmul2n[z] . . . . .	228
get_Fq_field . . . . .	213	gmulgs . . . . .	237
get_lex . . . . .	282	gmulgu . . . . .	237
get_modpr . . . . .	290	gmulsg . . . . .	237
get_nf . . . . .	289	gmulug . . . . .	237
get_nfpol . . . . .	289	gmulz . . . . .	238
get_nf_field . . . . .	213	gne . . . . .	231
get_prid . . . . .	290	gneg[z] . . . . .	236
get_Rg_algebra . . . . .	215	gneg_i . . . . .	236
gexpo . . . . .	30, 62	gnorml1 . . . . .	240
gexpo_safe . . . . .	62	gnorml1_fake . . . . .	240
gfloor . . . . .	227	gnorml2 . . . . .	239
gfrac . . . . .	227	gnot . . . . .	231
ggamma1m1 . . . . .	252	gor . . . . .	231
ggcd . . . . .	235	<i>GP prototype</i> . . . . .	76
gge . . . . .	231	gphelp_keyword_list . . . . .	59
ggt . . . . .	231	gpininstall . . . . .	58
ghalf . . . . .	13	gpow . . . . .	237
gidentical . . . . .	191, 229	gpowers . . . . .	238
gimag . . . . .	243	gpowgs . . . . .	237
ginv . . . . .	237	gprec . . . . .	218
ginvmod . . . . .	234	gprecision . . . . .	63
gisdoube . . . . .	217	gprec_w . . . . .	218
gisexactzero . . . . .	229	gprec_wensure . . . . .	218

gprec_wtrunc . . . . .	218	group_abelianSNF . . . . .	258
gprimepi_lower_bound . . . . .	178	group_domain . . . . .	257
gprimepi_upper_bound . . . . .	178	group_elts . . . . .	257
gp_alarm_handler . . . . .	59	group_export . . . . .	258
gp_call . . . . .	283	group_export_GAP . . . . .	259
gp_call2 . . . . .	283	group_export_MAGMA . . . . .	259
gp_callbool . . . . .	284	group_ident . . . . .	259
gp_callprec . . . . .	283	group_ident_trans . . . . .	259
gp_callvoid . . . . .	284	group_isA4S4 . . . . .	258
gp_context_restore . . . . .	59	group_isabelian . . . . .	257
gp_context_save . . . . .	59	group_leftcoset . . . . .	258
gp_echo_and_log . . . . .	59	group_order . . . . .	257
gp_eval . . . . .	283	group_perm_normalize . . . . .	258
gp_evalbool . . . . .	283	group_quotient . . . . .	258
gp_evalprec . . . . .	283	group_rightcoset . . . . .	258
gp_evalupto . . . . .	283	group_set . . . . .	257
gp_evalvoid . . . . .	283	group_subgroups . . . . .	258
gp_filter . . . . .	58	group_subgroup_isnormal . . . . .	258
gp_format_prompt . . . . .	58	group_subgroup_is_faithful . . . . .	257
gp_format_time . . . . .	58	group_to_cc . . . . .	257
gp_handle_exception . . . . .	57	gshift[z] . . . . .	228
gp_help . . . . .	58	gsigne . . . . .	30, 62
gp_load_gprc . . . . .	58	gsincos . . . . .	253
gp_meta . . . . .	58	gsizebyte . . . . .	26
gp_read_file . . . . .	38, 58	gsizeofword . . . . .	26
gp_read_str . . . . .	36, 37, 58, 79	gsmith . . . . .	331
gp_read_stream . . . . .	38	gsmithall . . . . .	331
gp_read_str_bitprec . . . . .	38	gsprintf . . . . .	262
gp_read_str_multiline . . . . .	37	gsqr . . . . .	237
gp_read_str_prec . . . . .	38	gsqrpowers . . . . .	238
gp_sigint_fun . . . . .	57	GSTR . . . . .	34
Gram matrix . . . . .	188	gsub . . . . .	237
gram_matrix . . . . .	188	gsubgs . . . . .	237
greal . . . . .	243	gsubsg . . . . .	237
gred_rfac2 . . . . .	34	gsubst . . . . .	240
grem . . . . .	234	gsubz . . . . .	238
grndtoi . . . . .	227	gsupnorm . . . . .	240
grootsof1 . . . . .	238	gsupnorm_aux . . . . .	240
ground . . . . .	227	gtocol . . . . .	221
groupelts_abelian_group . . . . .	258	gtodouble . . . . .	28, 217
groupelts_center . . . . .	258	gtofp . . . . .	27, 218
groupelts_conjclasses . . . . .	257	gtolong . . . . .	28, 217
groupelts_conj_set . . . . .	257	gtomat . . . . .	221
groupelts_exponent . . . . .	258	gtomp . . . . .	218
groupelts_quotient . . . . .	258	gtopoly . . . . .	219
groupelts_set . . . . .	257	gtopolyrev . . . . .	219
groupelts_solvablesubgroups . . . . .	258	gtos . . . . .	217
groupelts_to_group . . . . .	257	gtoser . . . . .	221
group_abelianHNF . . . . .	257	gtoser_prec . . . . .	221

gtou	217
gtovec	221
gtovecsmall	221
gtrans	275
gtrunc	227
gtrunc2n	91, 228
gunclone	27, 73
guncloneNULL	74
guncloneNULL_deep	74
gunclone_deep	74
gval	228
gvaluation	228
gvar	32, 35, 62
gvar2	62
gvsprintf	262
G_ZGC_mul	191
G_ZG_mul	191

## H

halfgcdii	102
hamming1	83
hashentry	271
hashtable	271
hash_create	272
hash_create_str	272
hash_create_ulong	272
hash_dbg	273
hash_destroy	273
hash_GEN	273
hash_haskey_GEN	272
hash_haskey_long	272
hash_init	272
hash_init_GEN	272
hash_init_ulong	272
hash_insert	272
hash_insert2	272
hash_insert_long	272
hash_keys	273
hash_remove	273
hash_remove_select	273
hash_search	272, 273
hash_search2	272
hash_select	272
hash_str	272, 273
hash_str_len	273
hash_values	273
hash_zv	273
hclassno	323

hclassno6	324
hclassno6u	324
hclassno6u_no_cache	324
hclassnoF_fact	324
heap	14
Hermite_bound	332
hexadecimal tree	41
HIGHBIT	66
HIGHEXPBIT	66
HIGHMASK	66
HIGHVALPBIT	66
HIGHWORD	66
hilbertii	108
hnf	330
hnfall	330
hnfddivide	328
hnfl11	330
hnfmerge_get_1	304
hnfmod	330
hnfmodid	330
hnfperm	330
hnf_CENTER	327
hnf_divscale	328
hnf_invimage	328
hnf_invscale	328
hnf_MODID	327
hnf_PART	327
hnf_solve	328
hqfeval	241
hyperell_locally_soluble	353
h_APROPOS	59
h_LONG	58
h_REGULAR	58

## I

icopy	89
icopyifstack	73
icopyspec	89
icopy_avma	72
idealadd	303
idealaddmultoone	305
idealaddtoone	304, 305
idealaddtoone_i	304
idealaddtoone_raw	304
idealappr	305, 321
idealappr0	321
idealapprfact	305
idealchinese	305

idealchineseinit . . . . .	305	idealred0 . . . . .	315
idealcoprime . . . . .	305	idealred_elt . . . . .	315
idealcoprimefact . . . . .	305	idealsqr . . . . .	303
idealdiv . . . . .	303	idealstar . . . . .	310
idealdivexact . . . . .	303	idealstar0 . . . . .	320
idealdivpowprime . . . . .	304	Idealstarprk . . . . .	316
idealfactor . . . . .	302, 305, 306	ideals_by_norm . . . . .	304
idealfactor_limit . . . . .	306	idealtyp . . . . .	290
idealfactor_partial . . . . .	306	identity_perm . . . . .	255
idealfrobenius_aut . . . . .	307	identity_ZV . . . . .	180
idealhnf . . . . .	302, 303	identity_zv . . . . .	184
idealhnf0 . . . . .	303	id_MAT . . . . .	290
idealHNF_inv . . . . .	304	id_PRIME . . . . .	290
idealHNF_inv_Z . . . . .	304	id_PRINCIPAL . . . . .	290
idealHNF_mul . . . . .	304	ifac_isprime . . . . .	177
idealhnf_principal . . . . .	303	ifac_next . . . . .	177
idealhnf_shallow . . . . .	303	ifac_read . . . . .	177
idealhnf_two . . . . .	303	ifac_skip . . . . .	177
idealHNF_Z_factor . . . . .	306	ifac_start . . . . .	177
idealHNF_Z_factor_i . . . . .	306	image . . . . .	193
idealinv . . . . .	303	image2 . . . . .	193
ideallog . . . . .	302	imag_i . . . . .	243
ideallog_units . . . . .	317	indexlexsort . . . . .	231
ideallog_units0 . . . . .	317	indexpartial . . . . .	312
idealmoddivisor . . . . .	315	indexsort . . . . .	231
idealmul . . . . .	303	indexvecsort . . . . .	231
idealmulpowprime . . . . .	304	indices_to_vec01 . . . . .	309
idealmulred . . . . .	303, 312	infinity . . . . .	34
idealpow . . . . .	303	inf_get_sign . . . . .	34
idealpowred . . . . .	303	initprimes . . . . .	68
idealpows . . . . .	303	initprimetable . . . . .	68
idealprimedec . . . . .	302, 305, 306	init_Flxq . . . . .	141
idealprimedec_degrees . . . . .	306	init_Fq . . . . .	136
idealprimedec_galois . . . . .	306	init_primepointer_geq . . . . .	68
idealprimedec_kummer . . . . .	306	init_primepointer_gt . . . . .	68
idealprimedec_limit_f . . . . .	306	init_primepointer_leq . . . . .	68
idealprimedec_limit_norm . . . . .	306	init_primepointer_lt . . . . .	68
idealprincipalunits . . . . .	310	input . . . . .	37
idealprod . . . . .	304	install . . . . .	36, 41, 78, 80
idealprodprime . . . . .	304	int2n . . . . .	88
idealprodval . . . . .	304	int2u . . . . .	88
idealpseudomin . . . . .	314	int2um1 . . . . .	88
idealpseudominvec . . . . .	314	integer . . . . .	29
idealpseudomin_nonscalar . . . . .	314	integser . . . . .	246
idealpseudored . . . . .	314	int_LSW . . . . .	29
idealramfrobenius . . . . .	307	int_MSW . . . . .	29
idealramfrobenius_aut . . . . .	307	int_nextW . . . . .	29
idealramgroups_aut . . . . .	307	int_normalize . . . . .	29
idealred . . . . .	314, 315	int_precW . . . . .	29



int_W	29	is_recursive_t	65
int_W_lg	29	is_scalar_t	65
invmod	104	is_universal_constant	217
invmod2BIL	84	is_vec_t	65
invr	98	is_Z_factor	176
inv_content	236	is_Z_factornon0	176
isclone	28	is_Z_factorpos	176
iscomplex	230	itor	89
isexactzero	229	itos	27, 90, 217
isinexact	230	itostr	261
isinexactreal	230	itos_or_0	90
isint	230	itou	90, 217
isint1	229	itou_or_0	90
isintm1	229		
isintzero	229	<b>K</b>	
ismpzero	229	killblock	73
isonstack	73	krois	105
isprime	178	kroiui	105
isprimeAPRCL	178	Kronecker symbol	105
isprimeECPP	178	kronecker	105
isprimepower	108	Kronecker_to_F2xqX	155
isprincipal	313	Kronecker_to_FlxqX	148
isprincipalfact	313	Kronecker_to_FlxqX_pre	148
isprincipalfact_or_fail	313	Kronecker_to_FpXQX	133
isprincipalforce	321	Kronecker_to_mod	209
isprincipalgen	321	Kronecker_to_ZXQX	198
isprincipalgenforce	321	Kronecker_to_ZXX	198
isprincipalraygen	321	krosi	105
isrationalzero	229	kross	105
isrationalzeroscalar	230	kroui	105
isrealappr	230	krouu	105
issmall	230		
is_357_power	107, 172, 173	<b>L</b>	
is_bigint	90	lcmii	102
is_const_t	65	ldata_get_an	355
is_entry	75	ldata_get_conductor	355
is_extscalar_t	65	ldata_get_degree	355
is_gchar_group	336	ldata_get_dual	355
is_intreal_t	65	ldata_get_gammavec	355
is_linit	355	ldata_get_k	355
is_matvec_t	65	ldata_get_k1	355
is_nf_extfactor	290	ldata_get_residue	355
is_nf_factor	290	ldata_get_rootno	355
is_noncalc_t	65	ldata_get_type	355
is_pm1	230	ldata_isreal	355
is_pth_power	173	ldata_newprec	356
is_qfb_t	65	ldata_vecan	356
is_rational_t	65	leading_coeff	32, 65
is_real_t	65		



map_proto_GL . . . . .	106	mftocol . . . . .	363
map_proto_lG . . . . .	106	mfvecembed . . . . .	363
map_proto_lGL . . . . .	106	mfvectomat . . . . .	363
matbrute . . . . .	264	MF_get_basis . . . . .	361, 362
matdet . . . . .	183	MF_get_CHI . . . . .	361
mathnf . . . . .	302	mf_get_CHI . . . . .	362
matid . . . . .	222	MF_get_dim . . . . .	361
matid_F2m . . . . .	119	MF_get_E . . . . .	361
matid_F2xqM . . . . .	155	mf_get_field . . . . .	362
matid_Flm . . . . .	116	MF_get_fields . . . . .	361
matid_FlxqM . . . . .	122	MF_get_gk . . . . .	361
matpermanent . . . . .	183	mf_get_gk . . . . .	362
matrix . . . . .	34	MF_get_gN . . . . .	361
matrixqz . . . . .	330	mf_get_gN . . . . .	362
matslice . . . . .	275	MF_get_k . . . . .	361
maxdd . . . . .	94	mf_get_k . . . . .	362
maxomegaoddu . . . . .	107	MF_get_M . . . . .	362
maxomegau . . . . .	107	MF_get_Mindex . . . . .	362
maxprime . . . . .	13, 67	MF_get_Minv . . . . .	362
maxprimeN . . . . .	67	MF_get_N . . . . .	361
maxprime_check . . . . .	67	mf_get_N . . . . .	362
maxss . . . . .	94	MF_get_newforms . . . . .	361
maxuu . . . . .	94	mf_get_NK . . . . .	362
MAXVARN . . . . .	66	MF_get_r . . . . .	361
MEDDEFAULTPREC . . . . .	16, 66	mf_get_r . . . . .	362
merge_factor . . . . .	232	MF_get_S . . . . .	361, 362
merge_sort_uniq . . . . .	232	MF_get_space . . . . .	361
mfcharmodulus . . . . .	363	mf_get_type . . . . .	362
mfcharorder . . . . .	363	millerrabin . . . . .	178
mfcharpol . . . . .	363	mindd . . . . .	94
mfcuspdim . . . . .	363	minss . . . . .	94
MFcusp_get_vMjd . . . . .	362	minuu . . . . .	94
mfdiv_val . . . . .	363	mkcol . . . . .	224
mfeisensteindim . . . . .	363	mkcol2 . . . . .	224
mfeisensteinspaceinit . . . . .	363	mkcol2s . . . . .	223
mfembed . . . . .	363	mkcol3 . . . . .	224
mffulldim . . . . .	363	mkcol3s . . . . .	223
mfiscuspidal . . . . .	363	mkcol4 . . . . .	224
mfmatembed . . . . .	363	mkcol4s . . . . .	223
mfnewdim . . . . .	363	mkcol5 . . . . .	224
MFnew_get_vj . . . . .	362	mkcol6 . . . . .	224
mfnumcuspsu . . . . .	359	mkcolcopy . . . . .	223
mfnumcuspsu_fact . . . . .	359	mkcoln . . . . .	25, 226
mfnumcusps_fact . . . . .	359	mkcols . . . . .	223
mfolddim . . . . .	363	mkcomplex . . . . .	224
mfsturmNgk . . . . .	363	mkerr . . . . .	225
mfsturmNk . . . . .	363	mkfrac . . . . .	224
mfsturm_mf . . . . .	363	mkfraccopy . . . . .	223
mftobasisES . . . . .	363	mkfracss . . . . .	223

mkintmod	224	mod8	100
mkintmodu	222	modinv_good_disc	352
mkintn	25, 26, 90, 226	modinv_good_prime	352
mkmat	225	modinv_height_factor	352
mkmat2	225	modinv_is_double_eta	352
mkmat22	225	modinv_is_Weber	352
mkmat22s	223	modpr_genFq	308
mkmat3	225	modpr_get_p	307
mkmat4	225	modpr_get_pr	307
mkmat5	225	modpr_get_T	307
mkmatcopy	223	modreverse	208
mkmoo	34	modRr_safe	243
mkoo	34	moebiusu	107
mkpolmod	224	moebiusu_fact	107
mkpoln	25, 226	monomial_F2x	153
mkqfb	225	monomial_Flx	141
mkquad	224	mpabs	94
mkrrfrac	224	mpabs_shallow	94
mkrrfraccopy	223	mpadd	15
mkvec	225	mpaff	89
mkvec2	225	mpbern	255
mkvec2copy	223	mpceil	90
mkvec2s	223	mpcmp	94
mkvec3	225	mpcopy	89
mkvec3s	223	mpcos[z]	251
mkvec4	225	mpeint1	251
mkvec4s	223	mpeuler	255
mkvec5	225	mpexpm1	251
mkveccopy	223	mpexpo	62
mkvecn	25, 226	mpexp[z]	251
mkvecs	223	mpfloor	90
mkvecsmall	223	mplambertW	252
mkvecsmall12	223	mplambertX	252
mkvecsmall13	223	mplambertxlogx_x	252
mkvecsmall14	224	mplambertx_logx	252
mkvecsmall15	224	mplog2	255
mkvecsmalln	224	mplog[z]	251
Mod16	100	mpneg	94
mod16	101	mpodd	100
Mod2	100	mppi	255
mod2	100	mpround	91
mod2BIL	101	mpshift	91
Mod32	100	mpsincos	252
mod32	101	mpsincosm1	251
Mod4	100	mpsinhcosh	252
mod4	100	mpsin[z]	251
Mod64	100	mpsqr	94
mod64	101	mptrunc	91
Mod8	100	mpveceint1	251



<code>nfsign_from_logarch</code>	310	<code>nf_PARTIALFACT</code>	311
<code>nfsign_fu</code>	309	<code>nf_pV_to_prV</code>	306
<code>nfsign_tu</code>	310	<code>nf_rnfeq</code>	318, 319
<code>nfsign_units</code>	309	<code>nf_rnfeqsimple</code>	318
<code>nfsqr</code>	297	<code>nf_ROUND2</code>	311
<code>nfsqri</code>	298	<code>nf_to_Fp_coprime</code>	315
<code>nfsqsub</code>	297	<code>nf_to_Fq</code>	307, 308
<code>nftrace</code>	297	<code>nf_to_Fq_init</code>	307
<code>nftyp</code>	289	<code>nf_to_scalar_or_alg</code>	298
<code>nfval</code>	297, 308	<code>nf_to_scalar_or_basis</code>	297
<code>nfV_cxlog</code>	310	<code>nmV_chinese_center</code>	162
<code>nfV_to_FqV</code>	308	<code>nmV_chinese_center_tree</code>	162
<code>nfV_to_scalar_or_alg</code>	298	<code>nm_Z_mul</code>	171
<code>nfX_disc</code>	298	<code>nonsquare_Fl</code>	85
<code>nfX_resultant</code>	298	<code>normalizpol</code>	33, 219
<code>nfX_to_FqX</code>	308	<code>normalizpol_approx</code>	219
<code>nfX_to_monice</code>	298	<code>normalizpol_lg</code>	219
<code>nf_cxlog</code>	310	<code>normalizeser</code>	221
<code>nf_cxlog_normalize</code>	310	<code>normalize_frac</code>	64
<code>nf_deg1_prime</code>	306	<code>NO_VARIABLE</code>	32, 35, 62, 66
<code>nf_FORCE</code>	312	<code>numberofconjugates</code>	319
<code>nf_GEN</code>	312	<code>numdivu</code>	107
<code>nf_GENMAT</code>	312, 313	<code>numdivu_fact</code>	107
<code>nf_GEN_IF_PRINCIPAL</code>	312	<code>numerr_name</code>	271
<code>nf_get_allroots</code>	291	<code>numer_i</code>	235
<code>nf_get_degree</code>	291	<code>nv_fromdigits_2k</code>	92
<code>nf_get_diff</code>	291	<code>nxCV_chinese_center</code>	162
<code>nf_get_disc</code>	291	<code>nxCV_chinese_center_tree</code>	163
<code>nf_get_G</code>	291	<code>nxMV_chinese_center</code>	162
<code>nf_get_Gtwist</code>	314, 315, 332	<code>nxV_chinese_center</code>	162
<code>nf_get_Gtwist1</code>	314	<code>nxV_chinese_center_tree</code>	163
<code>nf_get_index</code>	291		
<code>nf_get_invzk</code>	291		
<code>nf_get_M</code>	291, 313		
<code>nf_get_pol</code>	291		
<code>nf_get_prec</code>	291		
<code>nf_get_r1</code>	291		
<code>nf_get_r2</code>	291		
<code>nf_get_ramified_primes</code>	291		
<code>nf_get_roots</code>	291		
<code>nf_get_roundG</code>	291, 314, 315		
<code>nf_get_sign</code>	291		
<code>nf_get_Tr</code>	291		
<code>nf_get_varn</code>	291		
<code>nf_get_zk</code>	291		
<code>nf_get_zkden</code>	291		
<code>nf_get_zkprimpart</code>	291		
<code>nf_hyperell_locally_soluble</code>	353		
<code>nf_nfzk</code>	319		
		<b>O</b>	
		<code>obj</code>	287
		<code>obj_check</code>	287
		<code>obj_checkbuild</code>	288
		<code>obj_checkbuild_padicprec</code>	288
		<code>obj_checkbuild_prec</code>	288
		<code>obj_checkbuild_realprec</code>	288
		<code>obj_free</code>	288, 341
		<code>obj_init</code>	287
		<code>obj_insert</code>	287, 288
		<code>obj_insert_shallow</code>	288
		<code>obj_reinit</code>	287
		<code>odd</code>	84
		<code>odd_prime_divisors</code>	174
		<code>omega</code>	176
		<code>omegau</code>	107

ONLY_DIVIDES	110, 205
ONLY_REM	110, 204
outmat	39
output	39
output	39, 41, 264
out_printf	263
out_putc	263
out_puts	263
out_term_color	264
out_vprintf	263

## P

p-adic number	31
padicprec	167
padicprec_relative	167
padic_to_Fl	168
padic_to_Fp	112
padic_to_Q	167
padic_to_Q_shallow	167
parfor	45
parforeach	46
parforeach_init	46
parforeach_next	46
parforeach_stop	46
parforprime	46
parforprimestep	46
parforprimestep_init	46
parforprime_init	46
parforprime_next	46
parforprime_stop	46
parforvec	46
parforvec_init	46
parforvec_next	46
parforvec_stop	46
parfor_init	45
parfor_next	45
parfor_stop	45
paricfg_buildinfo	81
paricfg_compileddate	81
paricfg_datadir	81
paricfg_gphelp	81
paricfg_mt_engine	81
paricfg_vcsversion	81
paricfg_version	81
paricfg_version_code	81
pariErr	263
PariOUT	262
pariOut	263

paristack_newsize	56
paristack_resize	55
paristack_setsize	55
parivstack_reset	55
parivstack_resize	56
pari_add_defaults_module	56
pari_add_function	56
pari_add_hist	59
pari_add_module	56
pari_alarm	58
pari_ask_confirm	58
pari_calloc	17
pari_CATCH	47
pari_CATCH_reset	47
pari_center	58
pari_close	53
pari_close_opts	55
pari_community	58
pari_compile_str	58
pari_daemon	55
pari_ENDCATCH	47
pari_err	34, 40, 47, 266, 286
pari_err2str	271
pari_errfile	263
pari_err_last	48
pari_err_TYPE	340
pari_fclose	265
pari_flush	39, 263
pari_fopen	265
pari_fopengz	265
pari_fopen_or_fail	265
pari_fprintf	39
pari_fread_chars	264
pari_free	17, 69
pari_get_hist	59
pari_get_histrttime	59
pari_get_histtime	59
pari_get_homedir	265
pari_histtime	59
pari_hit_return	58
pari_infile	58
pari_init	13, 14, 53
pari_init_opts	53
pari_init_primes	54, 55
pari_is_default	284
pari_is_dir	264
pari_is_file	264
pari_kernel_close	54
pari_kernel_init	54

pari_kernel_version . . . . .	81	pari_unlink . . . . .	264
pari_kill_plot_engine . . . . .	365	pari_var_close . . . . .	75
pari_last_was_newline . . . . .	263	pari_var_create . . . . .	75
pari_library_path . . . . .	58	pari_var_init . . . . .	75
pari_malloc . . . . .	17, 69, 269	pari_var_next . . . . .	75
pari_mt_close . . . . .	55	pari_var_next_temp . . . . .	75
pari_mt_init . . . . .	54	PARI_VERSION . . . . .	81
pari_nb_hist . . . . .	59	pari_version . . . . .	81
PARI_OLD_NAMES . . . . .	14	PARI_VERSION_SHIFT . . . . .	81
pari_outfile . . . . .	39, 263	pari_vfprintf . . . . .	40
PARI_plot . . . . .	365	pari_vprintf . . . . .	40
pari_plot_by_file . . . . .	367	pari_vsprintf . . . . .	40
pari_printf . . . . .	39, 40, 41, 76, 263, 264	pari_warn . . . . .	40
pari_print_version . . . . .	58	parser code . . . . .	79
pari_putc . . . . .	39, 76, 263	path_expand . . . . .	265
pari_puts . . . . .	39, 76, 263, 264	perm_commute . . . . .	255
pari_rand . . . . .	103	perm_conj . . . . .	255
pari_realloc . . . . .	17, 269	perm_cycles . . . . .	256
pari_realloc_ip . . . . .	17	perm_inv . . . . .	255
pari_RETRY . . . . .	47	perm_mul . . . . .	255
pari_safefopen . . . . .	265	perm_order . . . . .	256
pari_set_last_newline . . . . .	263	perm_orderu . . . . .	256
pari_set_plot_engine . . . . .	365	perm_pow . . . . .	255
pari_sighandler . . . . .	55	perm_powu . . . . .	256
pari_sig_init . . . . .	55	perm_sign . . . . .	256
pari_sp . . . . .	17	perm_sqr . . . . .	255
pari_sprintf . . . . .	39, 40, 261	perm_to_GAP . . . . .	256
pari_stackcheck_init . . . . .	55	perm_to_Z . . . . .	256
pari_stack_alloc . . . . .	274	pgener_Fl . . . . .	85
pari_stack_base . . . . .	274	pgener_Fl_local . . . . .	85
pari_stack_delete . . . . .	274	pgener_Fp . . . . .	105
pari_stack_init . . . . .	274	pgener_Fp_local . . . . .	106
pari_stack_new . . . . .	274	pgener_Zl . . . . .	85
pari_stack_pushp . . . . .	274	pgener_Zp . . . . .	105
pari_stdin_isatty . . . . .	265	Pi2n . . . . .	255
pari_str . . . . .	262	PiI2 . . . . .	255
pari_strdup . . . . .	261	PiI2n . . . . .	255
pari_strndup . . . . .	261	plotbox . . . . .	365
pari_thread_alloc . . . . .	371	plotclip . . . . .	365
pari_thread_close . . . . .	371	plotcolor . . . . .	365
pari_thread_free . . . . .	371	plotcopy . . . . .	365
pari_thread_init . . . . .	371	plotcursor . . . . .	365
pari_thread_start . . . . .	371	plotdraw . . . . .	365
pari_thread_valloc . . . . .	371	plotd . . . . .	365
pari_timer . . . . .	41	plotdraw . . . . .	365
pari_TRY . . . . .	47	plotd . . . . .	365
pari_unique_dir . . . . .	266	plotd . . . . .	365
pari_unique_filename . . . . .	266	plotd . . . . .	365
pari_unique_filename_suffix . . . . .	266	plotd . . . . .	365



plotlines	365	polx_zx	201
plotlinetype	366	polynomial	32
plotmove	366	pol_0	221
plotpoints	366	pol_1	221
plotpointsize	366	pol_x	221
plotpointtype	366	pol_xn	222
plotrbox	366	pol_xnall	222
plotrecth	365	pol_x_powers	222
plotrecthraw	366	pop_lex	78, 282
plotrline	366	pow2Pis	253
plotrmove	366	powcx	253
plotrpoint	366	powcx_prec	253
plotscale	366	power series	33
plotstring	366	powersr	101
point_to_a4a6	341	powgi	238
point_to_a4a6_Fl	341	powii	101
pol0_F2x	152	powis	101
pol0_Flx	141	powIs	101
pol1_F2x	152	powiu	101
pol1_F2xX	155	powPis	253
pol1_Flx	141	powfrac	101
pol1_FlxX	146	powrs	101
polclass	352	powrshalf	101
polcoef_i	244	powru	101
poldivrem	234	powruhalf	101
poleval	204, 240	powuu	101
polgalois	259	ppg	175
polhensellift	164, 166	ppi	175
polintspec	188	pple	175
polint_i	188	ppo	175
pollegendre_reduced	245	prec2nbits	61
polmod	32	prec2nbits_mul	61
polmodular	352	prec2ndec	61
polmodular_ZM	352	precdbl	61
polmodular_ZXX	352	precision	63
polmod_nffix	319	precision0	219
polmod_nffix2	319	precpr	31, 62
polmod_to_embed	245	PRECPBITS	66
Polred	321	PRECPSHIFT	66
polred0	321	preferences file	79
polredabs	321	<i>prid</i>	305
polredabs2	321	prime	178
polredabsall	321	primeform	325
Polrev	221	primeform_u	325
polxn_Flx	141	primepi_lower_bound	178
polx_F2x	153	primepi_upper_bound	177, 178
polx_F2xX	155	primes	178
polx_Flx	141	primes_interval	178
polx_FlxX	146	primes_interval_zv	178

primes_upto_zv . . . . .	178	qfbcompraw_i . . . . .	324
primes_zv . . . . .	178	qfbcomp_i . . . . .	324
prime_fact . . . . .	222	qfbforms . . . . .	244
primitive root . . . . .	85	qfbpow . . . . .	324
primitive_part . . . . .	235	qfbpowraw . . . . .	325
primpart . . . . .	235	qfbpows . . . . .	324
printf . . . . .	39, 76	qfbpow_i . . . . .	325
print_fun_list . . . . .	59	qfbred . . . . .	324
prV_lcm_capZ . . . . .	306	qfbred_i . . . . .	324
prV_primes . . . . .	307	qfbsolve . . . . .	325
pr_basis_perm . . . . .	308	qfbsqr . . . . .	324
pr_equal . . . . .	309	qfbsqr_i . . . . .	324
pr_get_e . . . . .	305	qfb_1 . . . . .	324
pr_get_f . . . . .	305	qfb_apply_ZM . . . . .	244
pr_get_gen . . . . .	305	qfb_disc . . . . .	244
pr_get_p . . . . .	305	qfb_disc3 . . . . .	244
pr_get_tau . . . . .	305	qfb_equal1 . . . . .	324
pr_hnf . . . . .	305	qfeval . . . . .	241
pr_inv . . . . .	305	qfevalb . . . . .	241
pr_inv_p . . . . .	305	qfiseven . . . . .	189
pr_is_inert . . . . .	305	qfisolvep . . . . .	325
pr_norm . . . . .	305	qfi_log . . . . .	325
pr_uniformizer . . . . .	307	qfi_order . . . . .	325
psdraw . . . . .	366	qfi_Shanks . . . . .	325
psilseries . . . . .	246	qfl1l10 . . . . .	331
psplot . . . . .	366	qfl1l1gram0 . . . . .	331
psplotdraw . . . . .	366	qfr3 . . . . .	326
pthread_join . . . . .	371	qfr3_comp . . . . .	326
push_lex . . . . .	78, 282	qfr3_compraw . . . . .	326
putch . . . . .	262	qfr3_pow . . . . .	326
puts . . . . .	262	qfr3_red . . . . .	326
p_to_FF . . . . .	247	qfr3_rho . . . . .	326
p_to_FF(p,0) . . . . .	247	qfr3_to_qfr . . . . .	326
		qfr5 . . . . .	326
		qfr5_comp . . . . .	326
		qfr5_compraw . . . . .	326
		qfr5_dist . . . . .	327
		qfr5_pow . . . . .	326
		qfr5_red . . . . .	326
		qfr5_rho . . . . .	326
		qfr5_to_qfr . . . . .	327
		qfrsolvep . . . . .	325
		qfr_data_init . . . . .	326
		qfr_to_qfr5 . . . . .	327
		qf_apply_RgM . . . . .	241
		qf_apply_ZM . . . . .	241
		QM_charpoly_ZX . . . . .	184
		QM_charpoly_ZX_bound . . . . .	184
		QM_det . . . . .	186

  

Q	
QabM_tracerel . . . . .	300
QabV_tracerel . . . . .	300
Qab_tracerel . . . . .	300
Qab_trace_init . . . . .	300
Qdivii . . . . .	235
Qdivis . . . . .	235
Qdiviu . . . . .	235
Qevproj_apply . . . . .	184
Qevproj_apply_vecei . . . . .	184
Qevproj_down . . . . .	184
Qevproj_init . . . . .	184
qfbcomp . . . . .	324
qfbcompraw . . . . .	324

QM_gauss	184	QXQC_to_mod_shallow	159
QM_gauss_i	184	QXQM_mul	200
QM_image	184	QXQM_sqr	200
QM_image_shallow	184	QXQM_to_mod_shallow	159
QM_ImQ	330	QXQV_to_FpM	308
QM_ImQ_all	330	QXQV_to_mod	158
QM_ImQ_hnf	330	QXQXV_to_mod	159
QM_ImQ_hnfall	330	QXQX_gcd	200
QM_ImZ	330	QXQX_homogenous_evalpow	200
QM_ImZ_all	330	QXQX_mul	200
QM_ImZ_hnf	330	QXQX_powers	200
QM_ImZ_hnfall	330	QXQX_QXQ_mul	200
QM_indexrank	184	QXQX_sqr	200
QM_inv	184	QXQX_to_mod_shallow	158
QM_ker	186	QXQ_charpoly	200
QM_minors_coprime	330	QXQ_div	200
QM_mul	186	QXQ_intnorm	199
QM_QC_mul	186	QXQ_inv	199
QM_rank	184	QXQ_mul	199
QM_sqr	186	QXQ_mul(A,QXQ_inv(B,T),T)	200
QpV_to_QV	167	QXQ_norm	199
Qp_agm2_sequence	254	QXQ_powers	200
Qp_ascending_Landen	254	QXQ_reverse	200
Qp_descending_Landen	254	QXQ_sqr	199
Qp_exp	254	QXQ_to_mod_shallow	158
Qp_exp_prec	254	QXV_QXQ_eval	200
Qp_gamma	254	QXY_QXQ_evalx	200
Qp_log	254	QX_complex_roots	199, 245
Qp_sqrt	254	QX_disc	199
Qp_sqrtn	254	QX_factor	199
Qp_zeta	254	QX_gcd	199
QR_init	190	QX_mul	199
Qtoss	223	QX_resultant	199
quadclassno	323	QX_sqr	199
quadclassnoF	323	QX_ZXQV_eval	200
quadclassnoF_fact	323	QX_ZX_rem	199
quadclassnos	323	Q_abs	235
quadnorm	244	Q_abs_shallow	235
quadpoly	32	Q_content	236
quadpoly_i	224	Q_content_safe	236
quadratic number	32	Q_denom	236
quadratic_prec_mask	167	Q_denom_safe	236
quadtofp	217	Q_div_to_int	236
quad_disc	244	Q_factor	175
quotient_group	258	Q_factor_limit	175
quotient_groupelts	258	Q_gcd	235
quotient_perm	258	Q_lval	235
quotient_subgroup_lift	258	Q_lvalrem	235
QV_isscalar	189	Q_muli_to_int	236

Q_mul_to_int	236
Q_primitive_part	236
Q_primpart	236
Q_pval	235
Q_pvalrem	235
Q_remove_denom	236, 296

## R

radicalu	107
random	103
randomi	103
randomr	103
random_bits	103
random_F2x	153
random_F2xqE	349
random_F2xqX	156
random_Fl	85, 103
random_Fle	348
random_Fle_pre	348
random_Flj_pre	348
random_Flv	116
random_Flx	141
random_FlxqE	350
random_FlxqX	148
random_FpC	113
random_FpE	346
random_FpV	113
random_FpX	126
random_FpXQE	351
random_FpXQX	133
random_zv	184
rational function	34
rational number	31
raw	261
rcopy	89
rdivii	99
rdiviiz	99
rdivis	99
rdivsi	99
rdivss	99
read	38
readseq	38
real number	30
real2n	88
realprec	61
real_0	88
real_0_bit	88
real_1	88

real_1_bit	88
real_i	243
real_m1	88
real_m2n	88
rect2ps	366
rect2ps_i	366
rect2svg	367
reducemodinvertible	333
reducemodlll	334
remi2n	98, 194
remlll_pre	86
remll_pre	86
remsBIL	66
residual_characteristic	245
<i>resultant (reduced)</i>	165
resultant	234, 245
resultant2	245
retconst_col	226
retconst_vec	225
retmkcol	226
retmkcol2	226
retmkcol3	226
retmkcol4	226
retmkcol5	226
retmkcol6	226
retmkcomplex	226
retmkfrac	226
retmkintmod	226
retmkmat	226
retmkmat2	226
retmkmat3	226
retmkmat4	226
retmkmat5	226
retmkpolmod	226
retmkquad	226
retmkfrac	226
retmkvec	225
retmkvec2	225
retmkvec3	225
retmkvec4	225
retmkvec5	225
rfracrecip	202
rfracrecip_to_ser_absolute	221
rfrac_deflate	202
rfrac_deflate_max	202
rfrac_deflate_order	202
rfrac_to_ser	220
rfrac_to_ser_i	221
RgC_add	186

RgC_fpnorml2	189	RgM_mul	187
RgC_gtofp	189	RgM_mulreal	187
RgC_gtomp	189	RgM_multosym	187
RgC_is_ei	189	RgM_neg	186
RgC_is_FFC	247	RgM_powers	187
RgC_neg	186	RgM_QR_init	190
RgC_RgM_mul	187	RgM_rescale_to_int	182
RgC_RgV_mul	187	RgM_RgC_invmage	190
RgC_RgV_mulrealsym	187	RgM_RgC_mul	187
RgC_Rg_add	186	RgM_RgC_type	112
RgC_Rg_div	187	RgM_RgV_mul	187
RgC_Rg_mul	187	RgM_RgX_mul	187
RgC_Rg_sub	186	RgM_Rg_add	186
RgC_sub	186	RgM_Rg_add_shallow	186
RgC_to_FpC	112	RgM_Rg_div	187
RgC_to_FqC	115	RgM_Rg_mul	187
RgC_to_nfC	298	RgM_Rg_sub	186
RgE_to_F2xqE	349	RgM_Rg_sub_shallow	186
RgE_to_FlxqE	350	RgM_shallowcopy	275
RgE_to_FpE	347	RgM_solve	190
RgE_to_FpXQE	352	RgM_solve_realimag	190
RgMrow_RgC_mul	187	RgM_sqr	187
RgMrow_zc_mul	171	RgM_sub	186
RgMs_structelim	192	RgM_sumcol	188
RgM_add	186	RgM_to_F2m	120
RgM_Babai	191	RgM_to_F3m	121
RgM_check_ZM	181	RgM_to_Flm	169
RgM_det_triangular	190	RgM_to_FpM	113
RgM_diagonal	189	RgM_to_FqM	115
RgM_diagonal_shallow	189	RgM_to_nfM	298
RgM_dimensions	186	RgM_to_RgXV	220
RgM_fpnorml2	189, 239	RgM_to_RgXV_reverse	220
RgM_Fp_init	113	RgM_to_RgXX	220
RgM_gram_schmidt	191	RgM_transmul	187
RgM_gtofp	189	RgM_transmultosym	187
RgM_gtomp	189, 190	RgM_type	112
RgM_Hadamard	190	RgM_type2	112
RgM_hnfall	331	RgM_zc_mul	171
RgM_inv	190	RgM_zm_mul	171
RgM_invmage	190	RgM_ZM_mul	187
RgM_inv_upper	190	RgV_add	186
RgM_isdiagonal	189	RgV_check_ZV	180
RgM_isidentity	189	RgV_dotproduct	188
RgM_isscalar	189	RgV_dotsquare	188
RgM_is_FFM	247	RgV_gtofp	189
RgM_is_FpM	112	RgV_isin	189
RgM_is_QM	189	RgV_isin_i	189
RgM_is_ZM	189	RgV_isscalar	188
RgM_minor	275	RgV_is_arithprog	180

RgV_is_FpV . . . . .	112	RgXQC_red . . . . .	208
RgV_is_prV . . . . .	290	RgXQM_mul . . . . .	209
RgV_is_QV . . . . .	180	RgXQM_red . . . . .	209
RgV_is_ZMV . . . . .	185	RgXQV_factorback . . . . .	209
RgV_is_ZV . . . . .	180	RgXQV_red . . . . .	209
RgV_is_ZVnon0 . . . . .	180	RgXQV_RgXQ_mul . . . . .	209
RgV_is_ZVpos . . . . .	180	RgXQX_div . . . . .	209
RgV_kill0 . . . . .	188	RgXQX_divrem . . . . .	209
RgV_neg . . . . .	186	RgXQX_mul . . . . .	209
RgV_nffix . . . . .	319	RgXQX_powers . . . . .	209
RgV_polint . . . . .	188	RgXQX_pseudodivrem . . . . .	205
RgV_prod . . . . .	187	RgXQX_pseudorem . . . . .	205
RgV_RgC_mul . . . . .	187	RgXQX_red . . . . .	209
RgV_RgM_mul . . . . .	187	RgXQX_rem . . . . .	209
RgV_Rg_mul . . . . .	187	RgXQX_RgXQ_mul . . . . .	209
RgV_sub . . . . .	186	RgXQX_sqr . . . . .	209
RgV_sum . . . . .	187	RgXQX_translate . . . . .	209
RgV_sumpart . . . . .	187	RgXQ_charpoly . . . . .	208
RgV_sumpart2 . . . . .	187	RgXQ_inv . . . . .	208
RgV_to_F2v . . . . .	120	RgXQ_matrix_pow . . . . .	208
RgV_to_F3v . . . . .	121	RgXQ_minpoly . . . . .	208
RgV_to_Flv . . . . .	169	RgXQ_mul . . . . .	208
RgV_to_FpV . . . . .	112	RgXQ_norm . . . . .	208
RgV_to_RgM . . . . .	220	RgXQ_pow . . . . .	208
RgV_to_RgX . . . . .	219	RgXQ_powers . . . . .	208
RgV_to_RgX_reverse . . . . .	220	RgXQ_powu . . . . .	208
RgV_to_ser . . . . .	220	RgXQ_ratlift . . . . .	208
RgV_to_str . . . . .	261, 262	RgXQ_reverse . . . . .	208
RgV_type . . . . .	112	RgXQ_sqr . . . . .	208
RgV_type2 . . . . .	112	RgXQ_trace . . . . .	208
RgV_zc_mul . . . . .	171	RgXV_maxdegree . . . . .	203
RgV_zm_mul . . . . .	171	RgXV_prod . . . . .	204
RgXnV_red_shallow . . . . .	208	RgXV_RgV_eval . . . . .	204
RgXn_div . . . . .	207	RgXV_to_FlxV . . . . .	169
RgXn_div_i . . . . .	207	RgXV_to_RgM . . . . .	220
RgXn_eval . . . . .	207	RgXV_unscale . . . . .	206
RgXn_exp . . . . .	207	RgXX_to_Kronecker . . . . .	133, 198, 203
RgXn_expint . . . . .	207	RgXX_to_Kronecker_spec . . . . .	203
RgXn_inv . . . . .	207	RgXX_to_RgM . . . . .	220
RgXn_inv_i . . . . .	207	RgXY_cxevalx . . . . .	240
RgXn_mul . . . . .	207	RgXY_degreeex . . . . .	220
RgXn_powers . . . . .	207	RgXY_derivx . . . . .	220
RgXn_powu . . . . .	207	RgXY_swap . . . . .	220
RgXn_powu_i . . . . .	207	RgXY_swapspec . . . . .	220
RgXn_recip_shallow . . . . .	207	RgX_act_G12Q . . . . .	206
RgXn_red_shallow . . . . .	207	RgX_act_ZG12Q . . . . .	207
RgXn_reverse . . . . .	207	RgX_add . . . . .	203
RgXn_sqr . . . . .	207	RgX_addmulXn . . . . .	205
RgXn_sqrt . . . . .	207	RgX_addmulXn_shallow . . . . .	205

RgX_addspec	205	RgX_mul_i	204
RgX_addspec_shallow	205	RgX_mul_normalized	204
RgX_affine	206	RgX_neg	203
RgX_blocks	142, 147, 201	RgX_nffix	319
RgX_check_QX	199	RgX_normalize	204
RgX_check_ZX	193	RgX_pseudodivrem	205
RgX_check_ZXX	198	RgX_pseudorem	205
RgX_chinese_coprime	206	RgX_recip	202
RgX_coeff	201	RgX_recip_i	202
RgX_copy	201	RgX_recip_shallow	202
RgX_cxeval	240	RgX_rem	205
RgX_deflate	202	RgX_renormalize	202
RgX_deflate_max	202	RgX_renormalize_lg	202
RgX_deflate_order	202	RgX_rescale	206
RgX_degree	201	RgX_rescale_to_int	202
RgX_deriv	206	RgX_resultant_all	206
RgX_digits	205	RgX_RgMV_eval	241
RgX_disc	206	RgX_RgM_eval	240
RgX_div	205	RgX_RgV_eval	204
RgX_divrem	204	RgX_RgXnV_eval	207
RgX_divs	204	RgX_RgXn_eval	207
RgX_div_by_X_x	205	RgX_RgXQV_eval	208
RgX_equal	201	RgX_RgXQ_eval	207, 208
RgX_equal_var	201	RgX_Rg_add	203
RgX_even_odd	153, 201	RgX_Rg_add_shallow	203
RgX_extgcd	206	RgX_Rg_div	204
RgX_extgcd_simple	206	RgX_Rg_divexact	204
RgX_fpnorml2	206	RgX_Rg_eval_bk	204
RgX_gcd	205, 206	RgX_Rg_mul	204
RgX_gcd_simple	206	RgX_Rg_sub	203
RgX_gtofp	206	RgX_Rg_type	111
RgX_halfgcd	206	RgX_rotate_shallow	203
RgX_homogenize	202	RgX_shift	153, 203
RgX_homogenous_evalpow	202	RgX_shift_inplace	203
RgX_inflate	202	RgX_shift_inplace_init	203
RgX_integ	206	RgX_shift_shallow	203
RgX_isscalar	201	RgX_splitting	142, 201
RgX_is_FpX	122	RgX_sqr	204
RgX_is_FpXQX	127	RgX_sqrhigh_i	207
RgX_is_monomial	201	RgX_sqrspec	205
RgX_is_QX	201	RgX_sqr_i	204
RgX_is_rational	201	RgX_sub	203
RgX_is_ZX	201	RgX_sylvestermatrix	227
RgX_mul	204	RgX_to_F2x	168
RgX_mul2n	204	RgX_to_Flx	169
RgX_mulhigh_i	207	RgX_to_FlxqX	169
RgX_muls	204	RgX_to_FpX	122
RgX_mulspec	205	RgX_to_FpXQX	128
RgX_mulXn	205	RgX_to_FqX	128







scalarcol_shallow . . . . .	227	sd_realprecision . . . . .	285
scalarmat . . . . .	223	sd_recover . . . . .	285
scalarmat_s . . . . .	223	sd_secure . . . . .	285
scalarmat_shallow . . . . .	227	sd_seriesprecision . . . . .	285
scalarpol . . . . .	222	sd_simplify . . . . .	285
scalarpol_shallow . . . . .	227	sd_sopath . . . . .	285
scalarser . . . . .	221	sd_strictargs . . . . .	285
scalar_Flm . . . . .	116	sd_strictmatch . . . . .	285
scalar_ZX . . . . .	193	sd_string . . . . .	287
scalar_ZX_shallow . . . . .	193	sd_TeXstyle . . . . .	284
sdivsi . . . . .	99	sd_threadsize . . . . .	285
sdivsi_rem . . . . .	99	sd_threadsizemax . . . . .	285
sdivss_rem . . . . .	99	sd_timer . . . . .	285
sdomain_isincl . . . . .	356	sd_toggle . . . . .	286
sd_breakloop . . . . .	284	sd_ulong . . . . .	286
sd_colors . . . . .	284	secure . . . . .	57
sd_compatible . . . . .	284	serchop0 . . . . .	221
sd_datadir . . . . .	284	serchop_i . . . . .	221
sd_debug . . . . .	284	sertoser . . . . .	246
sd_debugfiles . . . . .	284	ser_inv . . . . .	246
sd_debugmem . . . . .	284	ser_isexactzero . . . . .	246
sd_echo . . . . .	284	ser_normalize . . . . .	246
sd_factor_add_primes . . . . .	284	ser_unscale . . . . .	246
sd_factor_proven . . . . .	284	setabssign . . . . .	64
sd_format . . . . .	284	setalldebug . . . . .	41
sd_graphcolormap . . . . .	284	setdefault . . . . .	56, 284
sd_graphcolors . . . . .	284	setexpo . . . . .	30, 33, 64
sd_help . . . . .	285	setisclone . . . . .	28
sd_histfile . . . . .	285	setlg . . . . .	28, 64
sd_histsize . . . . .	285	setlgfint . . . . .	29, 64
sd_intarray . . . . .	286	setprec . . . . .	31, 64
sd_lines . . . . .	285	setrand . . . . .	103
sd_linewrap . . . . .	285	setrealprecision . . . . .	251
sd_log . . . . .	285	setsigne . . . . .	29, 32, 33, 64
sd_logfile . . . . .	285	settyp . . . . .	28, 64
sd_nbthreads . . . . .	285	setunion_i . . . . .	232
sd_new_galois_format . . . . .	285	setvalp . . . . .	31, 33, 64
sd_output . . . . .	285	setvarn . . . . .	25, 32, 33, 64, 226
sd_parisize . . . . .	285	set_avma . . . . .	70
sd_parisizemax . . . . .	285	set_lex . . . . .	282
sd_path . . . . .	285	set_sign_mod_divisor . . . . .	310
sd_plothsizes . . . . .	285	<i>shallow</i> . . . . .	53
sd_prettyprinter . . . . .	285	shallowconcat . . . . .	275
sd_primelimit . . . . .	285	shallowconcat1 . . . . .	275
sd_prompt . . . . .	285	shallowcopy . . . . .	27, 275
sd_prompt_cont . . . . .	285	shallowextract . . . . .	275
sd_psfile . . . . .	285	shallowmatconcat . . . . .	275
sd_readline . . . . .	285	shallowmatextract . . . . .	275
sd_realbitprecision . . . . .	285	shallowtrans . . . . .	275









ZabM_inv	300	zero_Flx	141
ZabM_inv_ratlift	300	zero_FlxC	143
ZabM_ker	300	zero_FlxM	143
ZabM_pseudoinv	300	zero_zm	185
zCs_to_ZC	192	zero_zv	185
ZC_add	180	zero_zx	201
ZC_copy	180	ZGCs_add	192
ZC_divexactu	180	ZGC_G_mul	191
ZC_hnfrem	333	ZGC_G_mul_inplace	191
ZC_hnfremdiv	333	ZGC_Z_mul	191
ZC_is_ei	184	ZG_add	191
ZC_lincomb	181	ZG_G_mul	191
ZC_lincomb1_inplace	181	ZG_mul	191
ZC_lincomb1_inplace_i	181	ZG_neg	191
ZC_neg	180	ZG_normalize	191
ZC_nfval	308	ZG_sub	191
ZC_nfvalrem	308	ZG_Z_mul	191
ZC_prdvd	309	Zideallog	335
ZC_Q_mul	183	zidealstar	320
ZC_reducemodlll	333	zidealstarinit	320
ZC_reducemodmatrix	333	zidealstarinitgen	320
ZC_sub	180	zkchinese	304
zc_to_ZC	170	zkchinese1	304
ZC_union_shallow	181	zkchineseinit	304
ZC_ZV_mul	181	zkC_multable_mul	299
ZC_Z_add	180	zkmodprinit	308
ZC_Z_div	181	zkmultable_capZ	299
ZC_Z_divexact	180	zkmultable_inv	299
ZC_z_mul	171	zk_inv	299
ZC_Z_mul	180	zk_multable	299, 303
ZC_Z_sub	180	zk_scalar_or_multable	299, 305
zerocol	222	zk_to_Fq	308
zeromat	222	zk_to_Fq_init	308
zeromatcopy	222	zlm_echelon	167
zeropadic	221	Zlm_gauss	167
zeropadic_shallow	227	zlxX_translate1	147
zeropol	222	zlx_translate1	139
zeroser	221	ZM2_mul	182
zerovec	222	ZMrow_equal0	182
zerovec_block	222	ZMrow_ZC_mul	182
zero_F2m	119	zMs_to_ZM	192
zero_F2m_copy	120	zMs_ZC_mul	192
zero_F2v	119	ZMV_to_FlmV	185
zero_F2x	152	ZMV_to_zmV	185
zero_F3m_copy	121	zmV_to_ZMV	185
zero_F3v	121	ZM_add	182
zero_Flm	117	ZM_charpoly	183
zero_Flm_copy	117	ZM_copy	182
zero_Flv	117	zm_copy	185

ZM_det	183	ZM_pseudoinv	183
ZM_detmult	183	ZM_Q_mul	183
ZM_det_triangular	183	ZM_rank	183
ZM_diag_mul	182	ZM_reducemodlll	333
ZM_divexactu	182	ZM_reducemodmatrix	333
ZM_equal	182	zm_row	185
ZM_equal0	182	ZM_snf	328
ZM_famat_limit	301	ZM_snfall	328
ZM_gauss	183	ZM_snfall_i	329
ZM_hnf	327, 330	ZM_snfclean	329
ZM_hnfall	327, 328, 330, 331	ZM_snf_group	329
ZM_hnfall_i	328	ZM_sqr	182
ZM_hnfcenter	328	ZM_sub	182
ZM_hnfdivrem	333	ZM_supnorm	183, 239
ZM_hnflll	328	ZM_togglesign	182
ZM_hnfmod	327, 330	ZM_to_F2m	120
ZM_hnfmodall	327	ZM_to_F3m	121
ZM_hnfmodall_i	327	ZM_to_Flm	169
ZM_hnfmodid	327, 330	zm_to_Flm	170
ZM_hnfmodprime	327	ZM_to_zm	170
ZM_hnfperm	328	zm_to_ZM	170
ZM_hnfrem	333	zm_to_zxV	171
ZM_hnf_knapsack	328	ZM_transmul	182
ZM_imagecompl	183	ZM_transmultosym	182
ZM_incremental_CRT	160	zm_transpose	185
ZM_indeximage	183	ZM_zc_mul	171
ZM_indexrank	183	ZM_ZC_mul	182
ZM_init_CRT	160	zm_zc_mul	185
ZM_inv	183	ZM_zm_mul	171
ZM_inv_ratlift	183	ZM_ZV_mod	182
ZM_isdiagonal	183	ZM_ZX_mul	182
ZM_ishnf	184	ZM_Z_div	182
ZM_isidentity	183	ZM_Z_divexact	182
ZM_isscalar	183	ZM_Z_mul	182
ZM_ker	183	zncharcheck	335
ZM_lll	331, 332	zncharconj	335
ZM_lll_norms	332	znchardiv	335
ZM_max_lg	183	znchareval	335
ZM_mul	182	zncharker	335
zm_mul	185	zncharmul	335
ZM_multosym	182	zncharorder	335
ZM_mul_diag	182	zncharpow	335
ZM_neg	182	znchar_quad	335
ZM_nm_mul	171	znconreyfromchar	335
ZM_nv_mod_tree	161	znconreyfromchar_normalized	335
ZM_permanent	183	znconreylog_normalize	335
zm_permanent	185	znconrey_check	335
ZM_pow	183	znconrey_normalized	335
ZM_powu	183	znstar_get_conreycyc	295

znstar_get_conreygen . . . . .	295	Zp_issquare . . . . .	105
znstar_get_cyc . . . . .	295	Zp_log . . . . .	164
znstar_get_faN . . . . .	295	Zp_sqrt . . . . .	163
znstar_get_gen . . . . .	295	Zp_sqrtlift . . . . .	164
znstar_get_N . . . . .	295	Zp_sqrtnlift . . . . .	164
znstar_get_no . . . . .	295	Zp_teichmuller . . . . .	164
znstar_get_pe . . . . .	295	ZqX_liftfact . . . . .	167
znstar_get_U . . . . .	295	ZqX_liftroot . . . . .	167
znstar_get_Ui . . . . .	295	ZqX_roots . . . . .	167
Zn_ispower . . . . .	105	ZqX_ZqXQ_liftroot . . . . .	167
Zn_issquare . . . . .	105	Zq_sqrtnlift . . . . .	166
Zn_quad_roots . . . . .	105	zvV_equal . . . . .	185
Zn_sqrt . . . . .	105	zv_abs . . . . .	184
ZpMs_ZpCs_solve . . . . .	193	ZV_abscmp . . . . .	180
ZpM_echelon . . . . .	167	ZV_allpnqn . . . . .	102
ZpM_invlift . . . . .	164	ZV_cba . . . . .	176
ZpXQM_prodFrobenius . . . . .	166	ZV_cba_extend . . . . .	175
ZpXQX_digits . . . . .	167	ZV_chinese . . . . .	161
ZpXQX_divrem . . . . .	167	ZV_chinesetree . . . . .	162
ZpXQX_liftfact . . . . .	166	ZV_chinese_center . . . . .	161
ZpXQX_liftroot . . . . .	166, 167	ZV_chinese_tree . . . . .	162
ZpXQX_liftroots . . . . .	166	ZV_cmp . . . . .	180, 233
ZpXQX_liftroot_vald . . . . .	166	zv_cmp0 . . . . .	185
ZpXQX_roots . . . . .	166	ZV_content . . . . .	181
ZpXQX_ZpXQXQ_liftroot . . . . .	167	zv_content . . . . .	185
ZpXQ_div . . . . .	165	zv_copy . . . . .	185
ZpXQ_inv . . . . .	165	zv_cyc_minimal . . . . .	335
ZpXQ_invlift . . . . .	165	zv_cyc_minimize . . . . .	335
ZpXQ_log . . . . .	166	zv_diagonal . . . . .	185
ZpXQ_sqrt . . . . .	166	ZV_dotproduct . . . . .	181
ZpXQ_sqrtnlift . . . . .	165	zv_dotproduct . . . . .	185
ZpX_disc_val . . . . .	165	ZV_dotsquare . . . . .	181
ZpX_Frobenius . . . . .	165	ZV_dvd . . . . .	181
ZpX_gcd . . . . .	165	ZV_equal . . . . .	180
ZpX_liftfact . . . . .	164	zv_equal . . . . .	185
ZpX_liftroot . . . . .	164, 166	ZV_equal0 . . . . .	180
ZpX_liftroots . . . . .	164	zv_equal0 . . . . .	185
ZpX_monic_factor . . . . .	165	ZV_extgcd . . . . .	102, 181
ZpX_primedec . . . . .	165	ZV_indexsort . . . . .	181
ZpX_reduced_resultant . . . . .	165	ZV_isscalar . . . . .	189
ZpX_reduced_resultant_fast . . . . .	165	ZV_lcm . . . . .	102
ZpX_resultant_val . . . . .	165	ZV_lval . . . . .	93
ZpX_roots . . . . .	164	ZV_lvalrem . . . . .	93
ZpX_ZpXQ_liftroot . . . . .	166	ZV_max_lg . . . . .	181
ZpX_ZpXQ_liftroot_ea . . . . .	166	zv_neg . . . . .	184
Zp_div . . . . .	163	ZV_neg_inplace . . . . .	180
Zp_exp . . . . .	164	zv_neg_inplace . . . . .	185
Zp_inv . . . . .	163	ZV_nv_mod_tree . . . . .	161
Zp_invlift . . . . .	163	ZV_prod . . . . .	181



zv_prod . . . . .	185	ZXQM_sqr . . . . .	198
ZV_producttree . . . . .	160, 162	ZXQX_dvd . . . . .	205
zv_prod_Z . . . . .	185	ZXQX_gcd . . . . .	198
ZV_pval . . . . .	93	ZXQX_mul . . . . .	198
ZV_pvalrem . . . . .	93	ZXQX_sqr . . . . .	198
ZV_search . . . . .	181	ZXQX_ZXQ_mul . . . . .	198
zv_search . . . . .	185	ZXQ_charpoly . . . . .	197
ZV_snfall . . . . .	328	ZXQ_minpoly . . . . .	197
ZV_snfclean . . . . .	329	ZXQ_mul . . . . .	197
ZV_snf_gcd . . . . .	102, 329	ZXQ_powers . . . . .	197
ZV_snf_group . . . . .	329	ZXQ_powu . . . . .	197
ZV_snf_rank . . . . .	330	ZXQ_sqr . . . . .	197
zv_snf_rank . . . . .	330	ZXT_remi2n . . . . .	197
ZV_snf_rank_u . . . . .	330	ZXT_to_FlxT . . . . .	169
ZV_snf_trunc . . . . .	329	ZXV_dotproduct . . . . .	197
ZV_sort . . . . .	181	ZXV_equal . . . . .	197
ZV_sort_inplace . . . . .	181	ZXV_remi2n . . . . .	197
ZV_sort_shallow . . . . .	181	ZXV_to_FlxV . . . . .	169
ZV_sort_uniq . . . . .	181	ZXV_ZX_fromdigits . . . . .	194
ZV_sort_uniq_shallow . . . . .	181	ZXV_Z_mul . . . . .	197
ZV_sum . . . . .	181	ZXXT_to_FlxXT . . . . .	169
zv_sum . . . . .	185	ZXXV_to_FlxXV . . . . .	169
zv_sumpart . . . . .	185	ZXX_evalx0 . . . . .	198
ZV_togglesign . . . . .	180	ZXX_max_lg . . . . .	198
ZV_to_F2v . . . . .	120	ZXX_mul_Kronecker . . . . .	199
ZV_to_F3v . . . . .	121	ZXX_nv_mod_tree . . . . .	161
ZV_to_Flv . . . . .	169	ZXX_Q_mul . . . . .	198
zv_to_Flv . . . . .	170	ZXX_renormalize . . . . .	198
ZV_to_nv . . . . .	170	ZXX_sqr_Kronecker . . . . .	199
ZV_to_zv . . . . .	170	ZXX_to_F2xX . . . . .	155
zv_to_ZV . . . . .	170	ZXX_to_FlxX . . . . .	169
zv_to_zx . . . . .	171	zxX_to_FlxX . . . . .	169
ZV_union_shallow . . . . .	181	zxX_to_Kronecker . . . . .	148
ZV_zc_mul . . . . .	171	ZXX_Z_add_shallow . . . . .	198
ZV_zMs_mul . . . . .	192	ZXX_Z_divexact . . . . .	198
zv_ZM_mul . . . . .	171	ZXX_Z_mul . . . . .	198
ZV_ZM_mul . . . . .	182	ZX_add . . . . .	194
ZV_ZV_mod . . . . .	181	ZX_affine . . . . .	195
ZV_Z_dvd . . . . .	93	ZX_compositum . . . . .	197
zv_z_mul . . . . .	185	ZX_compositum_disjoint . . . . .	197
ZXC_nv_mod_tree . . . . .	161	ZX_content . . . . .	195
ZXC_to_FlxC . . . . .	169	ZX_copy . . . . .	193
ZXM_incremental_CRT . . . . .	160	ZX_deflate_max . . . . .	195
ZXM_init_CRT . . . . .	160	ZX_deflate_order . . . . .	195
ZXM_nv_mod_tree . . . . .	161	ZX_deriv . . . . .	196
ZXM_to_FlxM . . . . .	170	ZX_digits . . . . .	194
ZXn_mul . . . . .	197	ZX_disc . . . . .	196
ZXn_sqr . . . . .	197	ZX_divuexact . . . . .	194
ZXQM_mul . . . . .	198	ZX_div_by_X_1 . . . . .	194

ZX_equal	193, 197	ZX_to_mononic	195
ZX_equal1	193	zx_to_zv	171
ZX_eval1	196	zx_to_ZX	170
ZX_factor	196	ZX_translate	195
ZX_gcd	194	ZX_unscale	195
ZX_gcd_all	194	ZX_unscale2n	195
ZX_graeffe	196	ZX_unscale_div	195
ZX_incremental_CRT	160	ZX_unscale_divpow	195
ZX_init_CRT	160	ZX_Uspensky	196
ZX_is_irred	196	ZX_val	195
ZX_is_mononic	194	ZX_valrem	195
ZX_is_squarefree	196	ZX_Zp_root	164
ZX_lval	93	ZX_ZXY_resultant	196
zx_lval	201	ZX_ZXY_rnfequation	197
ZX_lvalrem	93	ZX_Z_add	194
ZX_max_lg	193	ZX_Z_add_shallow	194
ZX_mod_Xnm1	194	ZX_Z_divexact	194
ZX_mul	194, 199	zx_z_divexact	201
ZX_mulspec	194	ZX_Z_eval	195
ZX_mulu	194	ZX_Z_mul	194
ZX_neg	194	ZX_Z_normalize	195
ZX_nv_mod_tree	161	ZX_Z_sub	194
ZX_primitive_to_mononic	195	ZX_z_unscale	195
ZX_pval	93	Z_cba	175
ZX_pvalrem	93	Z_chinese	159
ZX_Q_mul	195	Z_chinese_all	159
ZX_Q_normalize	195, 311	Z_chinese_coprime	159
ZX_radical	195	Z_chinese_post	159
ZX_realroots_irred	196	Z_chinese_pre	159
ZX_rem	194	Z_content	236
ZX_remi2n	194	Z_ECM	175
ZX_renormalize	193	Z_factor	173, 175
zx_renormalize	201	Z_factor_limit	173, 174, 175
ZX_rescale	195	Z_factor_listP	174
ZX_rescale2n	195	Z_factor_until	173
ZX_rescale_lt	195	Z_FF_div	248
ZX_resultant	196	Z_incremental_CRT	160
zx_shift	201	Z_init_CRT	160
ZX_shifti	194	Z_isanypower	172, 175
ZX_sqr	194, 199	Z_isfundamental	177
ZX_sqrspec	194	Z_ispow2	172
ZX_squff	196	Z_ispower	172
ZX_sturm	196	Z_ispowerall	172
ZX_sturmpart	196	Z_issmooth	173
ZX_sturm_irred	196	Z_issmooth_fact	173
ZX_sub	194	Z_issquare	172
ZX_to_F2x	152	Z_issquareall	172
ZX_to_Flx	169	Z_issquarefree	177
zx_to_Flx	170	Z_issquarefree_fact	177

Z_lsmoothen . . . . .	174
Z_lval . . . . .	93
z_lval . . . . .	93
Z_lvalrem . . . . .	93
z_lvalrem . . . . .	93
Z_lvalrem_stop . . . . .	93
Z_nv_mod . . . . .	160
Z_pollardbrent . . . . .	175
Z_ppgle . . . . .	175
Z_ppio . . . . .	175
Z_ppo . . . . .	175
Z_pval . . . . .	93
z_pval . . . . .	93
Z_pvalrem . . . . .	93
z_pvalrem . . . . .	93
Z_smooththen . . . . .	174
Z_to_F2x . . . . .	152
Z_to_Flx . . . . .	171
Z_to_FpX . . . . .	124
Z_to_perm . . . . .	256
Z_ZC_sub . . . . .	180
Z_ZV_mod . . . . .	160
Z_ZV_mod_tree . . . . .	161
Z_ZX_sub . . . . .	194

.

_evalexpo . . . . .	64
_evallg . . . . .	63
_evalprecp . . . . .	63
_evalvalp . . . . .	63